SCHOLAR Study Guide

SQA Higher Mathematics Unit 1

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First published 2001 by Heriot-Watt University

This edition published in 2007 by Heriot-Watt University

SCHOLAR Study Guide Unit 1: Mathematics

1. Mathematics

Printed and bound in Great Britain by Graphic and Printing Services, Heriot-Watt University, Edinburgh.

Acknowledgements

Thanks are due to the members of Heriot-Watt University's SCHOLAR team who planned and created these materials, and to the many colleagues who reviewed the content.

We would like to acknowledge the assistance of the education authorities, colleges, teachers and students who contributed to the SCHOLAR programme and who evaluated these materials.

Grateful acknowledgement is made for permission to use the following material in the SCHOLAR programme:

The Scottish Qualifications Authority for permission to use Past Papers assessments.

The Scottish Executive for financial support.

Contents

Topic 1

Properties of the straight line

Contents

Learning Objectives

- *Use the properties of the straight line*

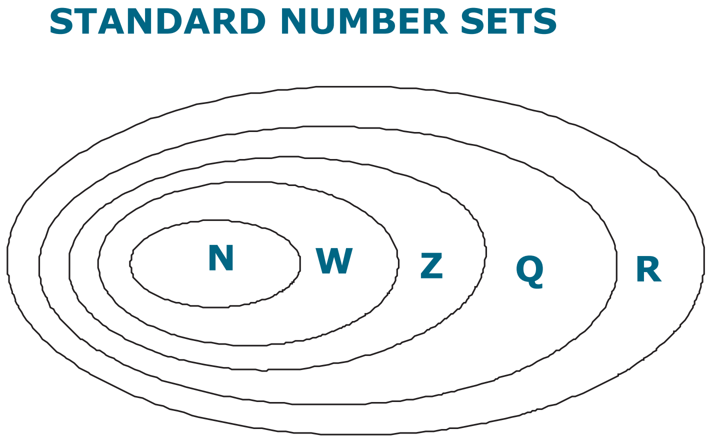
Minimum performance criteria:

- *Determine the equation of a straight line given two points on the line or one point and the gradient*
- *Find the gradient of a straight line using m = tan* θ
- *Find the equation of a line parallel to and a line perpendicular to a given line*

Prerequisites

A sound knowledge of the following subjects is required for this unit and this course in general:

- The symbols $\in$, $\notin$, {} in set theory
- The terms set, subset, empty set, member, element
- Number set convention:



where

$\mathbb{N}$ is the set of natural numbers, {1, 2, 3, ...}

$\mathbb{W}$ is the set of whole numbers, {0, 1, 2, 3, ...}

$\mathbb{Z}$ is the set of integers

$\mathbb{Q}$ is the set of rational numbers

$\mathbb{R}$ is the set of reals

1.1 Revision exercise

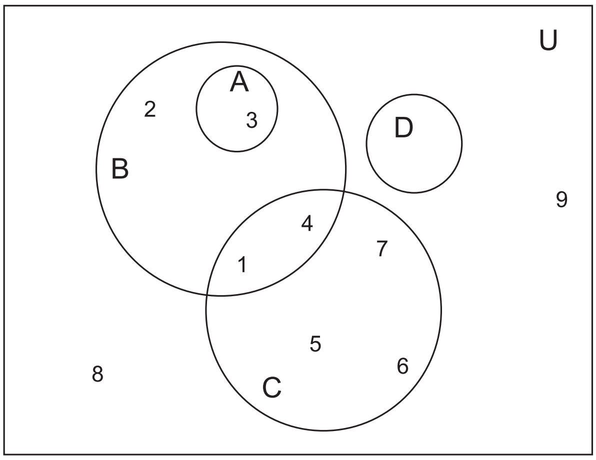
Learning Objective

Identify areas which need revision

Revision exercise

20 min

Q1: The following diagram represents a set relationship.

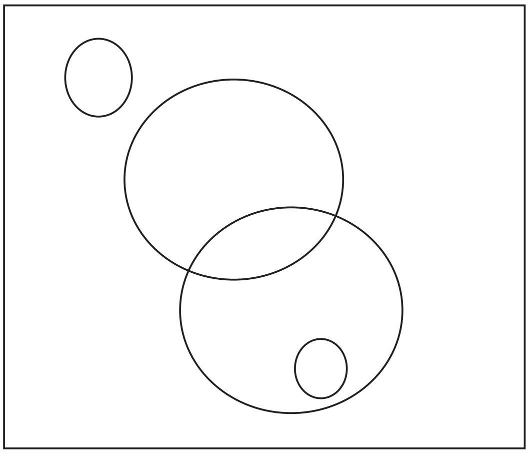


a) Which is the universal set and in set notation write out its members?
b) Name the empty set.
c) How many elements are in set B?
d) Which set is a subset of another set?
e) Identify the elements of set C.
f) Is 4 a member of set B?

Q2: Identify the smallest number set ($\mathbb{N}$, $\mathbb{W}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$) to which the following numbers belong:

a) $\sqrt{2}$
b) 0
c) 4
d) -3
e) $^2/_3$
f) -0.006

Q3: Enter the numbers and letters correctly in the following diagram using the information given.



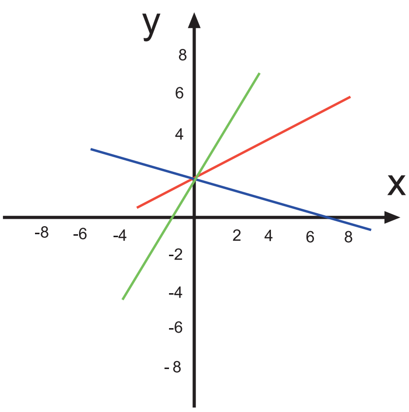
1. The universal set = {11, 13, 15, 17, 19, 21}
2. A is a subset of B
3. C = $\emptyset$
4. 15 $\in$ A
5. 13 $\in$ B and D
6. 17, 19 and 21 are members of D

1.2 Gradients of straight lines

Learning Objective

Find the gradient of a straight line

The following straight lines drawn on the graph have one feature in common and yet they are all different.

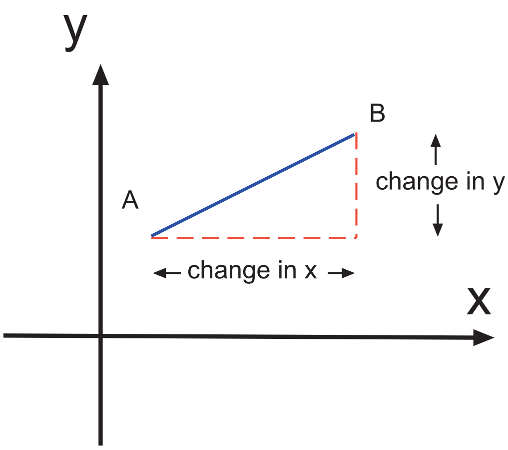


They all cross the y-axis at the point (0, 2) but the slopes of the lines are different.

Gradient

The slope of a straight line from the point A (x_1, y_1) to B (x_2, y_2) is called the gradient.

It is denoted m_{AB} and defined as $m_{AB} = \frac{y_2 - y_1}{x_2 - x_1}$ where $x_1 \neq x_2$



A simple way of remembering the formula is

$$m_{AB} = \frac{\text{change in y}}{\text{change in x}}$$

Example : Finding the gradient

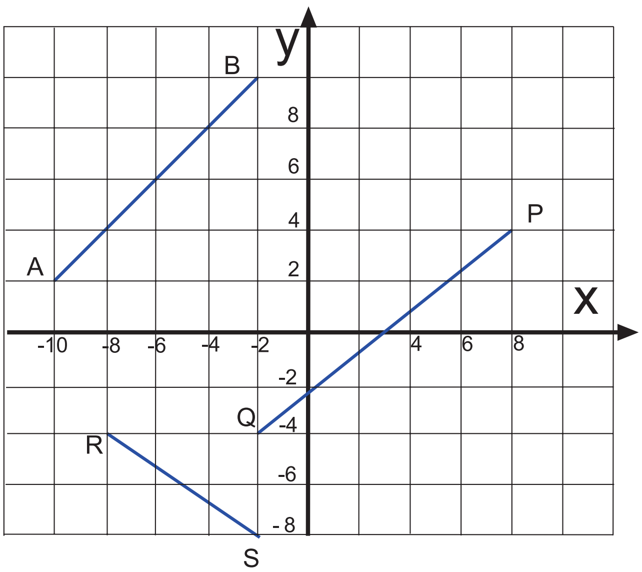
Find the gradient of the following straight lines:

a) AB

b) PQ

c) RS

d) QP



Answer:

The gradients are:

a) $m_{AB} = \frac{10 - 2}{- 2 - (- 10)} = \frac{8}{8} = 1$

b) $m_{PQ} = \frac{- 4 - 4}{- 2 - 8} = \frac{- 8}{- 10} = \frac{4}{5}$

c) $m_{RS} = \frac{- 8 - (- 4)}{- 2 - (- 8)} = \frac{- 4}{6} = - \frac{2}{3}$

d) $m_{QP} = \frac{4 - (- 4)}{8 - (- 2)} = \frac{8}{10} = \frac{4}{5}$

Notice that $m_{PQ} = m_{QP}$

It does not matter which point is taken first.

It is however, essential that the x-coordinates are taken in the same order as the y-coordinates.

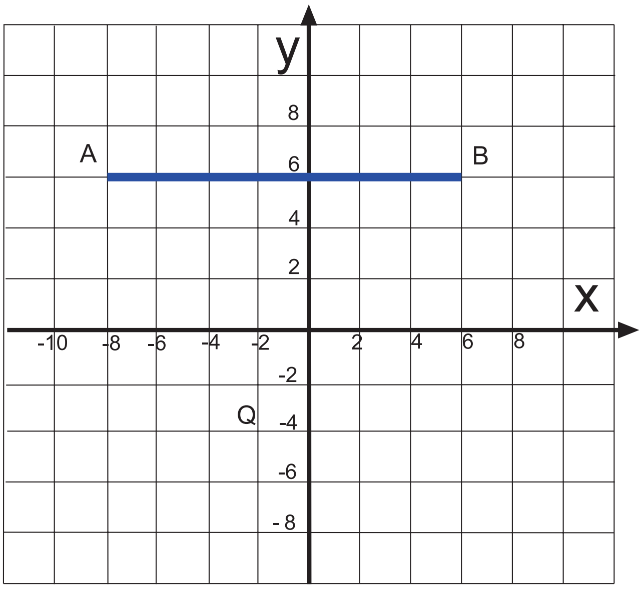
Gradients which have a plus value are called positive gradients and slope upwards from left to right. Negative gradients have a minus value and slope downwards from left to right.

There are two further sets of lines of particular interest:

- Those parallel to the x-axis and
- Those parallel to the y-axis

Example : Gradient of a straight line parallel to the x-axis

Find the gradient of the line AB shown:



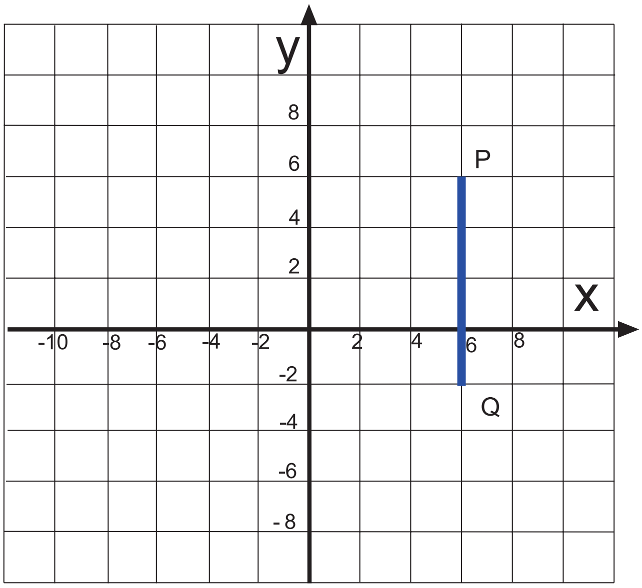
Answer:

$$m_{AB} = \frac{6 - 6}{6 - (-8)} = \frac{0}{14} = 0$$

In fact, all lines parallel to the x-axis have a gradient of 0 since the top line of the equation will always be 0

Example : Gradient of a straight line parallel to the y-axis

Find the gradient of PQ in the following diagram:



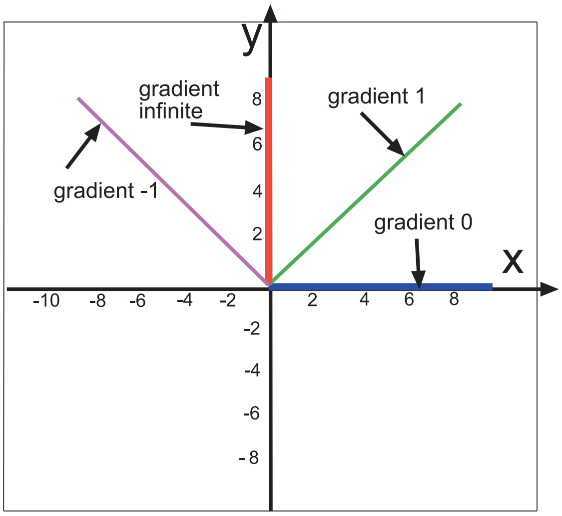
Answer:

$m_{PQ} = \frac{-2-6}{6-6} = \frac{-8}{0}$ but this is undefined by the definition that $x_1 \neq x_2$

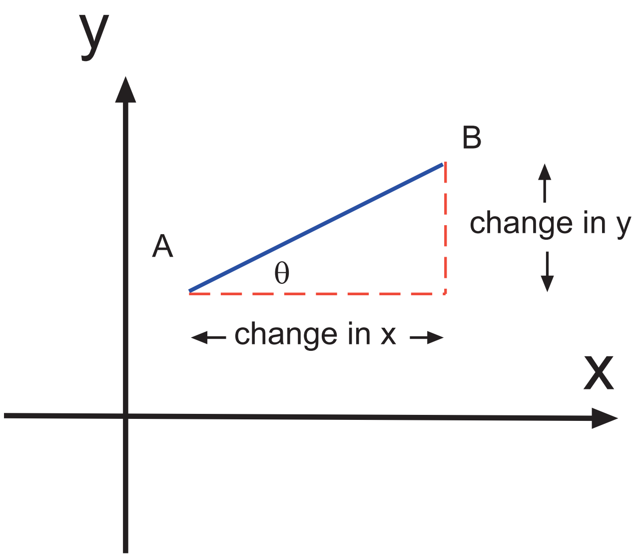
(dividing by zero is impossible).

This line has infinite gradient or has an undefined gradient.

All lines parallel to the y-axis have infinite gradient.



A further look at the gradient formula will give another useful way of calculating the gradient.



From the diagram $\tan \theta = \dfrac{\text{change in y}}{\text{change in x}}$

But $m_{AB} = \dfrac{\text{change in y}}{\text{change in x}}$ and so $m_{AB} = \tan \theta$

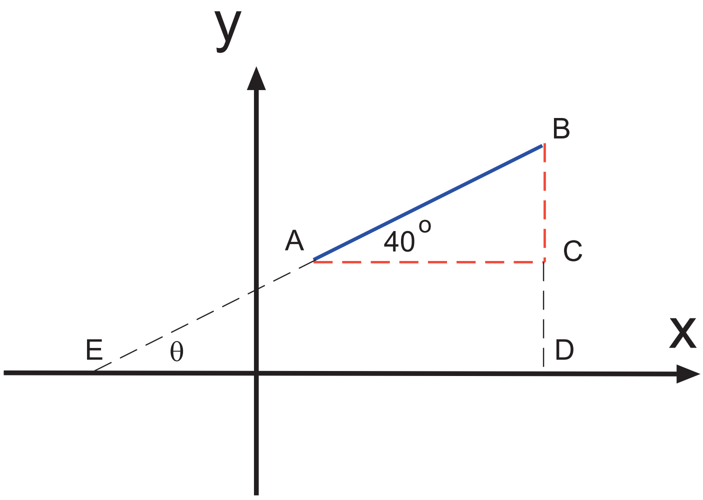
Gradient as a tangent

The gradient of a straight line is the tangent of the angle made by the line and the positive direction of the x-axis. (assuming that the scales are equal).

If A is the point (x_1, y_1) and B is the point (x_2, y_2) then $m_{AB} = \tan \theta$ where θ is the angle made by the line and the positive direction of the x-axis.

Example : Gradient of a straight line given the angle

Find the gradient of AB:



Answer:

Angle BAC = angle BED and so $\theta = 40°$

$m_{AB} = \tan \theta = \tan 40° = 0.8$ to one decimal place.

Gradients of straight lines exercise

10 min

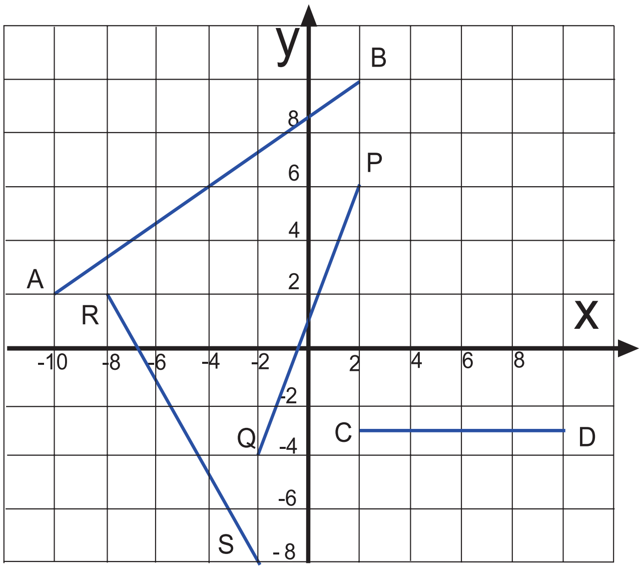
Q4: Find the gradients of the following straight lines:

a) AB

b) PQ

c) RS

d) CD



Q5: Find the gradient of the lines joining the points:

a) M (-4, 6) and N (3, 2)

b) L (-3, -3) and K (-3, 0)

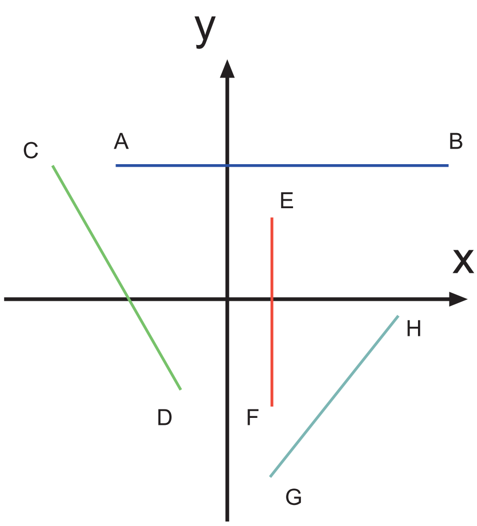
c) P (2, -6) and Q (-4, -6)

d) T (4, -1) and the origin O.

