Higher

Mathematics

UNIT 1 OUTCOME 3

Differentiation

Contents

HSN21300

OUTCOME 3

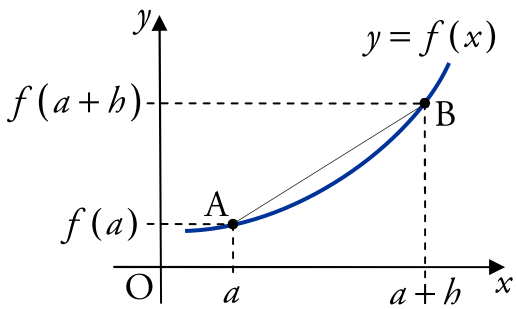
Differentiation

1 Introduction to Differentiation

From our work on Straight Lines, we saw that the gradient (or "steepness") of a line is constant. However, the "steepness" of other curves may not be the same at all points.

In order to measure the "steepness" of other curves, we can use lines which give an increasingly good approximation to the curve at a particular point.

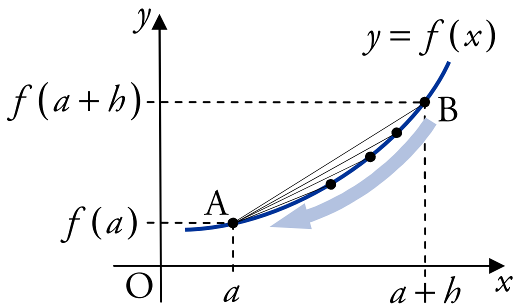
On the curve with equation $y = f(x)$, suppose point A has coordinates $(a, f(a))$.



At the point B where $x = a + h$, we have $y = f(a+h)$.

Thus the chord AB has gradient

$$\begin{aligned} m_{\text{AB}} &= \frac{f(a+h) - f(a)}{a+h-a} \\ &= \frac{f(a+h) - f(a)}{h}. \end{aligned}$$



If we let h get smaller and smaller, i.e. $h \to 0$, then B moves closer to A. This means that m_{AB} gives a better estimate of the "steepness" of the curve at the point A.

We use the notation $f'(a)$ for the "steepness" of the curve when $x = a$. So

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Given a curve with equation $y = f(x)$, an expression for $f'(x)$ is called the **derivative** and the process of finding this is called **differentiation**.

It is possible to use this definition directly to find derivates, but you will not be expected to do this. Instead, we will learn rules which allow us to quickly find derivatives for certain curves.

2 Finding the Derivative

The basic rule for differentiating $f(x)=x^n$, $n \in \mathbb{R}$, with respect to x is:

If $f(x)=x^n$ then $f'(x)=nx^{n-1}$.

Stated simply: the power (n) multiplies to the front of the x term, and the power lowers by one (giving $n-1$).

EXAMPLES

1. Given $f(x)=x^4$, find $f'(x)$.

2. Differentiate $f(x)=x^{-3}$, $x \neq 0$, with respect to x.

For an expression of the form $y=\ldots$, we denote the derivative with respect to x by $\dfrac{dy}{dx}$.

EXAMPLE

3. Differentiate $y=x^{-\frac{1}{3}}$, $x \neq 0$, with respect to x.

When finding the derivative of an expression with respect to x, we use the notation $\frac{d}{dx}$.

EXAMPLE

4. Find the derivative of $x^{\frac{3}{2}}$, $x \geq 0$, with respect to x.

Preparing to differentiate

It is important that before you differentiate, all brackets are multiplied out and there are no fractions with an x term in the denominator (bottom line).

For example:

$$\frac{1}{x^3}=x^{-3} \qquad \frac{3}{x^2}=3x^{-2} \qquad \frac{1}{\sqrt{x}}=x^{-\frac{1}{2}} \qquad \frac{1}{4x^5}=\frac{1}{4}x^{-5} \qquad \frac{5}{4\sqrt[3]{x}^2}=\frac{5}{4}x^{-\frac{2}{3}}.$$

EXAMPLES

1. Differentiate $\sqrt{x}$ with respect to x, where $x > 0$.

Note
It is good practice to tidy up your answer.