Q6: Triangle ABC is an equilateral triangle with base AB on the x-axis and apex C above the x-axis. Find the gradient of each of the sides correct to 1 decimal place.

Q7: Draw a line through the point (2, 4) which has a gradient of -1 and state where it crosses the x and y axes.

Q8: Identify the gradients of the lines as : positive, negative, zero or infinite:

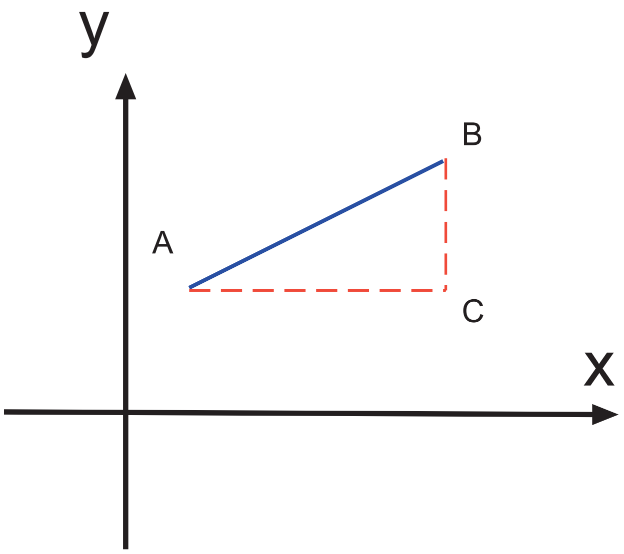


1.2.1 Distance between two points on a straight line

Learning Objective

Find the distance between two points on a straight line.

Given two points on a straight line, the gradient can be found but it can also be useful to know the distance between two points on the line.



By Pythagoras $AB^2 = AC^2 + BC^2$

Let A have coordinates (x_1, y_1) and B have coordinates (x_2, y_2)

Then C has coordinates (x_2, y_1)

The Pythagoras equation becomes

$AB^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$

and the length of AB = $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

The distance between the two points A $(x_1, y_1)$and B (x_2, y_2) is

$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Example : Distance between two points

Find the distance between the points (3, 4) and (-2, 3)

Answer:

Using the formula,

$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} =$

$\sqrt{(-2 - 3)^2 + (3 - 4)^2} =$

$\sqrt{26} = 5.1$ to 1d.p.

Distance between two points exercise

10 min

Q9: Find the distance between the points correct to 1d.p:

a) A (2, -1) and B (3, -5)
b) C (-2, -3) and D (1, 1)
c) E (-2, 0) and F (1, 1)

Calculator investigation and program

10 min

A program for a TI 83 graphics calculator follows which will provide questions on the gradients of straight lines. Try this if you have time and access to one.

To use the program calculate the gradient of the line shown, press enter, type in the gradient, press enter and the calculator will mark your answer.

Alternatively, investigate straight lines on a graphics calculator by plotting the equation in the form
y = mx + c for various values of m and c

```
:randInt(-5,5)⇒N
:Lbl 99
:"NX"⇒Y1
:ZDecimal
:Pause
:Disp "GRADIENT="
:Input M
:If M=N
:Then
:Disp "WELL DONE"
:Else
:Disp "TRY AGAIN"
:Goto 99
```

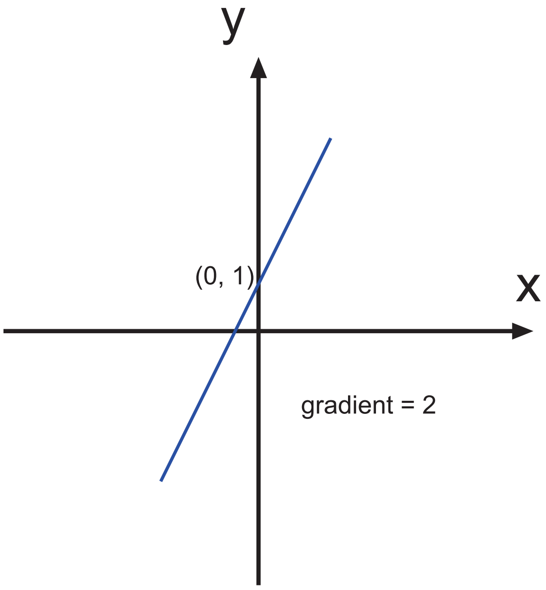
1.3 Equation of a straight line

Learning Objective

Determine the equation of a straight line

A straight line can be identified uniquely from its gradient and the point at which it crosses the y-axis (called the y-intercept).

Look at the line in the diagram:



This line is the only line with gradient 2 which passes through the point (0, 1).

Activity

Confirm that there are no other lines by trying to find one.

Equation of a straight line - standard form

The equation of a straight line takes the form y = mx + c where m is the gradient of the line and c is the y-intercept.

Examples

1. The equation of a straight line

Find the equation of the line with gradient -3 and which passes through the point (0, -4)

Answer:

The equation takes the form y = mx + c

m = -3 and c = -4 since the point (0, -4) is on the y-axis.

The equation is y = -3x - 4

2. Find where the line 6x - 2y = 8 crosses the y-axis and state its gradient.

Answer:

6x - 2y = 8 ⇒ 2y = 6x - 8 ⇒ y = 3x - 4

Thus the gradient is 3 and it crosses the y-axis at -4, that is, at the point (0, -4)

3. The point P (a, 3) lies on the line $y = 2x - 5$, find the value of a

Answer:

Substitute $y = 3$ in the equation to give

$3 = 2x - 5 \Rightarrow 2x = 8 \Rightarrow x = 4$

So when $x = 4$, $y = 3$ and the point P is (4, 3)

The value of a is 4

When a line is parallel to the x-axis the gradient is zero and the equation of the line becomes $y = c$ where c is the y-intercept.

When a line is parallel to the y-axis the line does not cross y and the equation becomes $x = k$ where k is value of x at the point where the line crosses the x-axis.

Equation of a line as y = mx + c exercise

10 min

Q10: Find the equation of the lines:

a) AB with gradient -4 and crossing the y-axis at $y = -2$
b) CD with gradient 0 and crossing the y-axis at 4
c) EF which is parallel to the y-axis and crosses the x-axis at $x = 1$
d) GH which has gradient -2 and passes through the point (0, -3)

Q11: State the gradients and y-intercepts of the lines:

a) $y = -2x - 1$
b) $y = 3x$
c) $y = -5x - 5$
d) $x = 2$
e) $y = x - 1$

The equation $6x - 2y = 8$ was previously rearranged as $y = 3x - 4$ to represent the equation of a straight line.

This equation can also be rearranged as $6x - 2y - 8 = 0$

These two equations are equal but give different forms of the equation of this straight line.

$y = 3x - 4$ is of the form $y = mx + c$

$6x - 2y - 8 = 0$ gives the general form of the equation of a straight line.

Equation of a straight line - general form

The general form of the equation of a straight line is

$Ax + By + C = 0$ (A, B not both zero).

Either form of the equation of a straight line can be used in calculations.

Conversion between equation forms exercise

10 min

Q12: Convert the following equations of a straight line into the general form and state the values for A, B and C:

a) y = -2x - 3
b) y = 3x
c) y = 2x -1
d) y = 4
e) x = -5

Q13: Convert the following equations from the general form into the form

y = mx + c and state the gradient and the y-intercept.

a) x - 4 = 0
b) y + 3 = 0
c) 2x + y - 3 = 0
d) 6x - 3y + 9 = 0

It is possible to find the equation of a straight line given either the gradient and one point on the line or alternatively given two points on the line.

The gradient of a line through two points A (x_1, y_1) and B (x, y) is given by

$$m_{AB} = \frac{y - y_1}{x - x_1}$$

Rearranging this gives $y - y_1 = m(x - x_1)$

If the gradient m and one of these points, say A is known then substitution of the values into the new equation will give the equation of the straight line. For example, if the gradient m = -2 and A is the point (5, 3) then the equation becomes

$y - 3 = -2(x - 5) \Rightarrow y = -2x + 13$

This alternative formula is useful to remember.

Equation of a straight line - alternative form
Given one point and the gradient, the equation of the straight line can be found by using the formula $y - y_1 = m(x - x_1)$ where (x_1, y_1) is the known point and m is the gradient.

Examples

1. The equation given one point and the gradient

Find the equation of the line with gradient -2 and which passes through the point (3, 4)

Answer:

$y - y_1 = m(x - x_1)$

Substituting the gradient of -2 and the point (3, 4) will give y - 4 = -2(x - 3)

That is, y = -2x + 10

The equation of the line is y = -2x + 10

2. The equation given two points

Find the equation of a line which passes through the points (2, -1) and (4, 3)

Answer:

The equation takes the form y = mx + c

The gradient m is

$$\frac{3 - (-1)}{4 - 2} = \frac{4}{2} = 2$$

Using the formula $y - y_1 = m(x - x_1)$ with the point (2, -1) gives $y + 1 = 2(x - 2) \Rightarrow y = 2x - 5$

Further equations exercise

15 min

Q14: Find the equations of the following lines:

a) AB which passes through the point (0, -2) and has gradient 4
b) GH which passes through the two points (3, 4) and (1, -2)
c) RS with gradient -4 and which passes through the point (2, -3)
d) CD which is parallel to the y-axis and passes through the point (-1, 0)
e) EF which passes through the points (0, 5) and (-3, 5)

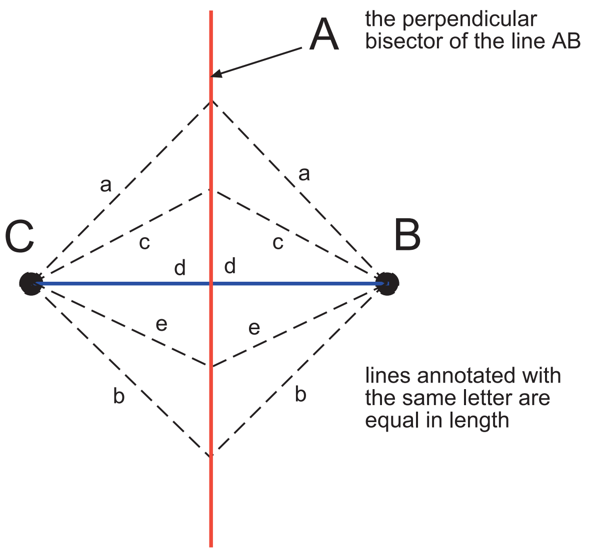
In some cases, there is more than one correct answer to a question on straight lines. For example instead of finding one particular point which satisfies the information given, the answer may be the set of points on a particular straight line. In these cases it is the locus of the point which is required.

The term locus therefore is the collection of possible answers for a particular point or line.

Example Find the locus of the point A such that the length of AB = the length of AC where BC is a straight line.

Answer:

The locus of A is the straight line which is the perpendicular bisector of the line BC and not just a point on the line.



Any of the three forms (standard, general or alternative) of the equation of a straight line can be used if the correct information is given. The standard and alternative forms are used most. A quick reference on which formula to use dependant on the information known is given in the table:

gradient	one point	y-intercept	EQUATION
yes		yes	standard
yes	yes		alternative
			look again

In some cases, of course, preliminary work may be needed to find some of the information required such as the gradient.

1.4 Properties of the gradients of straight lines

Learning Objective

Use the properties of the gradients of parallel and perpendicular lines

Q15: Find the gradients of the two lines y = 2x - 4 and y = 2x + 1

Plot them on a graph. What relationships exist between the two lines?

The answer to the last question leads to an important property of the gradients of parallel lines.

Two distinct lines are parallel if and only if the lines have equal gradients.

Example : Parallel lines

Which of the sets of lines are parallel?

a) Line A where y = -2x - 1 and line B where y = 2x + 1

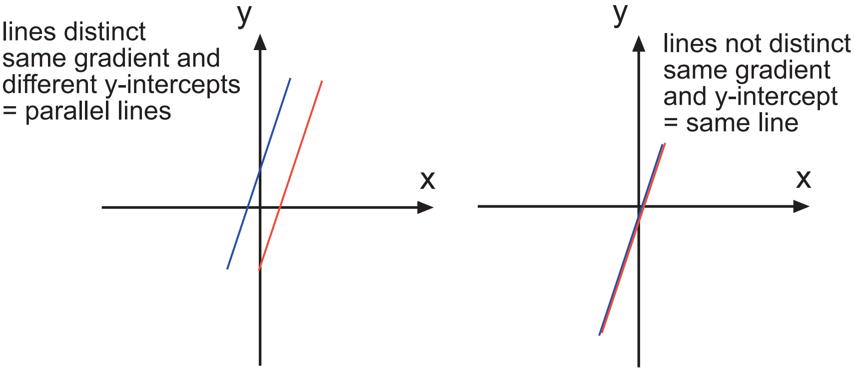
b) line AB where A = (1, 2) and B = (5, 6) and
line PQ where P = (-3, -1) and Q = (-1, 1)

Answer:

a) These are not parallel since line A has gradient -2 and line B has gradient 2

b) m_{AB} = 1, m_{PQ} = 1 The lines are parallel.

Note that if two lines have the same gradient and are not identified as distinct, inspection of the equation to find the y-intercept will reveal if they are in fact the same line.



This relationship between parallel lines and equal gradients can be used to determine whether points are collinear, lie on parallel but distinct lines, or neither.

Example : Collinearity

Identify which sets of points are collinear.

a) A (2, 4), B (4, 8) and C (3, 6)

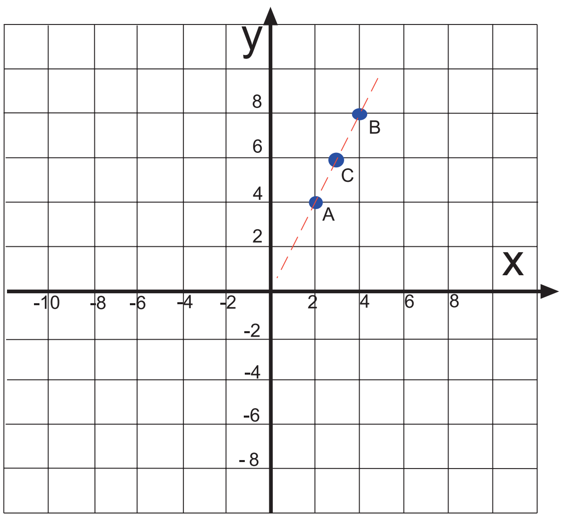
b) P (2, 3), Q (4, 5) and R (-1, -3)

c) K (-3, 2), L (-1, 1) and M (1, 1)

Answer:

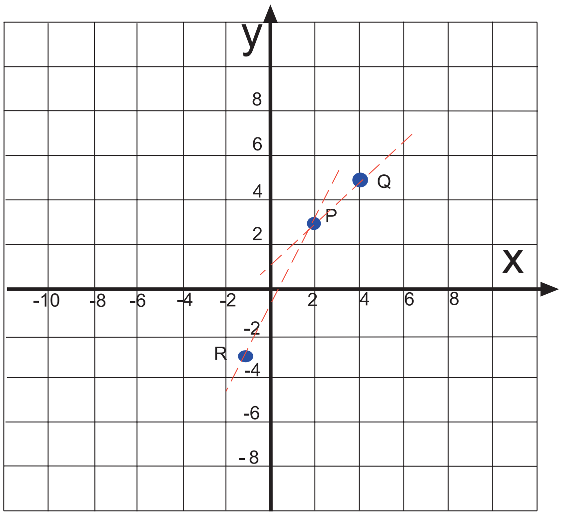
a) $m_{AB} = 2$, $m_{BC} = 2$

Since B is a common point on both lines and the gradients are the same the points are collinear. (Note: the gradients determine that the lines are parallel. The common point here in fact ensures that they are collinear.)



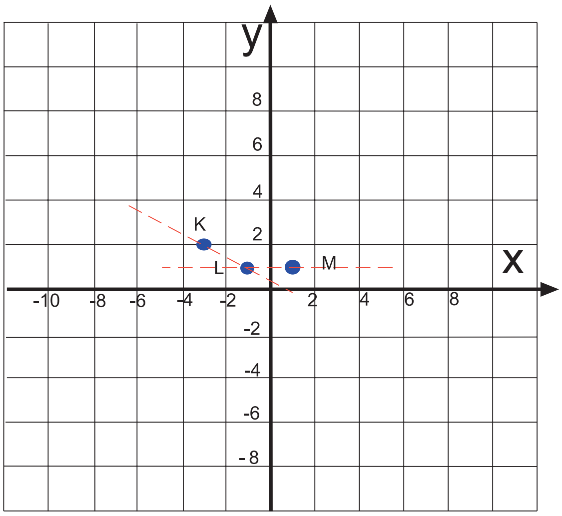
b) $m_{PQ} = 1$, $m_{QR} = {}^{8}/_{5}$

The gradients are not the same. The points are not collinear.



c) $m_{KL} = {}^{-1}/_2$, $m_{LM} = 0$

The points are not collinear since the gradients of the two lines are not the same.



Now consider the lines L with equation $y = {}^{1}/_{2}x - 3$ and M with equation $y = -2x + 4$

Q16: Plot the lines L with equation $y = {}^{1}/_{2}x - 3$ and M with equation $y = -2x + 4$ on a graph and determine the relationship between them.

The two lines in the last question are at right angles and there is a connection between this and their gradients giving another useful property of the gradients of straight lines.

Straight lines with gradients m_1 and m_2 are perpendicular if and only if $m_1m_2 = -1$

Example The gradients of a collection of lines are given. Which lines are perpendicular and which are parallel?

a) m_{AB} = -4

b) m_{CD} = $^1/_4$

c) m_{EF} = 4

d) m_{GH} = 2

e) m_{IJ} = -2

f) m_{KL} = $^1/_2$

g) m_{MN} = $-^1/_2$

h) m_{OP} = 4

Answer:

The parallel lines are OP and EF

The perpendicular lines are AB and CD, GH and MN, IJ and KL

Perpendicular and parallel lines exercise

10 min

Q17: Find the gradient of the line perpendicular to the line through the points A (3, 5) and B (1, -3)

Q18: The gradient of PQ is 4.7

The line VW is perpendicular to PQ. Find the angle to the nearest degree that VW makes with the positive x-axis.

Q19: The vertices of a quadrilateral are A (2, 3), B (3, -1), C (8, 4) and D (6, 7) Are any of the lines parallel or at right angles and what shape is the quadrilateral?

Q20: A submarine is travelling from a point P with coordinates (-3, 7) to another point Q (3, 1). A frigate is on patrol on a bearing of 135° from the point R (-1, 1). Will the frigate pass over the path of the submarine? Explain.

1.5 Concurrency properties of straight lines in triangles

Learning Objective

Use the geometrical concurrency properties of straight lines in triangles

There are several interesting properties involving straight lines and triangles. These properties are concerned with the intersection of two or more lines at a point.

Concurrency

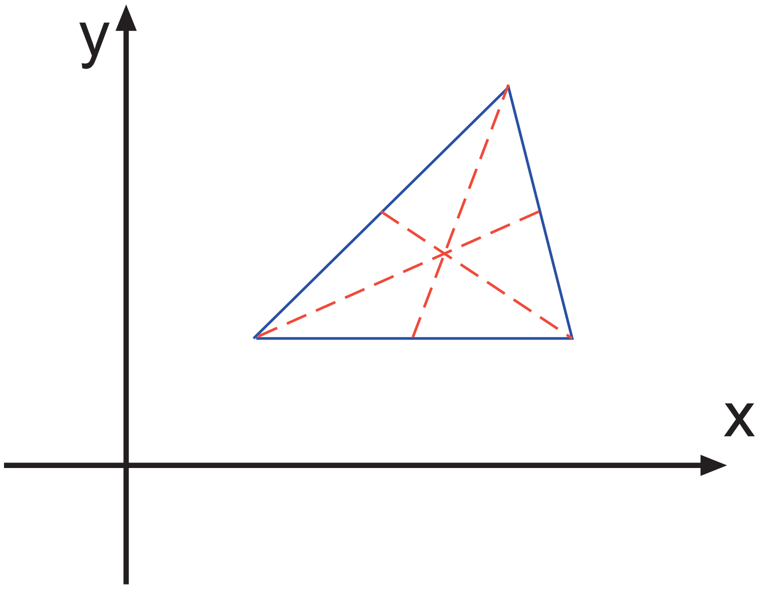
Lines which meet at a point are said to be concurrent.

1.5.1 Medians

Learning Objective

Know and use the properties of medians of a triangle.

A median of a triangle is a straight line from a vertex to the mid-point of the opposite side.



The medians of a triangle are concurrent and the point of intersection is called the centroid.


10 min

Construction activity

On graph paper construct a triangle with vertices A (2, 2), B (6, 6) and C (7, 2). Find and draw the medians. Confirm that the medians are concurrent.

The geometry is useful and clear to understand, but it is also important to understand the algebraic methods of finding the equation of a median and the point of intersection of the medians.

Strategy for finding the equation of a median

- Note the coordinates of the two end points of the line which the median meets.
- Find the midpoint coordinates of this line.
- Use the midpoint coordinates and the third vertex from where the median originates to determine the equation.

If the line AB has coordinates A (a_1, a_2) and B (b_1, b_2) the midpoint of this line is at the point with coordinates $\left(\frac{a_1 + a_2}{2}, \frac{b_1 + b_2}{2}\right)$

Example : The equation of a median

Find the equation of the median which touches the side AB in the triangle with vertices at A (3, 0), B (-3, 2) and C (2, 6)

Let the point at which the median touches AB be D.

Then the median is the line CD.

The coordinates of D which is the midpoint of AB are

$\left(\frac{a_1 + a_2}{2}, \frac{b_1 + b_2}{2}\right) = \left(\frac{3 - 3}{2}, \frac{0 + 2}{2}\right) = (0, 1)$The gradient of CD is

$\frac{6 - 1}{2 - 0} = \frac{5}{2}$

Using the equation of a straight line, $y - y_1 = m(x - x_1)$ with the point C (2, 6) gives

$y - 6 = \frac{5}{2}(x - 2) \Rightarrow y = \frac{5}{2}x + 1$

(this can also be written in the general form of $5x - 2y + 1 = 0$)

Remember: medians use midpoints. Midpoints are found from the two end points.

The point of intersection of the medians is very straighforward to find since it lies two thirds of the distance along each median measured from its vertex.

Intersection point of the medians

Let the points M (a, b), N (c, d) and P (e, f) be the vertices of a triangle. The point of intersection of the medians has coordinates

$\left(\frac{a + c + e}{3}, \frac{b + d + f}{3}\right)$

Example : The point of intersection of the medians

Find the point of intersection of the medians of a triangle which has vertices
A (2, 2), B (6, 6) and C (7, 1)

Answer:

Using the equation for the point of intersection gives

$\left(\frac{2 + 6 + 7}{3}, \frac{2 + 6 + 1}{3}\right) = (5, 3)$

The point of intersection is (5, 3)

Medians exercise

15 min

Q21: Find the equations of the medians of a triangle which has vertices
A (2, 2), B (6, 6) and C (7, 2)

Q22: Find the equations of the medians of the triangles with the following vertices:

a) A (3, 4), B (-3, 4) and C (1, 6)

b) P (-2, 0), Q (0, 4) and R (6, -2)

Q23: Find the equations of the medians and their point of intersection in the triangles with the following vertices:

a) C (1, 1), D(-3, -3) and E (5, -5)

b) K (-3, 1), L (1, -3) and M (1, 1)

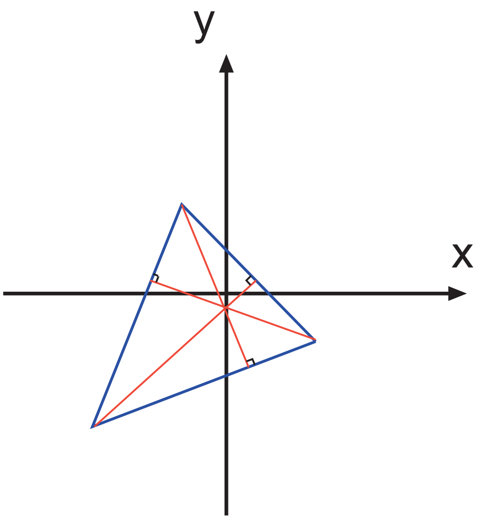
15 min

Construction activity

The point of intersection of the medians can also be found by using the coordinates of the midpoints of the sides in the formula instead of the vertices of the triangle. Investigate this for a variety of triangles and confirm the result.

1.5.2 Altitudes

An altitude of a triangle is a straight line from a vertex perpendicular to the opposite side.



The altitudes of a triangle are concurrent and the point of intersection is called the orthocentre.

The equation of an altitude is straightforward to find using the property of perpendicular lines shown earlier.

Strategy for the equation of an altitude

- Find the gradient of the line that it meets.
- Use $m_1m_2 = -1$ to determine the gradient of the altitude.
- Use the coordinates of the third vertex from where the altitude drops and the gradient to determine the equation of the altitude.

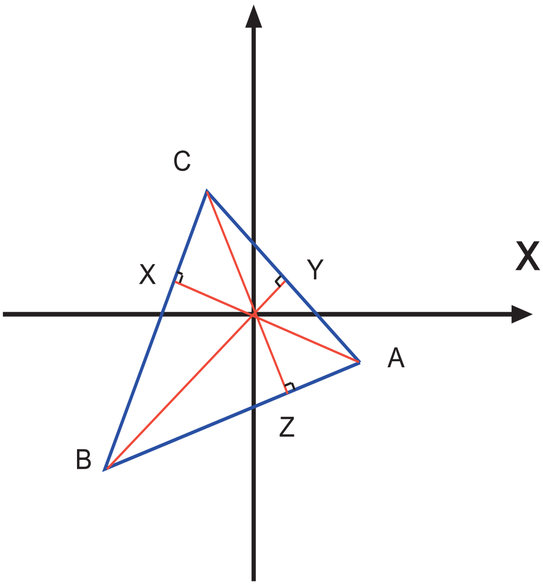
Example : Equation of an altitude

Find the equation of the altitudes of the triangle with vertices
A (2, -1), B (-3, -3) and C (-1, 2)

Answer:

Let the altitude from A to BC intersect BC at X
Let the altitude from B to AC intersect AC at Y

Let the altitude from C to AB intersect AB at Z



$m_{AB} . m_{CZ} = -1$
$m_{AC} . m_{BY} = -1$
$m_{BC} . m_{AX} = -1$

Since AB has gradient of $^2/_5$ then the gradient of CZ is $-^5/_2$

By similar calculations, the gradient of BY is 1 and the gradient of AX is $-^2/_5$

The equation of CZ is $y - 2 = -^5/_2(x + 1)$

That is, $y = -^5/_2 x - ^1/_2$

The equation of BY is $y + 3 = x + 3$

That is, $y = x$

The equation of AX is $y + 1 = -^2/_5(x - 2)$

That is, $y = -^2/_5 x - ^1/_5$

If the point of intersection of the altitudes is required this can be found by solving two equations for x or y and substituting this value in the third equation.

Altitude exercise

15 min

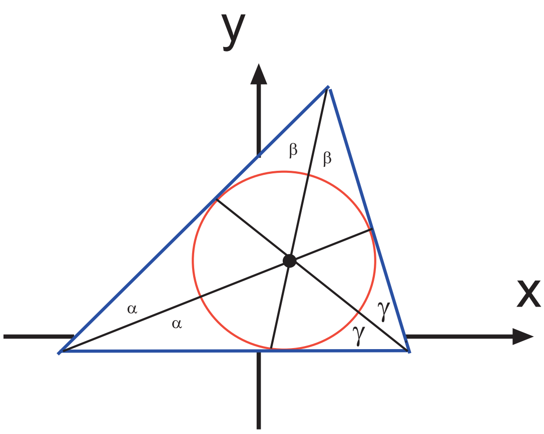
Q24: Find the equation of the altitudes in the triangles with vertices:

a) A (1, 4), B (-2, 5) and C (-1, 2)
b) K (2, 4), L (-1, -1) and M (3, 0)

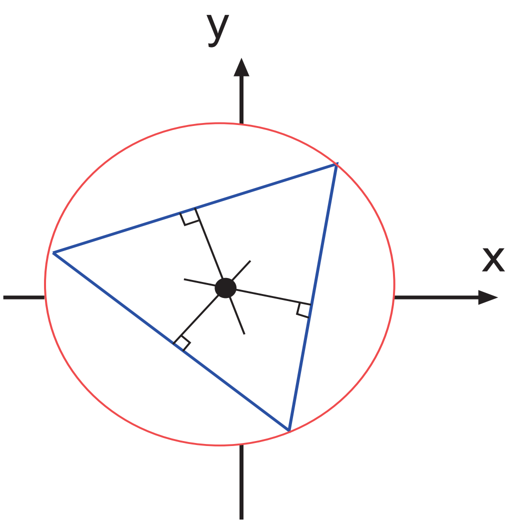
1.5.3 Bisectors of angles and sides

There are two further concurrency properties.

The bisectors of the angles of a triangle are concurrent and the point of intersection is the centre of an inscribed circle.



The perpendicular bisectors of the sides of a triangle are concurrent and the point of intersection is the centre of a circumscribed circle.



Example : Equation of a perpendicular bisector

Find the equation of the perpendicular bisector of the line AB in the triangle ABC with vertices at
A (-2, -2), B (4, -4) and C (0, 6)

Answer:

The gradient of AB is $-^1/_3$

The perpendicular bisector of AB has gradient 3 (using the property of perpendicular lines)

The mid point, say D, of AB is (1, -3)

The equation of the perpendicular bisector is therefore

y + 3 = 3(x - 1)

That is, y = 3x - 6

Bisector exercise

15 min

Q25: Find the equation of the perpendicular bisector of the line BC in the triangle with vertices A (-4, 3), B (0, 2), C (-5, -6).

State the equation in the general form Ax + By + C = 0

Q26: If the perpendicular bisector of the side AB of a triangle ABC has gradient of $-{}^{1}/_{2}$ and intersects AB at (3, -2) find the equation of the line AB

1.6 Summary

The following points and techniques should be familiar after studying this topic:

- Finding the gradient of a straight line.
- Finding the distance between two points.
- Using the different forms of the equation of a straight line dependent upon the information given.
- Using the properties of parallel and perpendicular lines.
- Using the properties of concurrency.

1.7 Extended information

Learning Objective

Display a knowledge of the additional information available on this subject

There are links on the web which give a selection of interesting sites to visit. These sites can lead to an advanced study of the topic but there are many areas which will be of passing interest.

Archimedes

Archimedes was a Greek mathematician of the 3rd century whose ideas and discoveries led to many of the geometrical results known nowadays.

Euclid

He was another famous mathematician of the same era as Archimedes. He is best known for his work on 3 dimensional solids but, in *The Elements* he laid down some fundamental beliefs concerning points and straight lines.

d'Oresme

Oresme, though not as well known as some of the prominent mathematicians and scientists is considered to be the inventor of the coordinate system. It was his work which inspired Descartes.

Descartes

An 18th century mathematician who formalised the concepts of coordinate geometry and provided the link to the algebra behind it.

Newton

In the 17th century Newton developed a new coordinate system (polar coordinates). He is however well known in the field of calculus and had many arguments with another leading mathematician (Leibniz) over this.

1.8 Review exercise

Review exercise

15 min

This is an exercise which reflects the work covered at 'C' level in this topic.

Q27: The triangle PQR has vertices at P (-2, -4), Q (4, 3) and R (1, -3). Find the equation of the perpendicular bisector of PR

Q28: The line EF is at an angle of 35° with the positive x-axis and crosses it at the point (2, 0) Find the equation of EF

Q29: Find the equation of a line through the origin and parallel to the line through the points (2, 4) and (5, -1)

1.9 Advanced review exercise

Advanced review exercise

15 min

Q30: ABC is a right angled triangle with base AC parallel to the x-axis and vertices A (2, 3) and B (6, 8). Find the equation of the altitude CD from the vertex C to the side AB in the form Ax + By + C = 0

Q31: ABCD is a parallelogram.

The coordinates of A are (1, 2) and of B are (-1, -3) The gradient of AD is 5. Find

a) the equation of AD
b) the equation of AB
c) the equation of BC
d) If DC crosses y at y = -8 find the equation of DC

1.10 Set review exercise

Set review exercise

15 min

Work out your answers and then access the web to input your answers in the online test called set review exercise.

Q32: Find the equation of the line through the points (-3, 4) and (-5, 2) and state the equation of the line which intersects it at right angles at the y-intercept.

Q33: Find the intersection of the medians of the triangle with vertices A (-1, 0), B (0, 4) and C (2, 1)

Q34: The ship is at coordinates (-2, 7) and sees the lighthouse at coordinates (3, 4).

a) If each unit is 1 nautical mile, how far away from the lighthouse is the ship?
b) The ship is heading on a bearing of 125°. On what side of the ship will the lighthouse be when the ship passes it?
c) Find the equation of the path of the ship.

Q35: A kite has a short diagonal with vertices at (-2, 3) and (5, 8)

1. Find the equation of this diagonal.
2. Hence find the equation of the leading diagonal.

Topic 2

Functions and graphs

Contents

Learning Objectives

- *Associate functions and graphs*

Minimum performance criteria:

- *Sketch and identify related graphs and functions*
- *Identify exponential and logarithmic graphs*
- *Find composite functions of the form f (g (x)), given f (x) and g (x)*

Prerequisites

A sound knowledge of the following subjects is required for this unit:

- Plotting coordinates on a graph
- Algebraic manipulation
- Straight line graphs

2.1 Revision exercise

Learning Objective

Identify areas which need revision

Revision exercise

15 min

Q1: Factorise $x^2 - 11x + 18$

Q2: Expand $3(x - 2)^2 - 12$

Q3: Factorise $x^2 - {}^1/_4$

Q4: Plot a square with opposite corners of (-3, -4) and (1, 2) and state the other two vertices. What are the coordinates of the centre of the square?

Q5: Plot the line with gradient -3 and which passes through the point (3, 4)
Another line passes through the points (2, -1) and (5, 2)
Plot this line and find the intersection of the two lines.

2.2 Function definitions

Learning Objective

Understand the basic concepts of a function and the terminology relating to it.

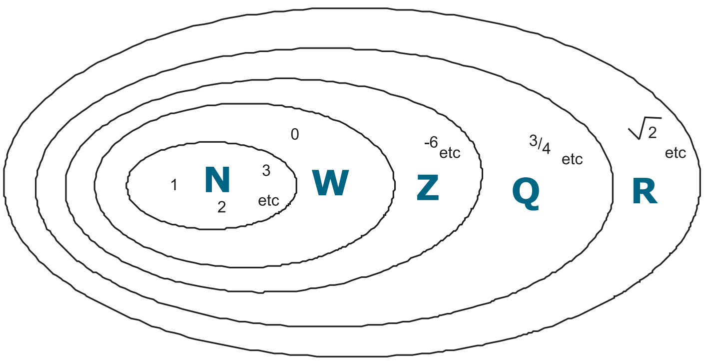
Standard number sets

The standard number sets are:

- $\mathbb{N}$ = {1, 2, 3, 4, 5, ...} the set of natural numbers.
- $\mathbb{W}$ = {0, 1, 2, 3, 4, 5, ...} the set of whole numbers.
- $\mathbb{Z}$ = {..., -3, -2, -1, 0, 1, 2, 3, ...} the set of integers.
- $\mathbb{Q}$ = the set of all numbers which can be written as fractions, called the set of rational numbers.
- $\mathbb{R}$ = the set of rational and irrational numbers (such as $\sqrt{2}$), called the set of real numbers.

These sets and their subsets (such as $\mathbb{Z}^+$ = {1, 2, 3, ...} or $\mathbb{R}^+$ = { x $\in \mathbb{R}$, x $>$ 0 }) are the only standard number sets used in this unit.

STANDARD NUMBER SETS



Function f

A function f from set A to set B is a rule which assigns to each element in A exactly one element in B. This is often written as f : A $\rightarrow$ B.

Domain of a function

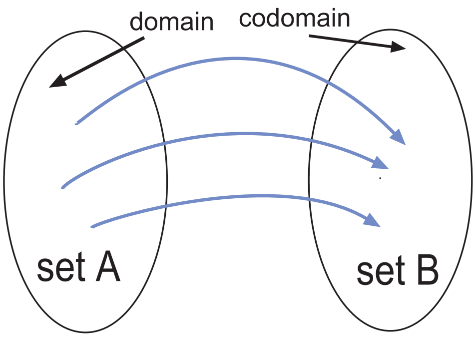
For a function f : A $\rightarrow$ B, A is called the domain of the function f.

Codomain of a function

For a function f : A $\rightarrow$ B, B is called the codomain of the function f.

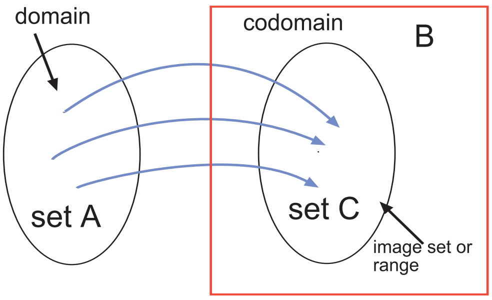
In this unit unless otherwise stated, the domain of a function will be the largest possible set of x values for which the rule defining f (x) makes sense.