2. Given $y = \dfrac{1}{x^2}$, where $x \neq 0$, find $\dfrac{dy}{dx}$.

Terms with a coefficient

For any constant a,

$$\text{if } f(x) = a \times g(x) \text{ then } f'(x) = a \times g'(x).$$

Stated simply: constant coefficients are carried through when differentiating.

So if $f(x) = ax^n$ then $f'(x) = anx^{n-1}$.

EXAMPLES

1. A function f is defined by $f(x) = 2x^3$. Find $f'(x)$.

2. Differentiate $y = 4x^{-2}$ with respect to x, where $x \neq 0$.

3. Differentiate $\dfrac{2}{x^3}$, $x \neq 0$, with respect to x.

4. Given $y = \dfrac{3}{2\sqrt{x}}$, $x > 0$, find $\dfrac{dy}{dx}$.

Differentiating more than one term

The following rule allows us to differentiate expressions with several terms.

If $f(x) = g(x) + h(x)$ then $f'(x) = g'(x) + h'(x)$.

Stated simply: differentiate each term separately.

EXAMPLES

1. A function f is defined for $x \in \mathbb{R}$ by $f(x) = 3x^3 - 2x^2 + 5x$.

   Find $f'(x)$.

2. Differentiate $y = 2x^4 - 4x^3 + 3x^2 + 6x + 2$ with respect to *x*.

Note

The derivative of an *x* term (e.g. $3x$, $\frac{1}{2}x$, $-\frac{3}{10}x$) is always a constant.

For example:

$$\frac{d}{dx}(6x) = 6, \qquad \frac{d}{dx}\left(-\tfrac{1}{2}x\right) = -\tfrac{1}{2}.$$

The derivative of a constant (e.g. 3, 20, π) is always zero.

For example:

$$\frac{d}{dx}(3) = 0, \qquad \frac{d}{dx}\left(-\tfrac{1}{3}\right) = 0.$$

Differentiating more complex expressions

We will now consider more complex examples where we will have to use several of the rules we have met.

EXAMPLES

1. Differentiate $y = \dfrac{1}{3x\sqrt{x}}$, $x > 0$, with respect to *x*.

Note
You need to be confident working with indices and fractions.

2. Find $\dfrac{dy}{dx}$ when $y = (x-3)(x+2)$.

Remember
Before differentiating, the brackets must be multiplied out.

3. A function *f* is defined for $x \neq 0$ by $f(x) = \dfrac{x}{5} + \dfrac{1}{x^2}$. Find $f'(x)$.

4. Differentiate $\dfrac{x^4 - 3x^2}{5x}$ with respect to *x*, where $x \neq 0$.

5. Differentiate $\dfrac{x^3+3x^2-6x}{\sqrt{x}}$, $x>0$, with respect to x.

Remember
$\dfrac{x^a}{x^b}=x^{a-b}$.

6. Find the derivative of $y=\sqrt{x}\left(x^2+\sqrt[3]{x}\right)$, $x>0$, with respect to x.

Remember
$x^a x^b=x^{a+b}$.

3 Differentiating with Respect to Other Variables

So far we have differentiated functions and expressions with respect to x. However, the rules we have been using still apply if we differentiate with respect to any other variable. When modelling real-life problems we often use appropriate variable names, such as t for time and V for volume.

EXAMPLES

1. Differentiate $3t^2-2t$ with respect to t.

2. Given $A(r)=\pi r^2$, find $A'(r)$.

Remember
π is just a constant.

When differentiating with respect to a certain variable, all other letters are treated as constants.

EXAMPLE

3. Differentiate px^2 with respect to p.

4 Rates of Change

The derivative of a function describes its "rate of change". This can be evaluated for specific values by substituting them into the derivative.