For example if f (x) = $\frac{1}{x}$ then the domain is {x $\in \mathbb{R}$: x $\neq$ 0}since the rule $\frac{1}{x}$ does not make sense if x = 0



The image set or range of a function

For a function f : A → B, the set C of elements in B which are images of the elements in A under the function f is called the image set or range of the function f. C is always contained in or equal to B. This is written C ⊆ B



Finding the range of a function is not always easy.

Is it a function exercise

10 min

Q6: Use the definition of a function to determine if the following are functions:

a) $y = \sqrt{x}$ where $x \in \mathbb{R}^+$

b) $y^2 = x^2$ where $x \in \mathbb{Z}$

c) $k = x^2 - 2x$ where $x \in [0, 4]$

d) $t^2 = x$; $x \in \mathbb{Z}^+$

e) x = 3

To define a function fully it is necessary to specify:

- The domain.
- The codomain.
- The rule.

A function can be defined using different notation.

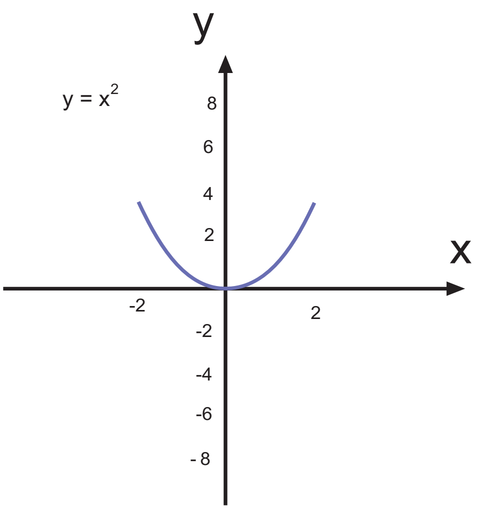
Other function notations may appear so it is important to understand the concepts. The terminology used will then be easier to understand.

When dealing with functions it is common practice to use the letter defining the function such as f instead of f (x).

Here is a selection of ways of doing this:

- Let f be the function defined by f (x) = x^2 where x $\in$ [-2, 2]
- f : [-2, 2] $\rightarrow \mathbb{R}$
  f : x $\rightarrow x^2$
- f is defined by f : x $\rightarrow x^2$ where x $\in$ [-2, 2]
- f is defined on the set T where T = {x: x $\in$ [-2, 2]} by f : x $\rightarrow x^2$
- f is defined by f (x) = x^2 with domain { x : -2 $\leq$ x $\leq$ 2}

The graph of a function f (x) is usually drawn as the curve y = f (x).



Examples

1. Function notation

Write the function f defined by

f : [0, 180°] $\rightarrow \mathbb{R}$

f : x $\rightarrow$ sin x

using two other methods of notation for a function.

Answer:

f is defined by f : x $\rightarrow$ sin x where x $\in$ [0, 180°] or

f is defined by f (x) = sin x with domain { x : 0 $\leq$ x $\leq$ 180°} are two possible solutions.

2. Finding the largest domain

Determine the largest possible domain for the following rule (note that this is NOT a function.

$k(z) = \sqrt{\frac{1}{z}}$

Answer:

$\sqrt{\frac{1}{z}}$ is undefined when z = 0 since $^1/_0$ is not defined and it is also undefined when z is negative since there is no solution in the reals (or smaller number sets) for the square root of a negative number.

This means that z must be greater than zero.

The largest possible domain is therefore $\mathbb{R}^+$ (the set of all positive real numbers).

3. Finding the range of a function

Find the range of the function f (x) = x - 7 with a domain of {1, 2, 3}

Answer:

This function has a codomain of $\mathbb{Z}$ and a range of (-6, -5, -4}

This shows that the range $\subseteq$ codomain.

4. Find the range of the function f (x) = x^2 where x $\in$ [-3, 3]

Answer:

It has a domain of [-3, 3] and a codomain of the real numbers. The rule is that every value of x maps to x^2

The range is actually [0, 9]

This is all possible values of y for x $\in$ [-3, 3]

Care must be taken to cover all situations.

Here, if the end points of the domain are used then f (x) = 9 in both cases and the fact that $0^2 = 0$ could be missed.

Notice that the range [0, 9] lies inside the codomain of $\mathbb{R}$ as required.

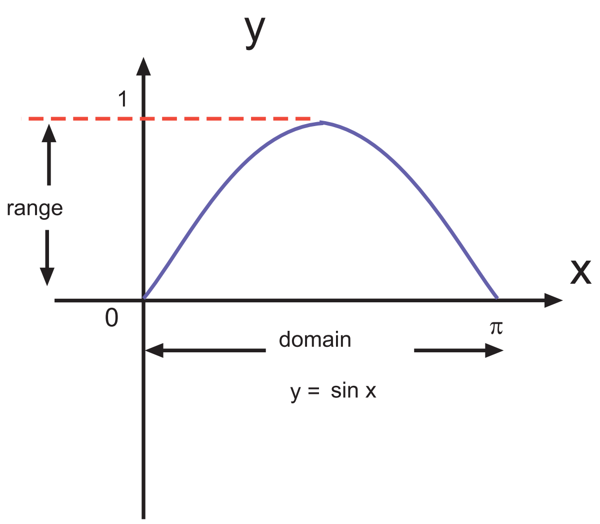
5. What is the range of the function f which maps x to sin x where x has a domain of [0, 180°] and codomain of $\mathbb{R}$?

Answer:

Note again that only part of this codomain of $\mathbb{R}$ will actually be the range. This range however will be totally included in $\mathbb{R}$.

The following points need to be considered when determining the range of the function where it is not practical to examine every value in the domain:

- SHAPE: Think about the shape of a sin graph and its maximum and minimum values. It may be that this is enough to determine the range.
- GRAPH VALUES: Consider the values which can be given to x.
  For example 0, 30°, 45°, 90°, and so on. Work out the values of sin x. Sketch the graph if it helps.



Here the range is [0, 1]

This can also be written as $0 \leq f(x) \leq 1$

or $\{f(x) \in \mathbb{R} : 0 \leq f(x) \leq 1\}$

10 min

Domain and range exercise

Q7: What is the range of the function $f(x) = x^2 - 2x$ where $\{x \in \mathbb{Z}: 0 \leq x \leq 4\}$?

Q8: What is the domain for the function $f(x) = \sqrt{x - 2}$ where $f(x) \in \mathbb{R}^+$?

Q9: What is the domain for the function $g(y) = \frac{1}{y}$?

Q10: Find the domain for the function $h(z) = (3 - z)^2 - z^{-1}$?

Q11: Work out the range of the function f with domain $\mathbb{N}$ (the natural numbers) and defined by $f(x) = \frac{1}{x}$

Q12: Find the range for the function g defined by $g(y) = y - 4$ with domain $\{ y \in \mathbb{Z} : y > 4 \}$.

Q13: Find the range of the function $h(w) = w^2 - 3$ with domain $\{w \in \mathbb{Z} : w > 1 \}$

2.3 Function inverses

Learning Objective

Find and sketch the inverse of a function

To understand when an inverse of a function exists, the following small section is needed as it explains the idea of one-to-one and onto functions.

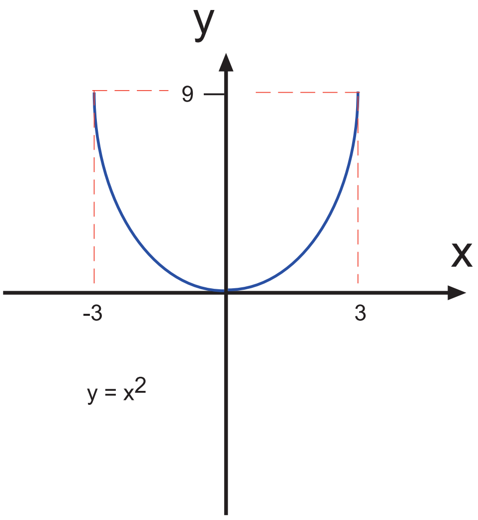
2.3.1 One-to-one and onto functions

Learning Objective

Display graphically the concept of a one-to-one or an onto function

The definition of a function states that the elements in set A must map to one and only one element of set B.

However, in some cases two elements in set A (domain) will map to the same element in set B (codomain). An example of this is the function f (x) = x^2 where the two values for x of 3 and -3 both map to the element 9 in set B.



There are functions which only map each individual element of set A to a different individual element of set B.

One-to-one function

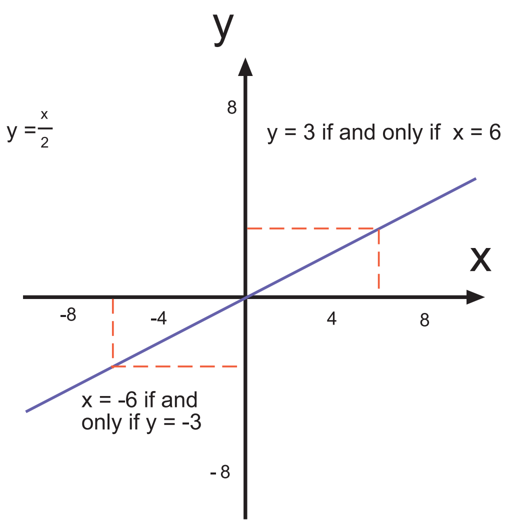
A function f : A $\rightarrow$ B is a one-to-one function if whenever f (s) = f (t)

then s = t where s $\in$ A and t $\in$ A

The function is said to be in one-to-one correspondence.

This means that each value of f (x) in the range is produced by one and only one value of x in the domain.

Consider the function f (x) = $\frac{x}{2}$



Here there is only one value in the domain, x = 6, which produces the value y = 3 in the codomain.

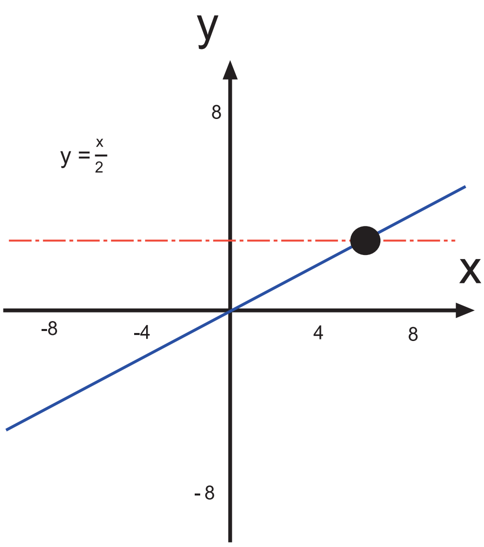
At any point on the graph of a one-to-one function this property will exist and can be checked as follows.

5 min

One-to-one demonstration

Take a horizontal line and move it up and down over the graph.

If at any point the line crosses the graph more than once then the function is not one-to-one.



The function $f(x) = \frac{x}{2}$ is a one-to-one function.

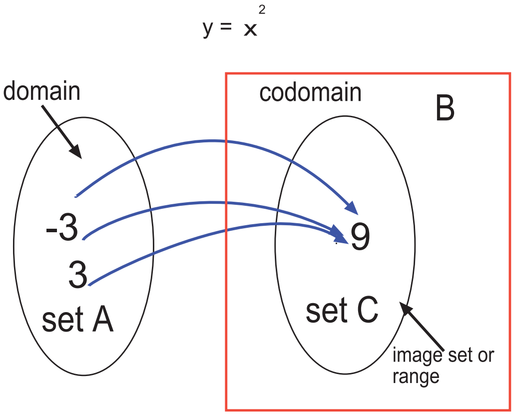
Functions which map more than one element of set A to the same element in set B also have a name.

Many-to-one function

A function which maps more than one element in the domain to the same element in the range or image set is called a many-to-one or a many-one function.

The function is said be in many-to-one correspondence.

It is also common to say that such a function is not one-to-one.



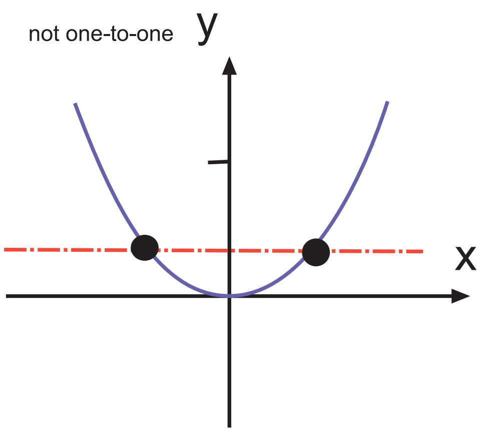
Note that f (3) = 3^2 = 9 and f (-3) = $(-3)^2$ = 9. So there are two different elements, 3 and -3, in the domain (set A) which map to 9 in the codomain (set B).

The function is not one-to-one.

Not one-to-one demonstration

Try the horizontal line approach again. Take a horizontal line and move it up and down over the graph.

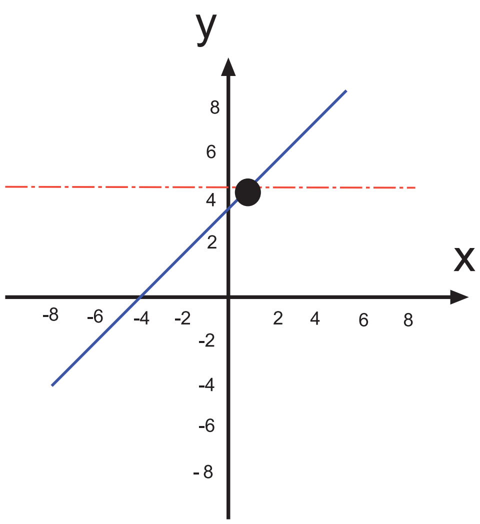
If at any point the line crosses the graph more than once then the function is not one-to-one.



Examples

1. Is f (x) = x + 4 a one-to-one function?

Answer:



This is a one-to-one function. The horizontal line only crosses it once at any point.

2. Is the function

G : $\mathbb{R} \rightarrow \mathbb{R}$
G (y) = $2y^2$ a one-to-one function?

Answer:

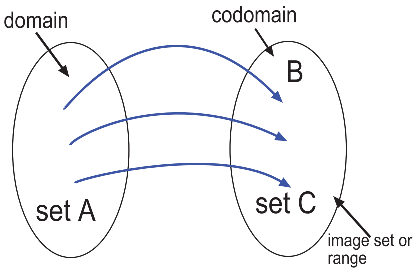
It is *not* a one-to-one function.
Here the range of the function is {G (y) $\in \mathbb{R}$: G (y) $\geq$ 0 } which is contained in but not equal to the codomain.

When the range is equal to the codomain the function has a special name.

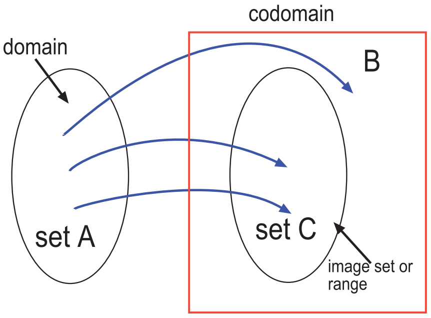
Onto function

An onto function is one in which the range is equal to the codomain.

The following diagram illustrates an onto function.



The next diagram illustrates a function which is *not* an onto function.



One-to-one and onto function exercise


10 min

Q14: Is h (z) = z^2 + 1 where z $\in \mathbb{R}$, codomain of { h (z) $\in \mathbb{R}$, h (z) $\geq$ 1} an onto function?

Q15: Is the function f (x) = -x where x $\in \mathbb{R}$ one-to-one?

Q16: Is the function g (y) = - y^2 where y $\in \mathbb{Z}$ one-to-one?

Q17: Is the function k (w) = 4w - w^2 where w $\in$ [0, 4] one-to-one?

Q18: Is the function h (z) = 4z - z^2 where z $\in \mathbb{Z}$: -2 $\leq$ z $\leq$ 2 one-to-one?

Q19: Is the function f (x) = -x where x $\in \mathbb{R}$, codomain of $\mathbb{R}$ an onto function?

Q20: Is g (y) = - y^2 where { y $\in \mathbb{Z}$: -2 $\leq$ y $\leq$ 2}, codomain of [-4, 4] an onto function?

Q21: Is g (y) = sin (y) where 0° $<$ y $<$ 90° , codomain of [-1, 1] an onto function?

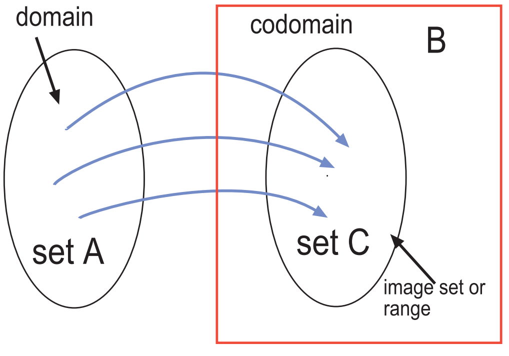
2.3.2 Inverse functions

Learning Objective

Sketch an inverse function given a graph of the function

Consider the function f (x) = y where x $\in$ A and y $\in$ B

This function maps the elements of A to elements of B



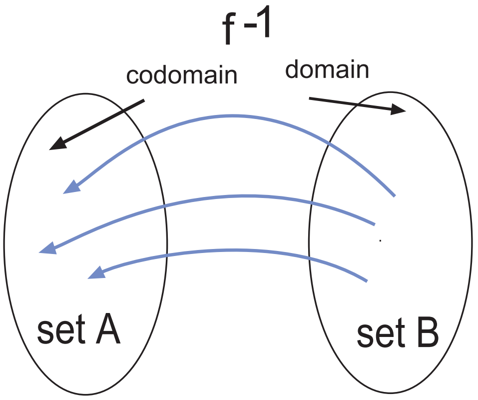
If a function exists which will map each element of B back to the element of A from which it came, this will be the inverse function of f and is denoted f^{-1}

One condition of a function is that it maps an element in the domain (set A) to only one element in the range (set B).

For an inverse of this function to exist, each element of the set B will have to map back to the element in set A from which it came.

However this only occurs when the image set of f is the whole of the codomain B.

Otherwise the range of the inverse function is not contained in the domain of the original function.

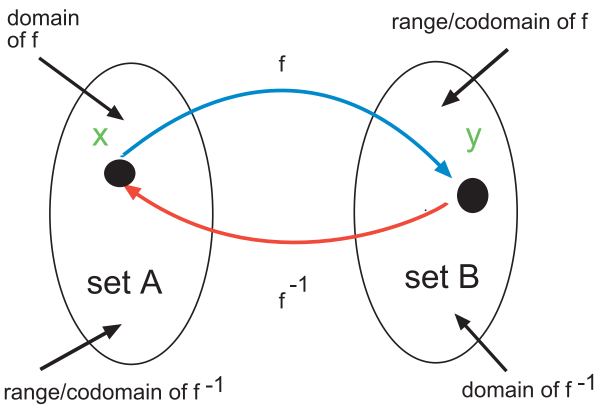


Inverse function

Suppose that f is a one-to-one and onto function. For each y ∈ B (codomain) there is exactly one element x ∈ A (domain) such that f (x) = y

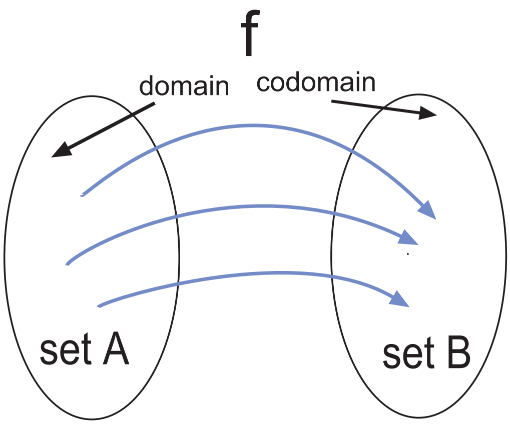
The inverse function is denoted f^{-1} (y) = x

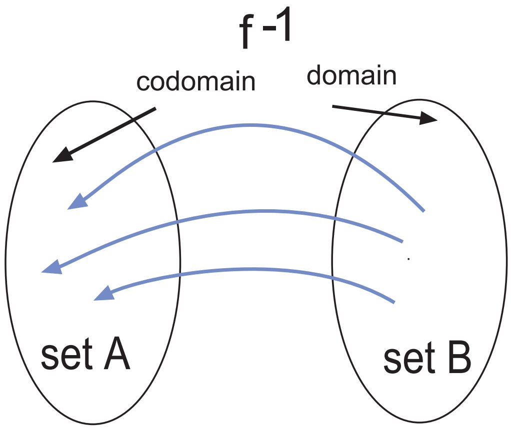
This means that each element in the domain of the function f maps to only one element of the range and in turn, this element is mapped back to the element from which it came.



The domain of the inverse function is the range of the original function.

The range of the inverse function is the domain of the original function.





The relationship between functions and their inverses is clearly seen using graphs.

The effect of interchanging x and y in the equation is the same as interchanging the axes on the graph.

This is the same as reflecting the graph of a function in the line y = x

To find the rule for an inverse function interchange y and x in the original rule, then rearrange to give a new equation for y = expression in x.

Example : The inverse of a function

Sketch the inverse function f^{-1} where f (x) = 2x is a one-to-one and onto function.

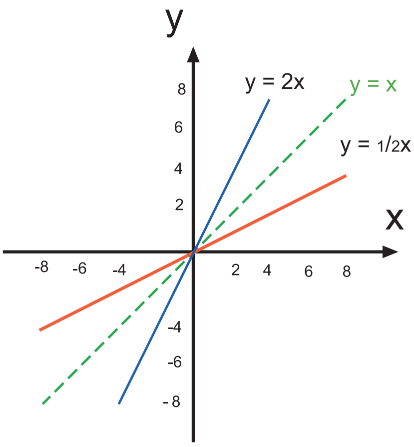
Answer:

First sketch the function f which is the line y = 2x

Reflect this graph in the line y = x to give the inverse function f^{-1}

By interchanging x and y the equation y = 2x becomes x = 2y and solving for y gives the line $y = {}^{1}/_{2}\, x$

The inverse function is $f^{-1}(x) = {}^{1}/_{2}x$



10 min

Inverse examples

There are some examples of inverses found by using reflection in the line y = x shown on the web.

When asked to find and sketch the inverse of a function it is important to check that the function in question is actually one-to-one and onto. It could be that the codomain of the original function will have to be restricted in order to find an inverse which actually exists.

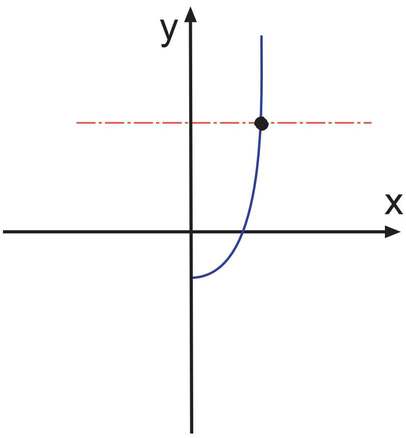
This is very important.

Example

Sketch the graph of the function $f : \mathbb{R}^+ \rightarrow [-3, \infty)$ where $f(x) = 4x^2 - 3$. Find and sketch its inverse if it exists.

Answer:

First of all check that the function is one-to-one.



Check now whether the function is onto.

The definition of onto means that the range equals the codomain. Here the codomain is $[-3, \infty)$ and the range is the same. The function is onto.

An inverse exists.

The equation of the inverse is found as follows:

Let f (x) = y then y = $4x^2 - 3$

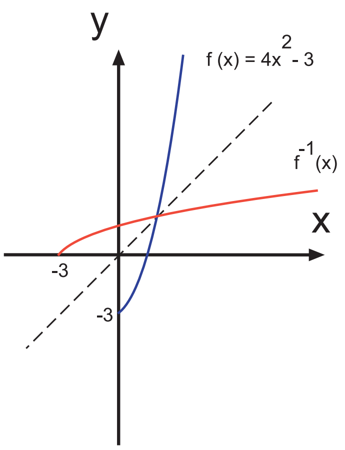
Interchange x and y to give

x = $4y^2 - 3$

Solve for y to obtain the inverse function with the equation y = $\sqrt{\frac{x + 3}{4}}$

Therefore $f^{-1}(x) = \sqrt{\frac{x + 3}{4}}$

Now draw the graphs and compare, using the reflection of y = $4x^2 - 3$ in the line y = x to graph the inverse.



Check the answers in a graphics calculator. If a graphics calculator is not available, an on-line calculator can be used at the following web site:

http://www.univie.ac.at/future.media/moe/fplotter/fplotter.html

This on-line tool is useful for the other exercises which follow if there is a shortage of graphic calculators.

Inverse functions exercise

15 min

Q22: Find and sketch the inverse of the function f (x) = 8 - 2x where $x \in \mathbb{R}$. A graphics calculator can be used to check your understanding of the concept.

Q23: Find and sketch the inverse of the function g (x) = $\frac{1}{2 - x}$ where $x \in \mathbb{R}$: $x \neq 2$

Q24: Find and sketch the inverse of the function h (x) = 4x - 6 where $x \in \mathbb{R}$

Q25: Plot the graph of y = $\log_{10} x$ for $x \in \mathbb{R}^+$ by choosing suitable values.

Use it to sketch the graph of y = 10^x by reflecting it in the line y = x

Note that 10^x is the inverse of the graph y = $\log_{10} x$

Q26: Plot the graph of y = 2^x and use it to sketch the graph of y = $\log_2 x$

Logarithmic and exponential graphs will be considered in more detail again in a later section.

2.4 Composite functions

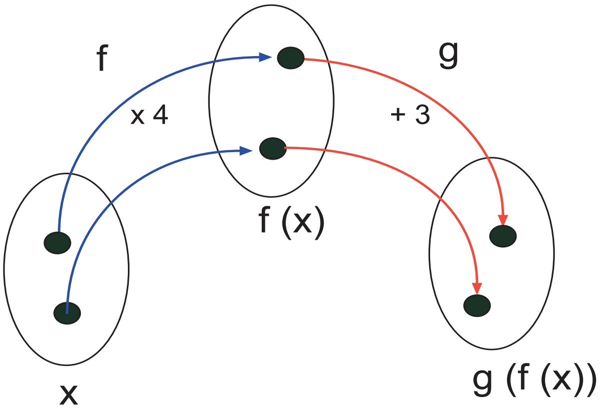
Learning Objective

Find composite functions using the notation f (g (x))

The function f (x) = 4x is a straightforward function with the rule 'times 4'

The function h (x) = 4x + 3, however, has a more complex rule which can be broken down into two parts: 'times 4' and 'add 3'

To obtain 4x + 3, therefore, involves taking the steps f (x) = 4x and then applying another function, say g (x) = x + 3 to this f (x) to give h (x) = 4x + 3 as follows:



The example demonstrates how h (x) can be expressed as h (x) = g (f (x))

The function letters are irrelevant and functions can be combined in different orders to produce new composite functions with the condition:

- The codomain of f (x) *must equal* the domain of g (f (x))

In this unit, this condition will hold for all the functions which are used.

Examples

1. Analysing a composite function

If h (x) = f (g (x)) find f (x) and g (x) for h (x) = $2x^3$

Answer:

To obtain h (x) from x, first cube x to give x^3 and then multiply by 2

Thus g (x) = x^3 and f (x) = 2x

2. Finding a composite function

If f (x) = $3x^2$ and g (x) = x + 1, find f (g (x))

Answer:

Start by substituting from the inside and work outwards.

f (g (x)) = f (x + 1) = $3 \times (x + 1)^2 = 3(x^2 + 2x + 1) = 3x^2 + 6x + 3$

Example Find the composite functions $f^{-1}(f(x))$ and $f(f^{-1}(x))$ when $f(x) = x + 3$

Answer:

$f^{-1}(x) = x - 3$ (interchange y and x: $y = x + 3$ becomes $x = y + 3$ and so $y = x - 3$)

$f^{-1}(f(x)) = f^{-1}(x + 3) = (x + 3) - 3 = x$

$f(f^{-1}(x)) = f(x - 3) = (x - 3) + 3 = x$

This illustrates two interesting properties of a function and its inverse.

- $f(f^{-1}(x)) = f^{-1}(f(x))$
- The composite function formed from $f(x)$ and $f^{-1}(x)$ in either order is x

Note that in general, $g(f(x)) \neq f(g(x))$

Example : $g(f(x)) \neq f(g(x))$

If $f(x) = 3x$ and $g(x) = x - 5$ then find $j(x) = f(g(x))$ and $k(x) = g(f(x))$

Answer:

$j(x) = f(g(x)) = f(x - 5) = 3(x - 5) = 3x - 15$

$k(x) = g(f(x)) = g(3x) = 3x - 5$

Composite function exercise

20 min

Q27: Find f (x) and g (x) for the following composite functions $h(x) = f(g(x))$

a) $1 - x^2$
b) $2^x - 2$
c) $\cos^2(x)$

Q28: Find j (x) and k (x) for the following composite functions $m(x) = k(j(x))$

a) $(x + 4)^2$
b) $x^3 - 2$
c) $\sin(4x)$

Q29: For the following pairs of functions find the composites $f(g(x))$ and $g(f(x))$

a) $f(x) = x^2 - 4x$ and $g(x) = 1 - 2x$
b) $f(x) = x(x - 2)$ and $g(x) = -5x$
c) $f(x) = \dfrac{1}{2 + x}$ and $g(x) = 1 - x^2$
d) $f(x) = \sin(x)$ and $g(x) = x - 1$

The method for composite functions is not restricted to only two base functions. It is possible to build up extremely complex functions using the same techniques.

Example If f (x) = g (h (j (x))) where f (x) = sin $(x + 4)^2$ find g (x), h (x) and j (x)

Answer:

f (x) is constructed from x + 4 which is then squared and finally the sine of this term is taken.

So working from the inside:

j (x) = x + 4

h (x) = x^2

f (x) = sin x

Note that when identifying the functions, use only the base term of x

Multiple composite functions exercise

15 min

Q30: If a (x) = sin x, b (x) = 2x + 1, c (x) = x^2 and d (x) = - x + $2x^2$, find the following composite functions:

a) f (x) = a (b (c (x)))
b) g (x) = d (c (a (x)))
c) h (x) = b (a (d (x)))

Q31: Identify p (x), q (x) and r (x) for the functions f (x) = p (q (r (x))) shown:

a) -cos $(2x^2 + 1)$
b) 2 $(x + 1)^3$
c) $2(x - 1)^2 - 1$

2.5 Related graphs

Learning Objective

Sketch the graphs of functions related to f (x)

If y = f (x) and k is a constant then the following functions can be related to it:

1. y = f (-x)
2. y = - f (x)
3. y = f (x) + k
4. y = k f (x)
5. y = f (x + k)
6. y = f (k x)

These related functions are known as transformations.

Note that more than one transformation may occur in a function.

For example the graph of $y = -2x^3$ is a combination of a type 2 transformation and a type 4 transformation of the function $f(x) = x^3$

Each of the transformation types shown is explained in the remainder of this section.

2.5.1 Type 1: y = - f (x)

The following exercise will ensure that the meaning of - f (x) is clear before drawing the graphs of the function - f (x)

Example : Finding - f (x)

Find - f (x) for the following functions:

a) $2x - 3$

b) $(x + 1)^2$

c) $4x^2 - 1$

d) $-2x^2 - 3x + 4$

Answer:

a) $-(2x - 3) = -2x + 3$

b) $-((x + 1)^2) = -(x^2 + 2x + 1) = -x^2 - 2x - 1$

c) $-(4x^2 - 1) = -4x^2 + 1$

d) $-(-2x^2 - 3x + 4) = 2x^2 + 3x - 4$

Finding - f (x) exercise

5 min

Q32: Find - f (x) for the following expressions:

a) $2x - 1$
b) sin (x)
c) -tan (x)
d) $3x^2$
e) $-x - 2$
f) $-x^2 + 1$

5 min

y = - f (x) calculator investigation

Either use a graphics calculator, the on-line calculator, or graph paper and sketch the graphs y = f (x) for the following functions. Determine and sketch y = - f (x) on the same graph as y = f (x) and find a relationship between the two functions :

a) f (x) = x + 2

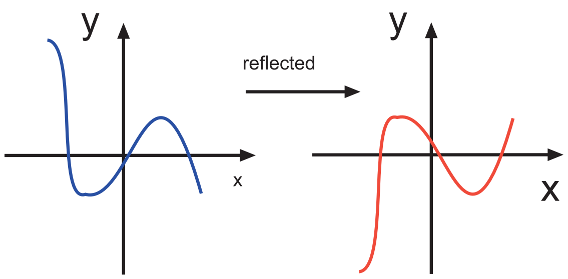
b) f (x) = x

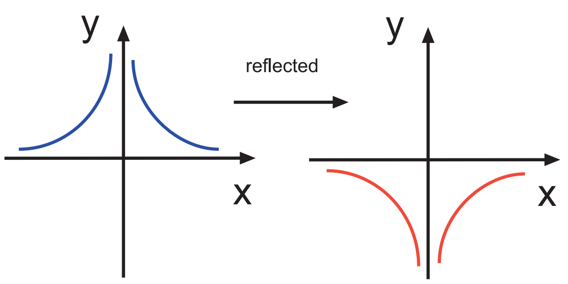
c) f (x) = sin (x)

d) f (x) = x^2

In all cases, the graph of y = - f (x) is a reflection in the x-axis of the graph of y = f (x)

Here are two examples:





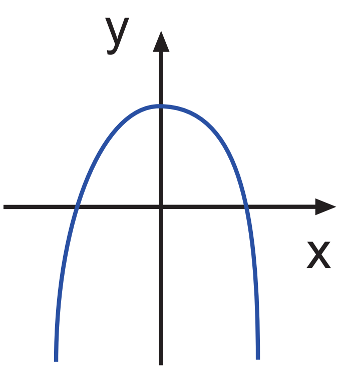
Rule for graphing y = - f (x)

To obtain the graph of y = - f (x) take the graph of y = f (x) and reflect it in the x-axis.

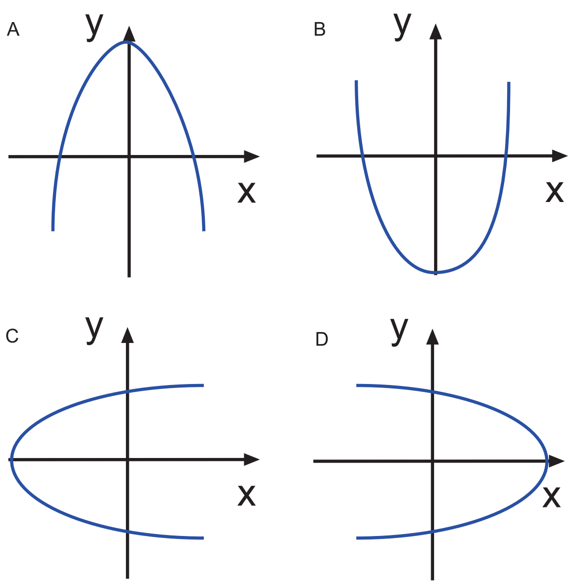
10 min

Identify - f (x) exercise

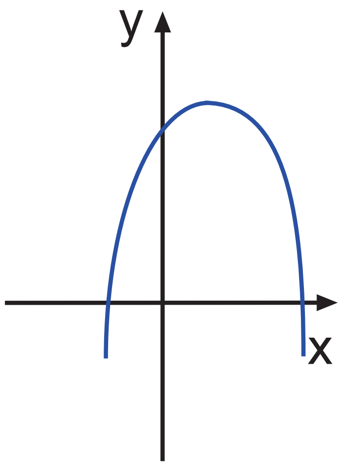
Q33: If y = f (x) has the graph



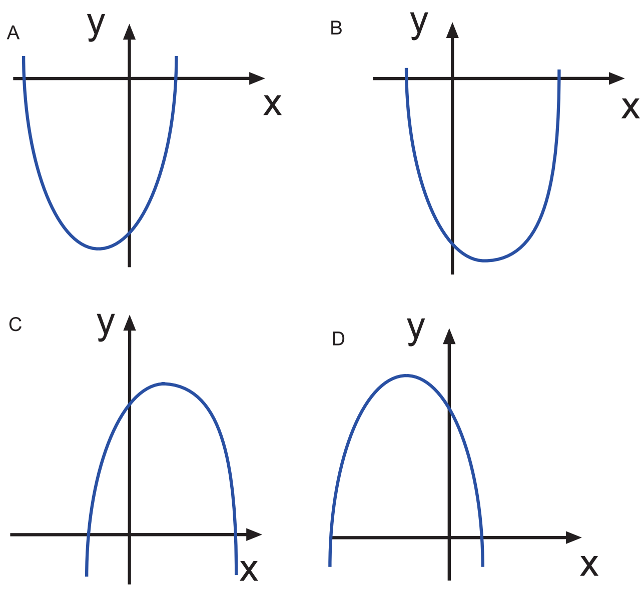
which of the following graphs represents y = - f (x)?



Q34: If y = f (x) has the graph



which of the following graphs represents y = - f (x)?



2.5.2 Type 2: y = f (-x)

Note that f (-x) is not necessarily the same as -f (x) and it is important to note the differences in the relationship with f (x) graphically.