EXAMPLES

1. Given $f(x) = 2x^5$, find the rate of change of f when $x = 3$.

2. Given $y = \frac{1}{x^{\frac{2}{3}}}$ for $x \neq 0$, calculate the rate of change of y when $x = 8$.

Displacement, velocity and acceleration

The velocity v of an object is defined as the rate of change of displacement s with respect to time t. That is:

$$v = \frac{ds}{dt}.$$

Also, acceleration a is defined as the rate of change of velocity with respect to time:

$$a = \frac{dv}{dt}.$$

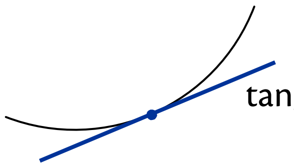
EXAMPLE

3. A ball is thrown so that its displacement s after t seconds is given by
   $s(t) = 23t - 5t^2$.

   Find its velocity after 2 seconds.

5 Equations of Tangents

As we already know, the gradient of a straight line is constant. We can determine the gradient of a curve, at a particular point, by considering a straight line which touches the curve at the point. This line is called a **tangent**.



The gradient of the tangent to a curve $y = f(x)$ at $x = a$ is given by $f'(a)$.

This is the same as finding the rate of change of f at a.

To work out the equation of a tangent we use $y - b = m(x - a)$. Therefore we need to know two things about the tangent:

- a point, of which at least one coordinate will be given;
- the gradient, which is calculated by differentiating and substituting in the value of x at the required point.

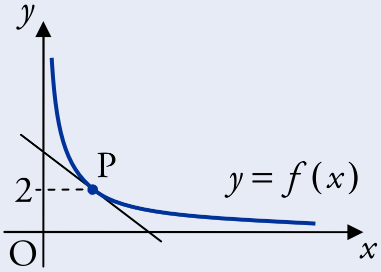
EXAMPLES

1. Find the equation of the tangent to the curve with equation $y = x^2 - 3$ at the point $(2, 1)$.

2. Find the equation of the tangent to the curve with equation $y = x^3 - 2x$ at the point where $x = -1$.

3. A function f is defined for $x > 0$ by $f(x) = \frac{1}{x}$.

Find the equation of the tangent to the curve $y = f(x)$ at P.



4. Find the equation of the tangent to the curve $y = \sqrt[3]{x}^2$ at the point where $x = -8$.

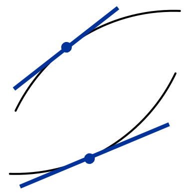
5. A curve has equation $y = \frac{1}{3}x^3 - \frac{1}{2}x^2 + 2x + 5$.

   Find the coordinates of the points on the curve where the tangent has gradient 4.

6 Increasing and Decreasing Curves

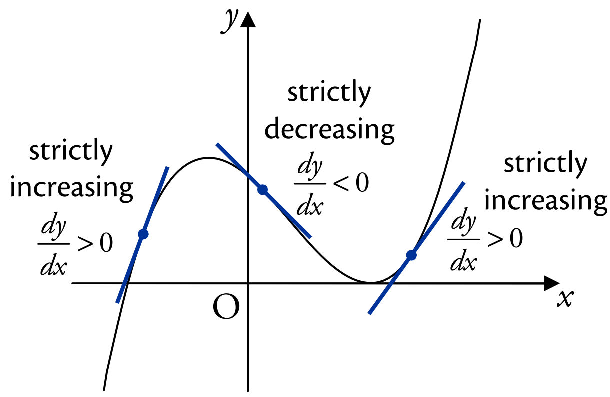
A curve is said to be **strictly increasing** when $\frac{dy}{dx} > 0$.

This is because when $\frac{dy}{dx} > 0$, tangents will slope upwards from left to right since their gradients are positive. This means the curve is also "moving upwards", i.e. strictly increasing.



Similarly:

A curve is said to be **strictly decreasing** when $\frac{dy}{dx} < 0$.



EXAMPLES

1. A curve has equation $y = 4x^2 + \frac{2}{\sqrt{x}}$.

   Determine whether the curve is increasing or decreasing at $x = 10$.