Example : Finding f (-x)

Find f (-x) for the following functions:

a) 2 - x

b) $3x^2$ - 2x

c) $(x + 1)^2$

Answer:

a) 2 - (-x) = 2 + x

b) $3 (-x)^2 - 2 (-x) = 3x^2 + 2x$

c) $((-x) + 1)^2 = (1 - x)^2 = x^2 - 2x + 1$

10 min

Finding f (-x) exercise

Q35: Find f (-x) for the following expressions:

a) 3x - 7
b) cos (x)
c) tan (x)
d) $3x^3$
e) -x - 2
f) $-x^2 + 4$

5 min

y = f (-x) calculator investigation

Either use a graphics calculator, the on-line calculator, or graph paper and sketch the graphs y = f (x) for the following functions. Determine and sketch y = f (-x) on the same graph as y = f (x) and find a relationship between the two functions :

a) f (x) = x - 3

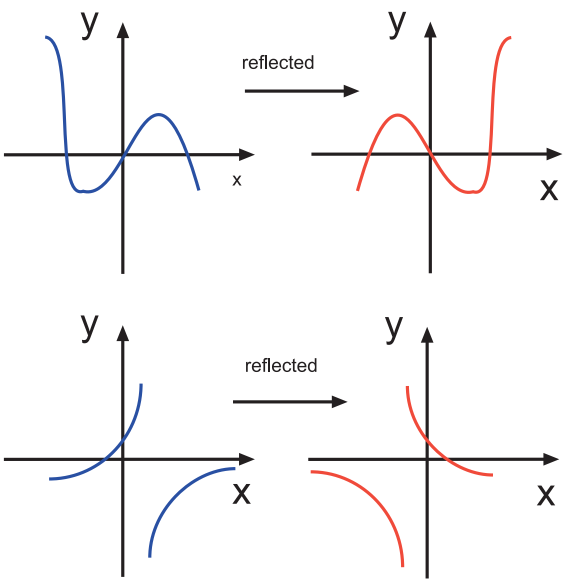
b) f (x) = 2x

c) f (x) = sin (x)

d) f (x) = x^3

In all cases, the graph of y = f (-x) is a reflection in the y-axis of the graph of y = f (x).

Here are two examples:

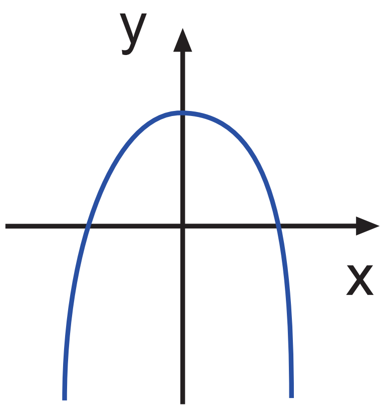


Rule for graphing y = f (-x)
To obtain the graph of y = f (-x) take the graph of y = f (x) and reflect it in the y-axis.

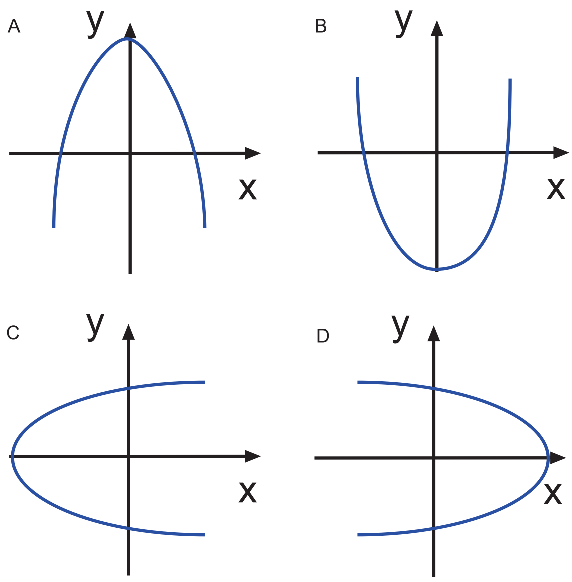
Identify f (-x) exercise

10 min

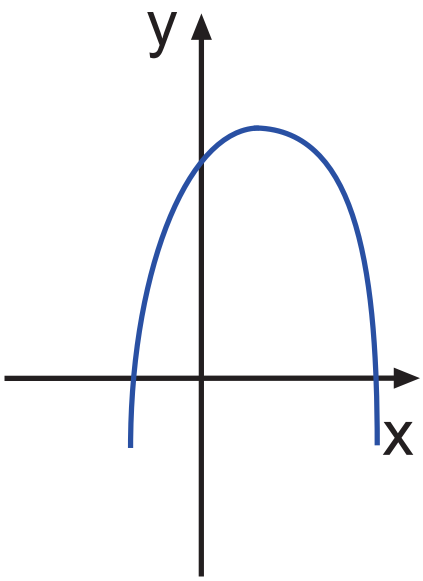
Q36: If y = f (x) has the graph



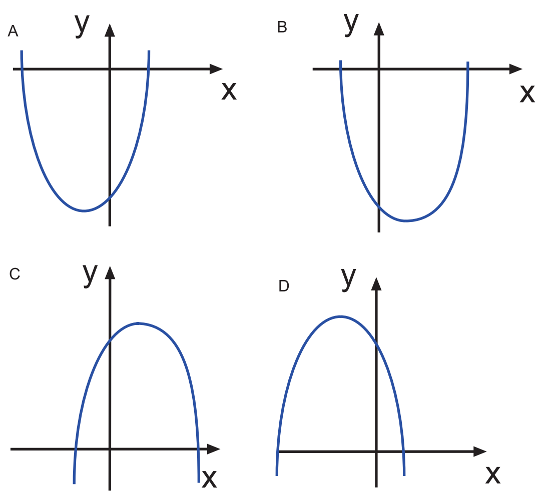
which of the following graphs represents y = f (-x)?



Q37: If y = f (x) has the graph



which of the following graphs represents y = f (-x)?



2.5.3 Type 3: y = f (x) + k

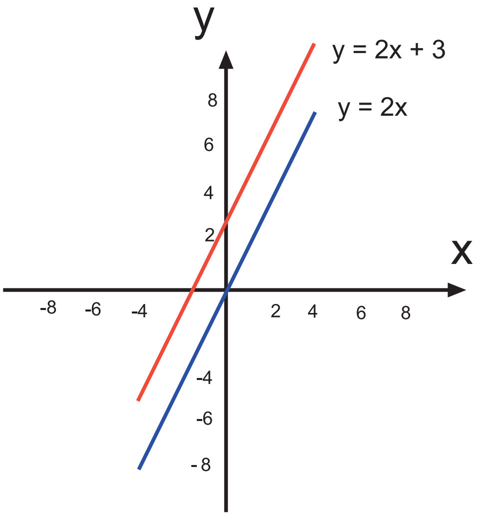
Example Sketch the graph of y = 2x + 3 for x ∈ [-2, 2]

Answer :

First of all complete a table of values with x from -2 to 2 as column headings and calculate values for the rows y = 2x and y = 2x + 3

values of x	-2	-1	0	1	2
y = 2x	-4	-2	0	2	4
y = 2x + 3	-1	1	3	5	7

Plotting the points for y = 2x and those for y = 2x + 3 gives:



The effect of adding 3 to y = 2x is clearly seen. The graph of y = 2x + 3 has moved three units up the y-axis.

y = f (x) + k calculator investigation


5 min

Either use a graphics calculator, the on-line calculator, or graph paper and sketch the graphs of y = f (x) for the following functions. For each y = f (x) on the same screen or page, sketch the second graph y = g (x) mentioned and determine a relationship between the functions:

a) f (x) = 3x and g (x) = 3x - 4

b) f (x) = sin (x) and g (x) = sin (x) + 5

c) f (x) = x^2 and g (x) = x^2 + 1

d) f (x) = -2x and g (x) = -2x + 4

e) f (x) = cos x and g (x) = cos (x) - 3

The activities should clearly demonstrate the following rule:

Rule for graphing y = f (x) + k

To obtain the graph of y = f (x) + k take the graph of y = f (x).

- For k $>$ 0 slide the graph UP the y axis by k units.
- For k $<$ 0 slide the graph DOWN the y axis by k units.

This type of transformation is known as a vertical translation.

10 min

Vertical translation exercise

Q38: What is the relationship between y = 6x and y = 6x - 3?

Q39: What is the relationship between y = 2x and y = 2x + 8?

Q40: Describe the relationships between the two graphs y = 3x - 6 and y = 3x - 9

Q41: Using a graphics calculator explore and describe the relationships between the two graphs y = x^2 + 3x and y = x^2 + 3x + 2

Q42: Using a graphics calculator explore and describe the relationships between the two graphs y = ln x and y = ln x + 4

Q43: Using a graphics calculator explore and describe the relationships between the two graphs y = cos (2x) and y = cos (2x) - 3

2.5.4 Type 4: y = k f (x)

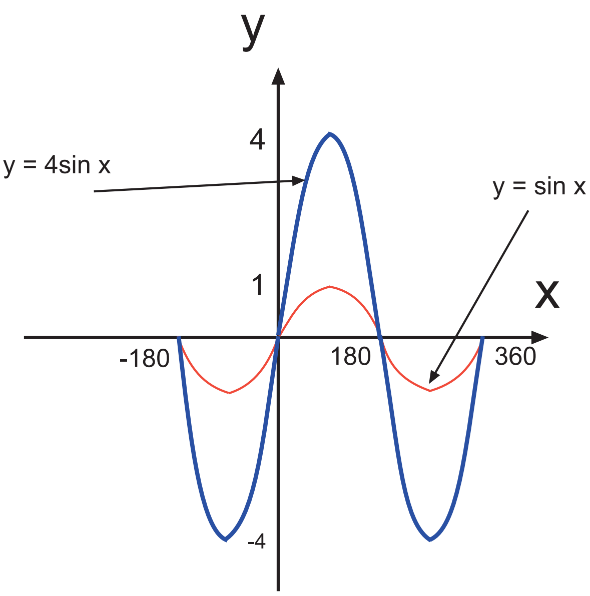
Example Sketch the graph of y = 4 sin x for x $\in$ [-180°, 360°]

Complete a table of values with x = -180° to 360° as column headings.

Calculate the values for the rows y = sin x and y = 4 sin x and plot the points for the two graphs.

values of x	-180°	-90°	0	90°	180°	270°	360°
y = sin x	0	-1	0	1	0	-1	0
y = 4 sin x	0	-4	0	4	0	-4	0

The relationship between the two graphs can be seen on the following diagram. This time the original graph is stretched vertically.



y = k f (x) calculator investigation


5 min

Either use a graphics calculator, the on-line calculator, or graph paper and sketch the graphs of y = f (x) for the following functions. For each y = f (x) on the same screen or page, sketch the second graph y = g (x) mentioned and determine a relationship between the functions:

a) f (x) = sin (x) and g (x) = 3 sin (x)

b) f (x) = cos (x) and g (x) = 2 cos (x)

c) f (x) = x^3 and g (x) = $2x^3$

Again the activities lead to a straightforward rule.

Rule for graphing y = k f (x)
To obtain the graph of y = k f (x) scale the graph of y = f (x) vertically by a factor of k.
This type of transformation is known as a vertical scaling.

Therefore if k $>$ 1 the scaling will in fact stretch the graph of f (x)

and if k $<$ 1 the scaling will shrink the graph of f (x)

This rule works for all functions but this subsection has concentrated on demonstrating it through sin and cos graphs which show the scaling more clearly than linear or quadratic functions.

Vertical scaling exercise


10 min

Q44: Sketch the graph of y = $^1/_2$ cos x

Q45: What is the relationship between the graph of y = cos x and y = 3 cos x?

Q46: What is the relationship between the graph of y = sin x and y = $^1/_4$ sin x?

Q47: Using a graphics calculator explore the relationship between the two graphs $y = \cos 2x$ and $y = {}^1/_2 \cos 2x$

Q48: Explore the relationship between $y = \ln x$ and $y = 3 \ln x$ using a graphics calculator.

Q49: Explore the relationship between $y = {}^1/_2 e^x$ and $y = 2e^x$ using a graphics calculator.

2.5.5 Type 5: y = f (x + k)

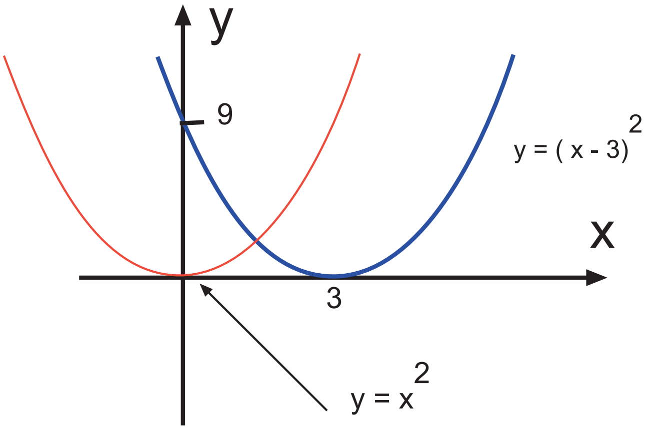
Example Sketch the graph of $y = (x - 3)^2$

This is of the form f (x + k) where k = -3 and $f(x) = x^2$

Complete a table of values with x = -1 to 5 as column headings and calculate the values for the rows $y = x^2$ and $y = (x - 3)^2$

values of x	-1	0	1	2	3	4	5
$y = x^2$	1	0	1	4	9	16	25
$y = (x - 3)^2$	16	9	4	1	0	1	4

The effect is a sideways shift of the graph.




5 min

y = f (x + k) calculator investigation

Either use a graphics calculator, the on-line calculator, or graph paper and sketch the graphs of y = f (x) for the following functions. For each y = f (x) on the same screen or page, sketch the second graph y = g (x) mentioned and determine a relationship between the functions:

a) $f(x) = x^2$ and $g(x) = (x - 2)^2$

b) $f(x) = x^3$ and $g(x) = (x + 1)^3$

c) f (x) = 2x and g (x) = 2 (x - 3)

The rule this time needs a bit more care.

Rule for graphing y = f (x + k)

To obtain y = f (x + k) take y = f (x)

- For k $>$ 0 slide the graph to the *left* by k units.
- For k $<$ 0 slide the graph to the *right* by k units.

This type of transformation is called a horizontal translation or in trigonometric terms it is also called a phase shift.

Horizontal translation exercise

10 min

Q50: Sketch the graph of y = 3(x - 2) and compare with graph of y = 3x

Q51: What is the relationship between y = cos x° and y = cos (x + 30)°?

Q52: Where will the graph of y = $(2x - 1)^2$ lie in relation to the graph of y = $(2x)^2$?

Q53: Explore the relationship between y = $\sqrt{x}$ and y = $\sqrt{(x + 2)}$ using a graphics calculator.

2.5.6 Type 6: y = f (k x)

Example Sketch the graph of y = cos (2x) for x $\in$ [-180°, 360°]

Answer :

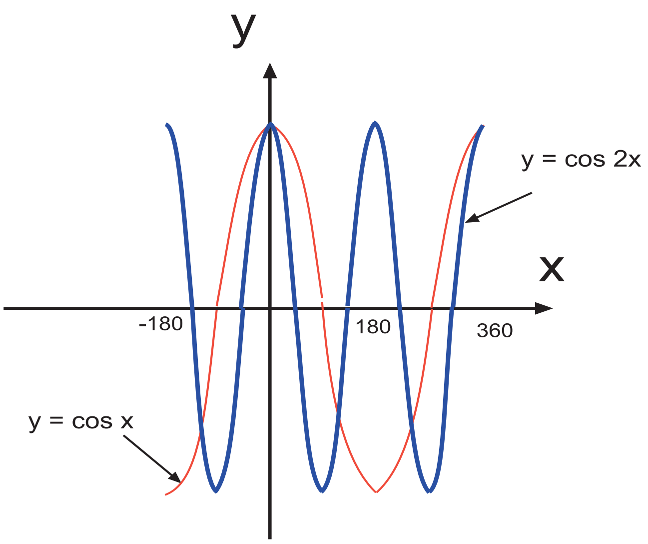
The function is of the form y = f (k x) so f (x) = cos x

Then f (k x) = cos (k x). So k = 2

Complete a table of values with x = -180° to 360° as column headings and calculate the values for the rows y = cos x and y = cos (2x)

values of x	-180°	-90°	0	90°	180°	270°	360°
y = cos x	-1	0	1	0	-1	0	1
y = cos (2x)	1	-1	1	-1	1	-1	1

The effect is clearly seen on the diagram.



5 min

y = f (kx) calculator investigation

Either use a graphics calculator, the on-line calculator, or graph paper and sketch the graphs of y = f (x) for the following functions. For each y = f (x) on the same screen or page, sketch the second graph y = g (x) mentioned and determine a relationship between the functions:

a) f (x) = sin (x) and g (x) = sin (3x)

b) f (x) = cos (x) and g (x) = cos (2x)

c) f (x) = x^3 and g (x) = $(2x)^3$

Another rule follows for this type of transformation but take care with this one.

Rule for graphing y = f (k x)

To obtain y = f (k x) scale the graph of y = f (x) horizontally by a factor of $^1/_k$

This type of transformation is called a horizontal scaling.

Therefore when k $>$ 1, the graph of y = f (kx) will *squash* horizontally.

When k $<$ 1, the graph will *stretch* horizontally.

Horizontal scaling exercise

15 min

Q54: What is the relationship between y = sin x and y = sin $^1/_3$ x?

Q55: Sketch the graph of y = tan 3x and compare with the graph of y = tan x

Q56: Using a graphics calculator explore and describe the relationship between the graph of y = cos x° and y = cos $^3/_4$x°

Q57: Using a graphics calculator explore and describe the relationship between the graph of y = $\sqrt{x}$ and y = $\sqrt{3x}$

2.5.7 Modulus function

There is a particular function of x which uses part reflection in the x-axis. This function is called the modulus function.

The modulus of a real number is the value of the number (called its absolute value) regardless of the sign. So the modulus of 3 (written $|\ 3\ |$) is 3 and the modulus of -3 (written $|$-3 $|$) is also 3.

This is the basis of the definition of the modulus function.

The modulus function

For $x \in \mathbb{R}$ the modulus function of f (x), denoted by $|\ f(x)\ |$ is defined by

$$|\ f(x)\ | = \begin{array}{l} f(x) \text{ if } f(x) \geqslant 0 \\ -f(x) \text{ if } f(x) < 0 \end{array}$$

Rule for graphing y = $|$ f (x) $|$

To sketch the graph of a modulus function $|$ f (x) $|$, first sketch the graph of the function y = f (x)

Take any part of it that lies below the x-axis and reflect it in the x-axis.

The modulus function y = $|$ f (x) $|$ is the combined effect of the positive part of the original function and the new reflected part.