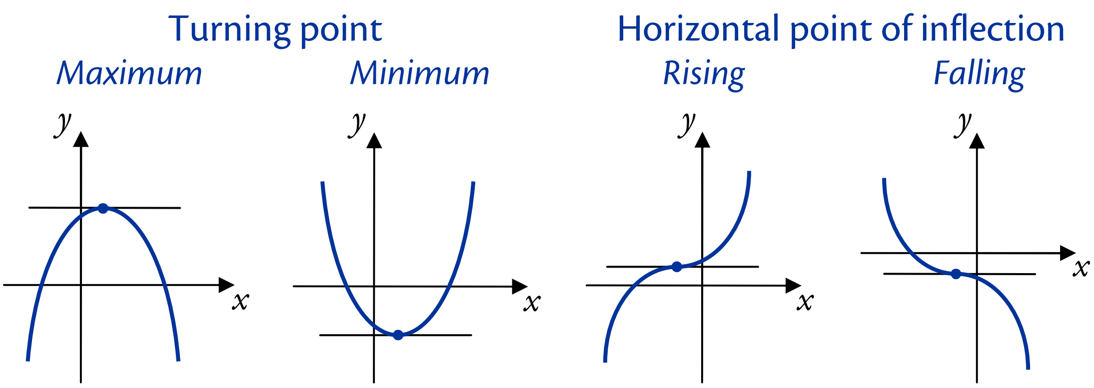
2. Show that the curve $y = \frac{1}{3}x^3 + x^2 + x - 4$ is never decreasing.

7 Stationary Points

At some points, a curve may be neither increasing nor decreasing – we say that the curve is **stationary** at these points.

This means that the gradient of the tangent to the curve is zero at stationary points, so we can find them by solving $f'(x) = 0$ or $\frac{dy}{dx} = 0$.

The four possible stationary points are:



A stationary point's nature (type) is determined by the behaviour of the graph to its left and right. This is often done using a "nature table".

8 Determining the Nature of Stationary Points

To illustrate the method used to find stationary points and determine their nature, we will do this for the graph of $f(x)=2x^3-9x^2+12x+4$.

Step 1

Differentiate the function.

$$f'(x)=6x^2-18x+12$$

Step 2

Find the stationary values by solving $f'(x)=0$.

$$\begin{aligned} f'(x)&=0 \\ 6x^2-18x+12&=0 \\ 6(x^2-3x+2)&=0 \quad (\div 6) \\ (x-1)(x-2)&=0 \\ x=1 \text{ or } x&=2 \end{aligned}$$

Step 3

Find the y-coordinates of the stationary points.

$f(1)=9$ so $(1,9)$ is a stat. pt.

$f(2)=8$ so $(2,8)$ is a stat. pt.

Step 4

Write the stationary values in the top row of the nature table, with arrows leading in and out of them.

x	$\rightarrow$ 1 $\rightarrow$	$\rightarrow$ 2 $\rightarrow$
$f'(x)$		
Graph		

Step 5

Calculate $f'(x)$ for the values in the table, and record the results. This gives the gradient at these x values, so zeros confirm that stationary points exist here.

x	$\rightarrow$ 1 $\rightarrow$	$\rightarrow$ 2 $\rightarrow$
$f'(x)$	0	0
Graph		

Step 6

Calculate $f'(x)$ for values slightly lower and higher than the stationary values and record the sign in the second row, e.g.: $f'(0.8)>0$ so enter $+$ in the first cell.

x	$\rightarrow$ 1 $\rightarrow$	$\rightarrow$ 2 $\rightarrow$
$f'(x)$	$+$ 0 $-$	$-$ 0 $+$
Graph		

Step 7

We can now sketch the graph near the stationary points:

$+$ means the graph is increasing and

$-$ means the graph is decreasing.

x	$\rightarrow$ 1 $\rightarrow$	$\rightarrow$ 2 $\rightarrow$
$f'(x)$	$+$ 0 $-$	$-$ 0 $+$
Graph	/ ‾ \	\ _ /

Step 8

The nature of the stationary points can then be concluded from the sketch.

$(1,9)$ is a max. turning point.

$(2,8)$ is a min. turning point.

EXAMPLES

1. A curve has equation $y = x^3 - 6x^2 + 9x - 4$.

   Find the stationary points on the curve and determine their nature.

2. Find the stationary points of $y = 4x^3 - 2x^4$ and determine their nature.

3. A curve has equation $y = 2x + \frac{1}{x}$ for $x \neq 0$. Find the x-coordinates of the stationary points on the curve and determine their nature.

9 Curve Sketching

In order to sketch a curve, we first need to find the following:

- x-axis intercepts (roots) – solve $y = 0$;
- y-axis intercept – find y for $x = 0$;
- stationary points and their nature.

EXAMPLE

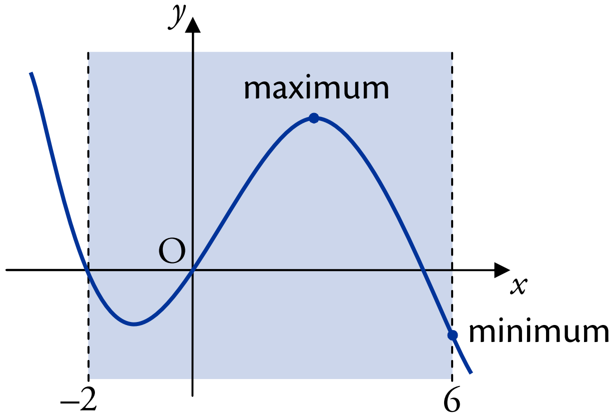
Sketch the curve with equation $y = 2x^3 - 3x^2$.

10 Closed Intervals

Sometimes it is necessary to restrict the part of the graph we are looking at using a **closed interval** (also called a restricted domain).

The maximum and minimum values of a function can either be at its stationary points *or* at the end points of a closed interval.

Below is a sketch of a curve with the closed interval $-2 \leq x \leq 6$ shaded.



Notice that the minimum value occurs at one of the end points in this example. It is important to check for this whenever we are dealing with a closed interval.

EXAMPLE

A function f is defined for $-1 \leq x \leq 4$ by $f(x) = 2x^3 - 5x^2 - 4x + 1$.

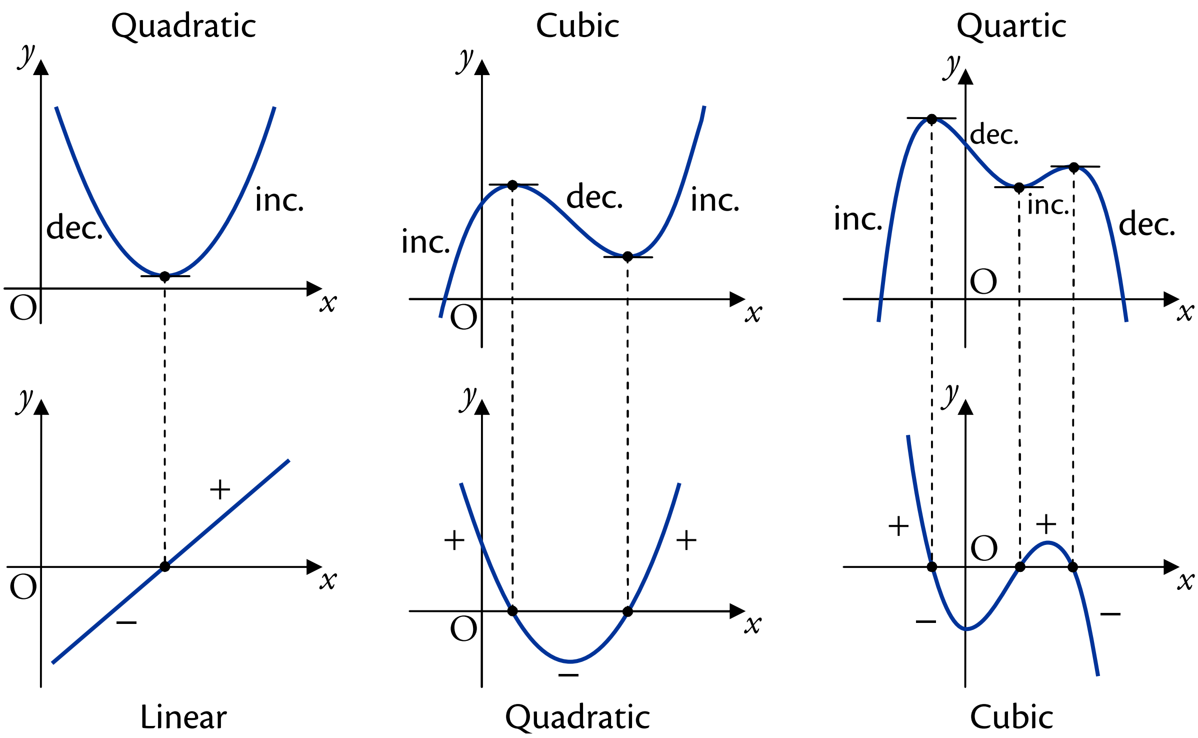
Find the maximum and minimum value of $f(x)$.