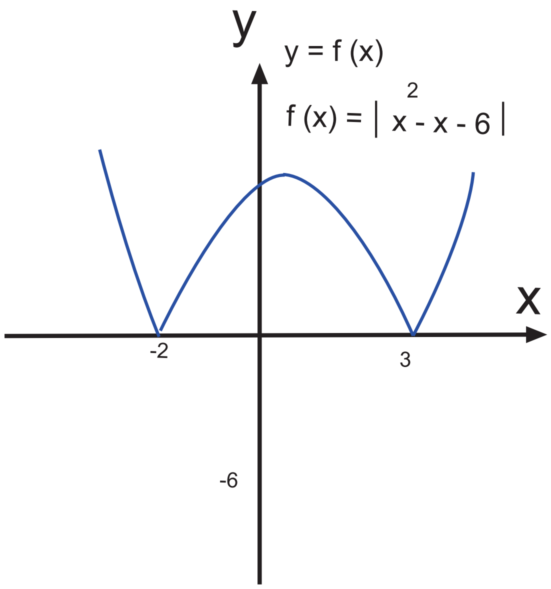
Example : The graph of the modulus

Sketch the graph of f (x) = $|\ x^2 - x - 6\ |$

The graph of y = x^2 - x - 6 crosses the x-axis at x = -2 and x = 3

It has a minimum turning point at ($^1/_2$, -$^{25}/_4$) Reflect the curve between x = -2 and x = 3

Keep the remainder of the curve y = f (x)



Q58: Sketch the graph of $y = \left|2x^2 - x - 10\right|$

Q59: Sketch the graph of $y = |\tan x|$ for $x \in [-270°, 270°]$

2.5.8 Combinations of relations

It is possible using the rules shown in this section, to sketch and identify more complicated functions and graphs which are combinations of relations.

Note that any movement along the axis should be done *before* any scaling. This is important particularly with trig graphs. The scaling is then carried out from the *new* origin.

Example Identify the functions shown as g (x) as combinations of f (x) and explain the relationship:

a) g (x) = 4 f (x) - 2

b) g (x) = -2 f (x) + 1

c) g (x) = 3 f (x - 2)

d) g (x) = f (3x + 6)

Answer:

a) g (x) is formed from f (x) by moving it down the y-axis by 2 units and stretching it vertically by a factor of four (Movement and then scaling around the new origin of (0, -2)

b) g (x) is formed from f (x) by reflecting it in the x-axis, moving it 1 unit up the y-axis and then stretching it by a factor of 2

c) g (x) is formed by moving f (x) 2 units along the x-axis to the right and then scaling this by a factor of 3 vertically around the new origin of (2, 0)

d) g (x) = f (3 (x + 2)) and is formed by moving f (x) 2 units to the left along the x-axis and then scaling horizontally by a factor of $^1/_3$. That is in this case, squashing the graph around its new origin of (-2, 0) to one third of its size.

Multiple relationships exercise

10 min

Q60: Identify the relationships to f (x) for the following functions:

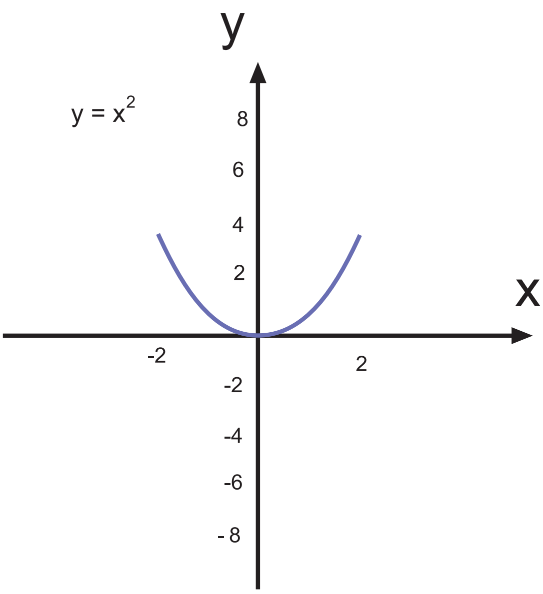
a) -4 f (x)
b) 6 f (x) - 2
c) 3 f (x - 30)°
d) f (2x - 4)

2.6 Quadratics and completing the square

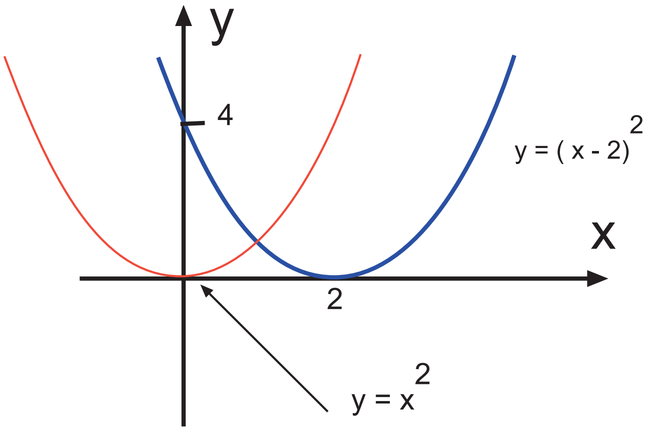
Learning Objective

Complete the square in a quadratic and find the turning points of the graph

The graph of $y = x^2$ is well known.



The graph of $x^2 - 4x + 4 = (x - 2)^2$ is also relatively easy to sketch by relating it to $y = x^2$ and using the transformation rules of the previous section.



A little more work is needed however, to sketch the graph of $y = x^2 - 2x + 4$ without reverting to completing a table of values.

The important points in sketching any graph are:

- the turning points, if any
- intersection with the x-axis, if any
- intersection with the y-axis, if any

For more complex functions, there are of course other areas of interest but these three points are basic to most.

The turning points can be found for quadratics by a method called 'completing the square'. It is this method which will now be examined in detail.

2.6.1 Completing the square

The quadratic $x^2 - 2x + 4$ does not factorise but it is possible to rearrange it into the form $(x - 1)^2 + 3$

This form is the result of the process called 'completing the square'.

In general terms this form is stated as $(x + k)^2 + m$

Completing the square

After completing the square,

a quadratic function has the form $(x + k)^2 + m$

An example will help to demonstrate the method.

Examples

1. Completing the square

Express x^2 - 10x + 2 in the form $(x + k)^2$ + m by completing the square.

Answer:

Step 1	k = -5 and p = 2	Express as $x^2 + 2kx + p$ and state k and p
Step 2	$= (x - 5)^2 - 25 + 2$	Use k and write the quadratic as $(x + k)^2 - k^2 + p$
Step 3	$(x - 5)^2 - 23$	Tidy up

2. Express x^2 + 6x -1 in the form $(x + k)^2$ + m by completing the square.

Answer:

Step 1	k = 3 and p = -1	Express as $x^2 + 2kx + p$ and state k and p
Step 2	$= (x + 3)^2 - 9 + 5$	Use k and write the quadratic as $(x + k)^2 - k^2 + p$
Step 3	$(x + 3)^2 - 4$	Tidy up

If, however, the coefficient of x^2 is greater than 1 then it is necessary to divide through initially by this coefficient before following the three steps.

Examples

1. Express $2x^2$ - 8x + 6 in the form $n (x + k)^2$ + m by completing the square.

Answer:

$2x^2 - 8x + 6 = 2 [x^2 - 4x + 3]$

$= 2 [(x - 2)^2 - 4 + 3]$

$= 2 (x - 2)^2 - 2$

2. Express $-4x^2$ + 24x - 4 in the form $n (x + k)^2$ + m by completing the square.

$-4x^2 + 24x - 4 = -4 [x^2 - 6x + 1]$

$= -4 [(x - 3)^2 - 9 + 1] = -4 (x - 3)^2 + 32$

This step by step approach to completing the square is worth remembering.

Strategy for completing the square

STEP 1	divide through by the coefficient of x^2 if it is not 1
STEP 2	Express as $x^2 + 2kx + p$ and note k and p
STEP 3	Use k and m and write the quadratic as $(x + k)^2 - k^2 + p$
STEP 4	Tidy up the new expression

Completing the square exercise

10 min

Q61: Write the following expressions in the form $(x + k)^2 + m$

a) $x^2 - 4x + 1$
b) $x^2 + 8x - 2$
c) $x^2 - 6x - 6$
d) $x^2 + 2x + 8$

Q62: Complete the square and rewrite the following in the form $(x + k)^2 + m$

a) $x^2 - 4x$
b) $x^2 - 2x + 1$
c) $x^2 - 4x + 7$

Q63: Complete the square and rewrite the following in the form $n (x + k)^2 + m$

a) $2x^2 - 4x$
b) $-2x^2 - 8x + 6$
c) $-3x^2 - 12x + 9$
d) $-2x^2 + 3x + 3$
e) $3x^2 + 6x - 1$

2.6.2 Using the completed square version of a quadratic

Learning Objective

Identify maximum and minimum values of a quadratic by examining the expression after completing the square

Completing the square can be used to find the maximum or minimum values of a function.

These minimum or maximum values of the function relate to the turning points of the graph of the function.

If a function has a minimum value of -2 when x = 4 then the point (4, -2) is a minimum turning point on the graph.

Examples

1. Minimum value of a function

Find the minimum value of the function $f (x) = x^2 - 4x + 1$

Answer:

First of all express the function after completing the square.

$x^2 - 4x + 1 = (x - 2)^2 - 3$

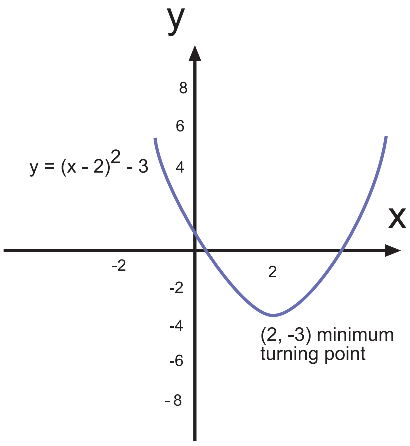
Notice that $(x - 2)^2$ is always positive since it is a square.

The minimum value that $(x - 2)^2$ can take is zero and this occurs if x = 2

The minimum value of the function is therefore 0 - 3 = -3

Note that there is no maximum value on this function unless x has a restriction placed on its values.

The graph of the function is shown.



2. Maximum value of a function

Find the maximum value of the function $f(x) = (\sin x + 1)^2 + 3$

Answer:

This is already in the completed square form.

$(\sin x + 1)^2$ is a square and so it is always positive. It has a maximum value when sin x is at its maximum. That is when sin x = 1.

So $(\sin x + 1)^2$ has a maximum of 4 and the function has a maximum of 4 + 3 = 7

Note that this function also has a minimum since sin x has a minimum of -1. At this value the function will then have a minimum of 0 + 3 = 3

There is an extremely useful shortcut which may be apparent from the first example. If the completed square version of a function is known in the form $(x + k)^2 + m$ then the turning point is (-k, m).

Turning point coordinates from completing the square

For a quadratic function in the form $(x + k)^2 + m$, the coordinates of the turning point are given by (-k, m)

Maximum/minimum values exercise

10 min

Q64: Complete the square if necessary and determine if the following functions have minimum or maximum values or both. Give the values.

a) $x^2 - 4x + 9$

b) $(2 \cos x - 1)^2 + 1$ (take care)

c) $-3 - 2(x + 3)^2$

d) $-4x^2 - 16x - 8$

Finding the turning points in this way gives an alternative to using related graph techniques and composite functions. It is particularly useful for the more complex quadratics. An easier example however, will demonstrate the method.

Recall the paragraph near the beginning of this section which stated:

The important points in sketching any graph are:

- the turning points, if any
- intersection with the x-axis, if any
- intersection with the y-axis, if any

The intersection with the x-axis, if it exists, can be found by equating the expression in x to zero and solving for x

The intersection with the y-axis, if it exists, can be found by substituting x = 0 into the expression for f (x) and finding this value.

Example Sketch the graph of the quadratic $x^2 + 4x + 3$

Answer:

Completing the square gives $(x + 2)^2 - 1$

The shortcut gives a turning point at (-2, -1)

A quick check by looking at the formula will reveal that it is a minimum.

It crosses the x-axis when y = 0

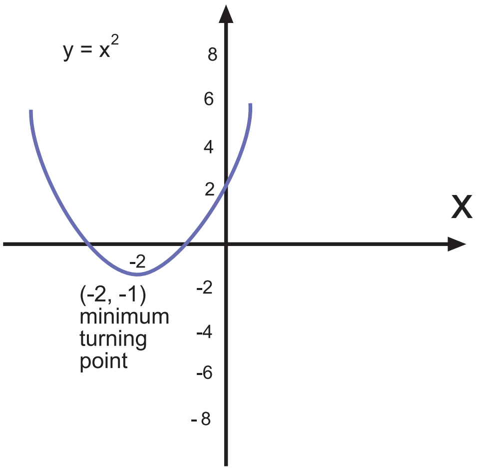
That is $x^2 + 4x + 3 = 0 \Rightarrow (x + 3)(x + 1) = 0 \Rightarrow x = -3$ or $x = -1$

This is the points (-1, 0) and (-3, 0)

It crosses the y-axis when x = 0

That is when $y = (0 + 2)^2 - 1 = 3$

This is the point (0, 3)



This technique will be used in the section on identifying functions.

2.7 Angles and exact values

Learning Objective

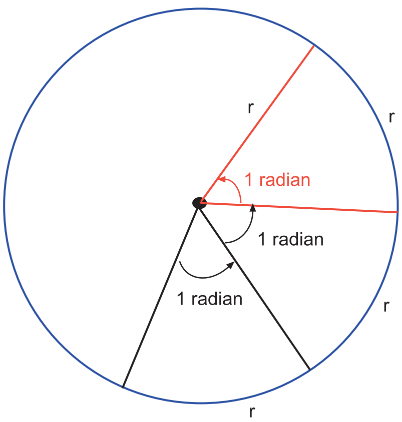
Find the exact trig ratio values for common angles in all four quadrants

There are two units of measurement which can be used for angles. Measuring in degrees will be familiar but it is also possible to measure in units called radians.

Radian

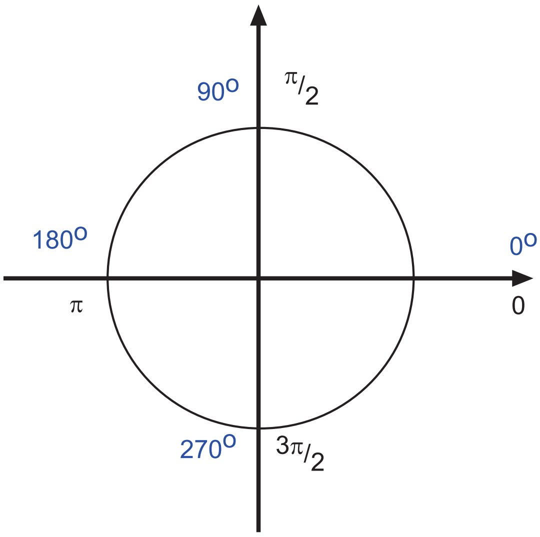
An angle subtended at the centre of a circle by an arc of length equal to the radius of the circle is called a radian.

There is no symbol for a radian. Therefore it is assumed that an angle with no units stated is measured in radians, otherwise if the angle is measured in degrees, it must have the degree symbol of $^\circ$ showing.



There are 2π radians in one complete revolution. Since there are 360° in one complete revolution it follows that 2π radians = 360°

It is more common however, to remember that π radians = 180° which makes each radian *roughly* equal to 60°



Sometimes it is necessary to convert angle measurements from one form into the other.

If possible leave radian measurements in the form of a fraction of π. This is an exact answer and is more accurate for problem solving.

Examples

1. Convert degrees to radians

Convert into radians the following angle measurements given in degrees :

a) 60°

b) 145°

c) 225°

d) 330°

Answer:

To convert, since 180° = π radians, divide by 180 and multiply by π

a) $\frac{60}{180} \times \pi = \frac{60\pi}{180} = \frac{\pi}{3}$

b) $\frac{145}{180} \times \pi = \frac{145\pi}{180} = \frac{29\pi}{36} = 2.53$

Note that in this case it makes more sense to give the radian answer as a decimal. Normally two decimal places is enough. However, remember to use exact fractions where possible.

c) $\frac{225}{180} \times \pi = \frac{5\pi}{4}$

d) $\frac{330}{180} \times \pi = \frac{11\pi}{6}$

2. Convert radians to degrees

Convert into degrees, the following angle measurements given in radians:

a) $\frac{2\pi}{3}$

b) $\frac{\pi}{12}$

c) $\frac{5\pi}{2}$

d) 3π

Answer:

This time, divide by π and multiply by 180

a) $\frac{2\pi}{3} \times \frac{180}{\pi} = \frac{360}{3} = 120^\circ$

b) $\frac{\pi}{12} \times \frac{180}{\pi} = \frac{180}{12} = 15^\circ$

c) $\frac{5\pi}{2} \times \frac{180}{\pi} = 450^\circ$

d) $\frac{3\pi}{1} \times \frac{180}{\pi} = 540^\circ$

	divide by	multiply by
degrees $\rightarrow$ radians	180	π
radians $\rightarrow$ degrees	π	180

Radian and degrees exercise

15 min

Q65: Convert the following angles from degrees to radians:

a) 90°

b) 135°

c) 270°

d) 105°

e) 315°

f) 30°

Q66: Convert the following angles from radians to degrees:

a) $\frac{3\pi}{4}$

b) $\frac{11\pi}{6}$

c) $\frac{5\pi}{3}$

d) $\frac{11\pi}{12}$

e) $\frac{7\pi}{5}$

f) $\frac{13\pi}{9}$

The exact values of the trig. ratios (sin, cos and tan) for certain angles can also make calculations quicker and easier.

When x = 0 the values of sin x, cos x and tan x are easily seen from their graphs. Also when x = 90° $\left(\frac{\pi}{2}\right)$ the values of sin x and cos x can be taken from the graphs.

The values are

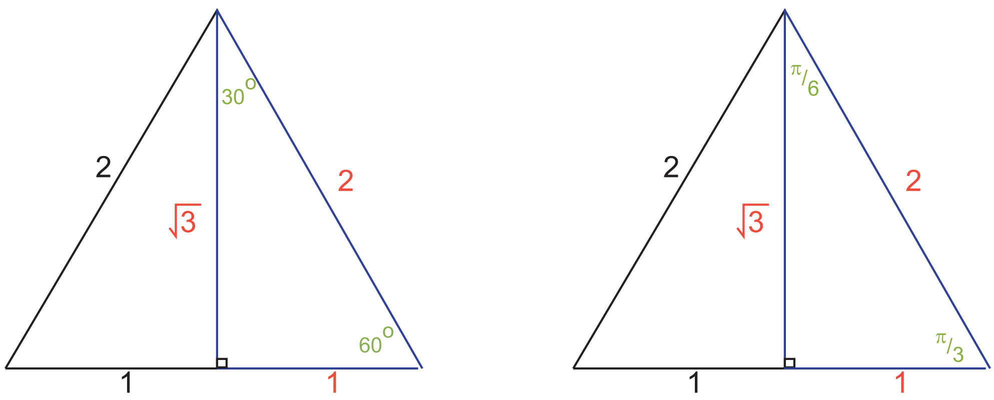
	sin x	cos x	tan x
x = 0	0	1	0
x = 90°	1	0	***

There are two constructions which will give the exact values of other common angles.