11 Graphs of Derivatives

The derivative of an x^n term is an x^{n-1} term – the power lowers by one. For example, the derivative of a cubic (where x^3 is the highest power of x) is a quadratic (where x^2 is the highest power of x).

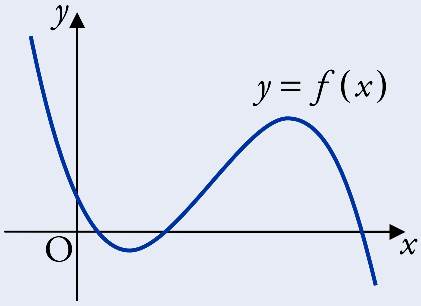
When drawing a derived graph:

- All stationary points of the original curve become roots (i.e. lie on the x-axis) on the graph of the derivative.
- Wherever the curve is strictly decreasing, the derivative is negative. So the graph of the derivative will lie below the x-axis – it will take negative values.
- Wherever the curve is strictly increasing, the derivative is positive. So the graph of the derivative will lie above the x-axis – it will take positive values.



EXAMPLE

The curve $y = f(x)$ shown below is a cubic. It has stationary points where $x = 1$ and $x = 4$.



Sketch the graph of $y = f'(x)$.

12 Optimisation

In the section on closed intervals, we saw that it is possible to find maximum and minimum values of a function.

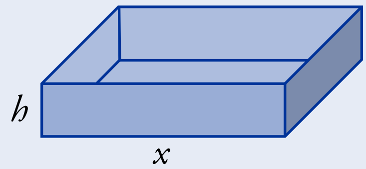
This is often useful in applications; for example a company may have a function $P(x)$ which predicts the profit if £x is spent on raw materials – the management would be very interested in finding the value of x which gave the maximum value of $P(x)$.

The process of finding these optimal values is called **optimisation**.

Sometimes you will have to find the appropriate function before you can start optimisation.

EXAMPLE

1. Small plastic trays, with open tops and square bases, are being designed. They must have a volume of 108 cubic centimetres.



The internal length of one side of the base is x centimetres, and the internal height of the tray is h centimetres.

(a) Show that the total internal surface area A of one tray is given by

$$A = x^2 + \frac{432}{x}.$$

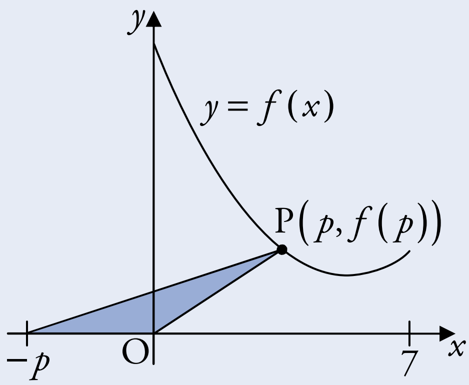
(b) Find the dimensions of the tray using the least amount of plastic.

Optimisation with closed intervals

In practical situations, there may be bounds on the values we can use. For example, the company from before might only have £100 000 available to spend on raw materials. We would need to take this into account when optimising.

Recall from the section on Closed Intervals that the maximum and minimum values of a function can occur at turning points *or* the endpoints of a closed interval.

2. The point P lies on the graph of $f(x) = x^2 - 12x + 45$, between $x = 0$ and $x = 7$.



A triangle is formed with vertices at the origin, P and $(-p, 0)$.

(a) Show that the area, A square units, of this triangle is given by

$$A = \tfrac{1}{2}p^3 - 6p^2 + \tfrac{45}{2}p.$$

(b) Find the greatest possible value of A and the corresponding value of p for which it occurs.