Construction for 30° $\left(\frac{\pi}{6}\right)$ and 60 ° $\left(\frac{\pi}{3}\right)$ angles

- Sketch an equilateral triangle with side length 2 units.
- Draw in the perpendicular bisector from the apex to the base.
- Take one of the triangles now formed.
- The angles of this triangle are: 30° $\left(\frac{\pi}{6}\right)$, 60° $\left(\frac{\pi}{3}\right)$, 90° $\left(\frac{\pi}{2}\right)$
- The sides of this triangle are 2 units, 1 units (base is halved by the bisector) and $\sqrt{3}$ units (by Pythagoras).
- Read off the exact values for sin, cos and tan of the angles of 30° $\left(\frac{\pi}{6}\right)$ and 60° $\left(\frac{\pi}{3}\right)$ using the SOH - CAH - TOA ratios.

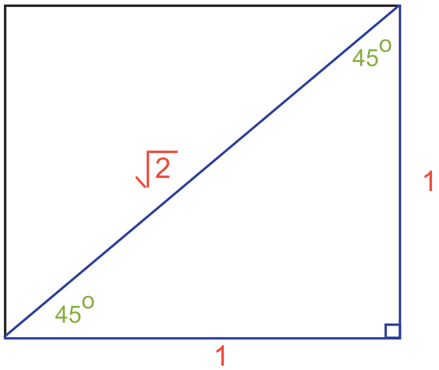
The two diagrams show this construction for degrees and radians.

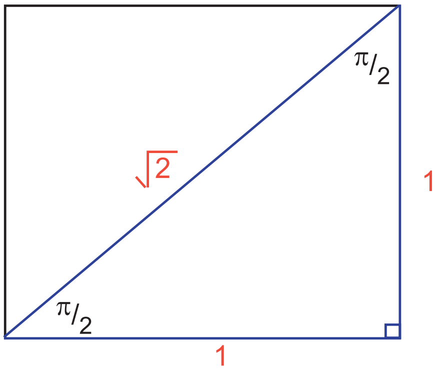


Construction for 45 $^\circ\left(\frac{\pi}{4}\right)$ angle

- Sketch a square of side length 1 unit.
- Draw in one of the diagonals.
- Take one of the triangles now formed.
- The angles of this triangle are: $45^\circ\left(\frac{\pi}{4}\right)$ twice and $90^\circ\left(\frac{\pi}{2}\right)$
- The sides of this triangle are 1 unit, 1 unit and $\sqrt{2}$ units (by Pythagoras).
- Read off the exact values for sin, cos and tan of the angle of $45^\circ\left(\frac{\pi}{4}\right)$ using the SOH - CAH - TOA ratios.

The two diagrams show this construction for degrees and radians.





These exact values for trig. ratios can be extended to angles in all four quadrants.

Recall the quadrant diagram.

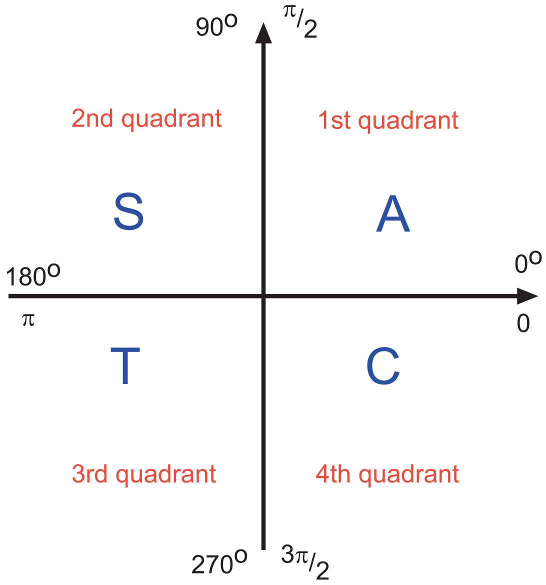
This can be remembered easily as All Students Talk Constantly and depicts the quadrants in which the various ratios are positive.

The first has all ratios (A) positive.

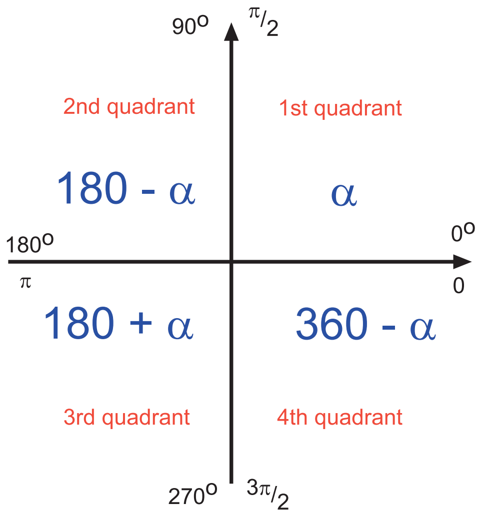
The second has sine (S) positive.

The third has cosine (C) positive.

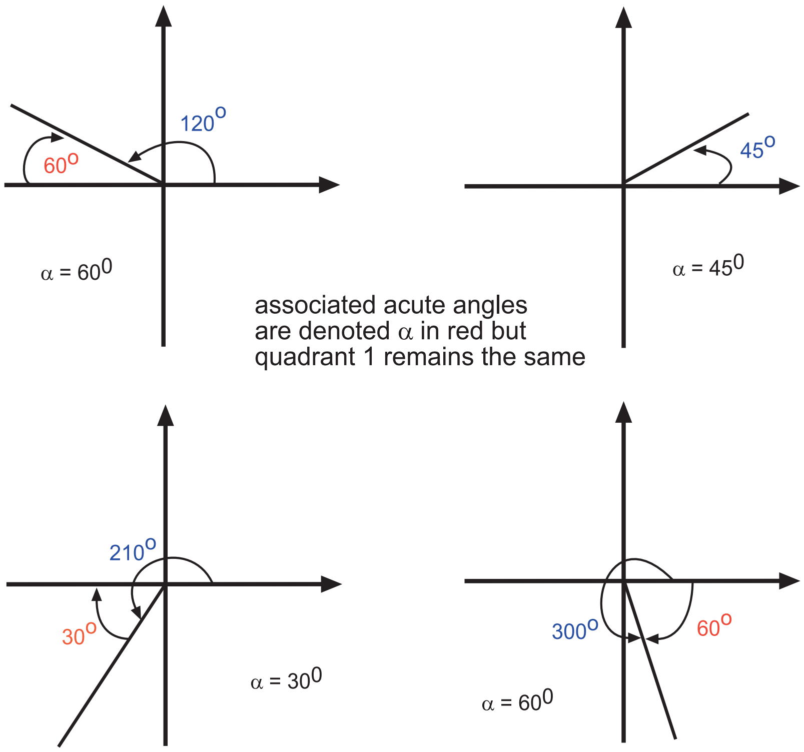
The fourth has tangent (T) positive.



To find an exact value for an angle greater than 90°, the angle is converted to the associated angle between 0 and 90° with the help of the following diagram.



Examples of this will help to make it clearer.



The associated angle exact value is then found and the sign of the value is given by the quadrant sign diagram.

Activity

Redraw similar diagrams to those above using radians.

Example : Exact angles

Find the exact values of the following:

a) $\sin 45°$

b) $\cos 210°$

c) $\tan 300$

d) $\sin \dfrac{2\pi}{3}$

e) $\cos \dfrac{11\pi}{6}$

f) $\tan \dfrac{3\pi}{4}$

Answer:

The answers are found by:

- referring to which quadrant the angle lies in.
- finding the associate acute angle if necessary.
- calculating the exact value.
- determining the sign of the answer by referring to the quadrant sign diagram.

a) This is a first quadrant angle.

Sine is positive in quadrant 1

The exact value of sin 45° is $\frac{1}{\sqrt{2}}$ and the answer is + $\frac{1}{\sqrt{2}}$

b) This is a third quadrant angle and cosine is positive in the third quadrant.

The associated acute angle α is found by solving 210 = 180 + α: thus α = 30°

The exact value of cos 30° = $\frac{\sqrt{3}}{2}$

Thus cos 210° = $\frac{\sqrt{3}}{2}$

c) This is a fourth quadrant angle and tangent is negative in the fourth quadrant.

The associated acute angle α is found by solving α = 360 - 300: thus α = 60°

The exact value of tan 60° = $\sqrt{3}$

Thus tan 300° = - $\sqrt{3}$

d) This is a second quadrant angle and sine is positive in the second quadrant.

The associated acute angle α is found by solving

$\alpha = \pi - \frac{2\pi}{3} \Rightarrow \alpha = \frac{\pi}{3}$

The exact value of sin $\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

Thus

sin $\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$

e) This is a fourth quadrant angle and cosine is positive in the fourth quadrant.

The associated acute angle α is found by solving

$\alpha = 2\pi - \frac{11\pi}{6} \Rightarrow \alpha = \frac{\pi}{6}$

The exact value of cos $\frac{\pi}{6} = \frac{1}{2}$

Thus

cos $\frac{11\pi}{6} = \frac{1}{2}$

f) This is a second quadrant angle and tangent is negative in the second quadrant.

The associated acute angle α is found by solving

$\alpha = \pi - \frac{3\pi}{4} \Rightarrow \alpha = \frac{\pi}{4}$

The exact value of tan $\frac{\pi}{4} = 1$

Thus

tan $\frac{3\pi}{4} =$ - 1

Exact values exercise

20 min

Q67: Find the exact values of the following:

a) tan 135°

b) cos $\frac{\pi}{2}$

c) sin 330°

d) tan $\frac{11\pi}{6}$

e) tan $\frac{2\pi}{3}$

f) cos 210°

g) sin $\frac{7\pi}{6}$

h) sin 90°

i) tan 0

Q68: Find the exact values of the following:

a) sin 135°

b) cos $\frac{3\pi}{4}$

c) cos 315°

d) tan 240°

e) sin $\frac{7\pi}{4}$

f) cos $\frac{5\pi}{3}$

g) sin π

h) cos 135°

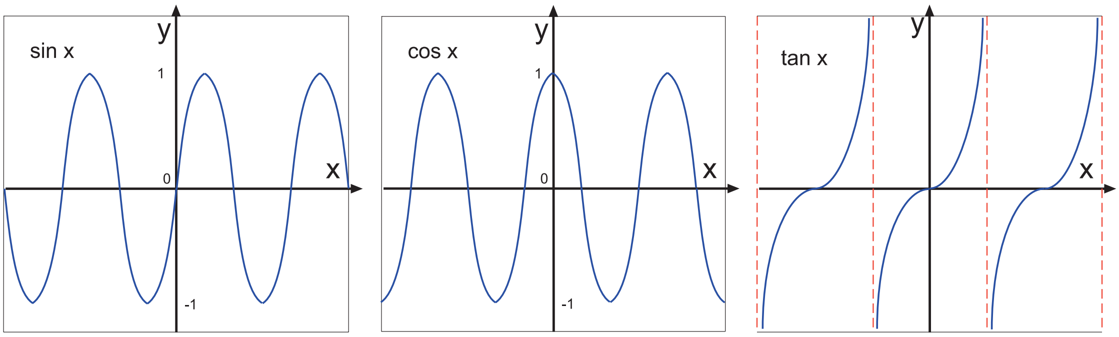
i) tan 330°

2.8 Graphs of trigonometric functions

Learning Objective

Recognise the equations features of trig functions form their graphs. Sketch trig functions from their equation in relation to the graphs fo sin x and cos x

The graphs of sine, cosine and tangent have some special features.



The graphs all show a repeating pattern.

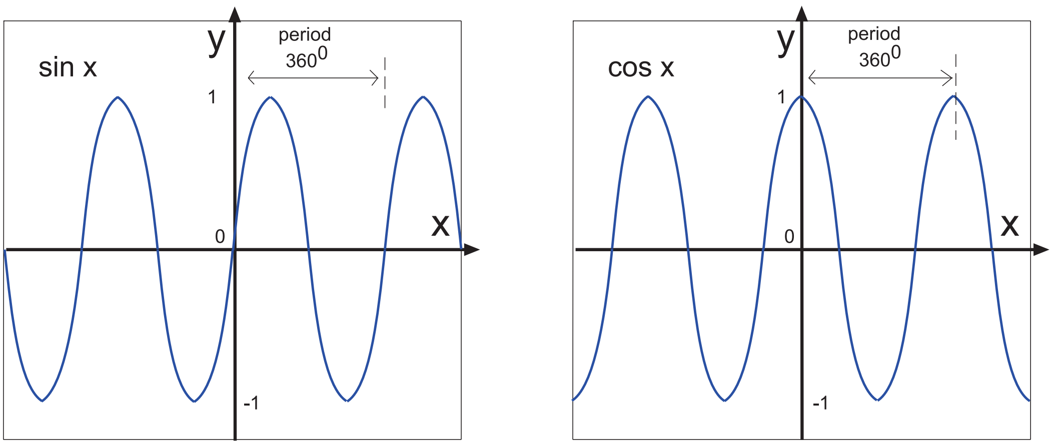
Activity

Using a graphics calculator, find out the length along the x-axis through which each graph demonstrates one full repeat. (Use the trace facility.)

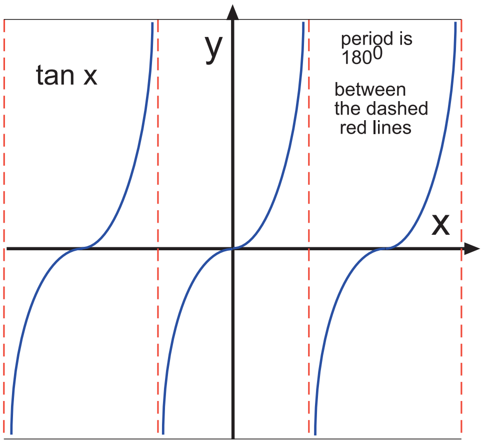
Period of a graph

The length along the x-axis over which the graph traces one full pattern is called the period of the graph.

For the graphs of sin x and of cos x the period is 360° or 2π



For the graph of tan x the period is 180° or π

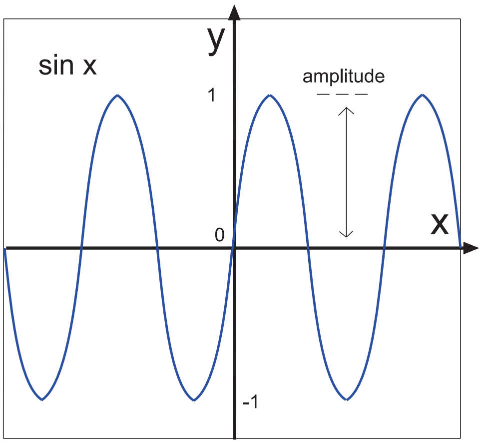


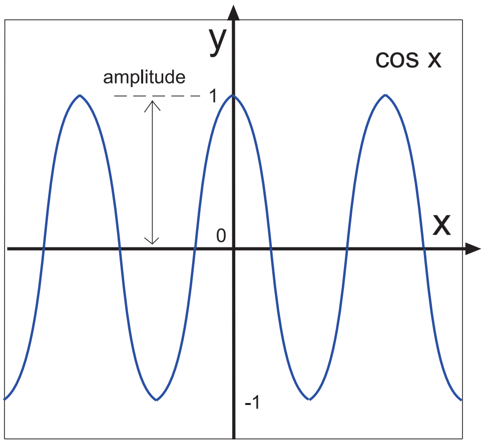
The sine and cosine graphs can also be described as having an amplitude.

Amplitude of a graph

The amplitude of a graph is half of the distance along the y-axis between the maximum and minimum values of the graph.

The two graphs, sin x and cos x both have an amplitude of 1. (Half the distance between the maximum of 1 and the minimum of -1).



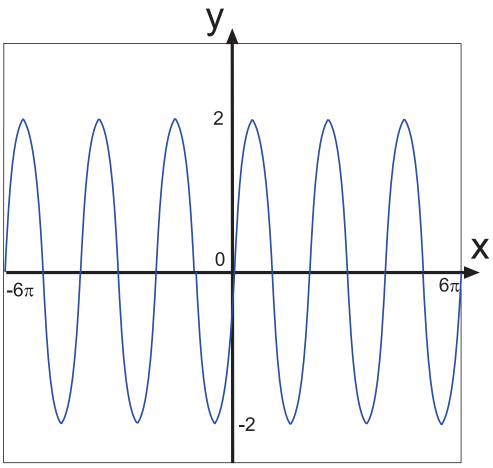


Using the techniques previously given in the section on related graphs it is now possible to determine the amplitude and period of a variety of trig. graphs. This can be done both from the function or from its graph.

Examples

1. Finding the amplitude and period of a trig. graph

Find the amplitude and the period (in radians) of the following graph:



Answer:

The amplitude is half the distance between the maximum and the minimum. That is $^1/_2(2 - (-2)) = 2$

The period is the length along the x-axis required to trace one full pattern. Since there are three full patterns in 6π, one full pattern takes 2π. The period is 2π

2. Finding the amplitude and period of a trig. function

Find the amplitude and the period (in radians) of the function 3 cos ($^1/_2$x)

Answer:

Starting from the simple function cos x this function can be examined in parts as follows.

The rule for graphing y = k f (x) states that the graph of y = f (x) is stretched vertically by a factor of k. Here k = 3

If the graph of y = cos x is stretched by a factor of 3, the maximum becomes $3 \times 1 = 3$ and the minimum becomes $3 \times -1 = -3$. The amplitude is then $^1/_2$ of 6 = 3

Also the rule for f (k x) states that when k is less than 1 the graph will stretch horizontally by a factor of $^1/_k$

In this case it will stretch by a factor of 2 and the period will be twice as large as the period for cos x. The period is therefore $2 \times 2\pi = 4$

Amplitude and period exercise

10 min

Q69: Find the amplitude and the period (in degrees) for the following functions:

a) 2 sin x
b) $^1/_4$ cos x
c) 3 sin (2x)
d) sin (4x)
e) cos ($^1/_2$x)
f) 3 sin 4x
g) 2 cos 3x

For a more complicated function such as f (x) = a cos (bx + c) - d (for example 3 cos (2x - 30)° - 4) the technique is almost the same.

The term - d will only move the graph down the axis by d units but will not affect the shape.

At this stage rearrange the equation f (x) = a cos (bx + c) - d to give
f (x) = a cos b(x + $^c/_b$) - d

Now, in a similar way to the effect which d has, the term + $^c/_b$ will move the graph to the left by $^c/_b$ units but will not affect the shape.

It is only the values of a and b which will affect the shape and so affect the amplitude and period of a function.

It is worth remembering that for cos and sin functions, the value of a indicates the number of repeats in 360°

Example : Amplitude and period from the equation

Find the amplitude and period (in degrees) of the function 2 cos (5x - 60)° - 7

Answer:

This is in the form a cos (bx + c) + d = a cos b(x + $^c/_b$) + d where

a = 2, b = 5, c = -60 (so $^c/_b$ = -12) and d = -7

Only a and b affect the period and amplitude.

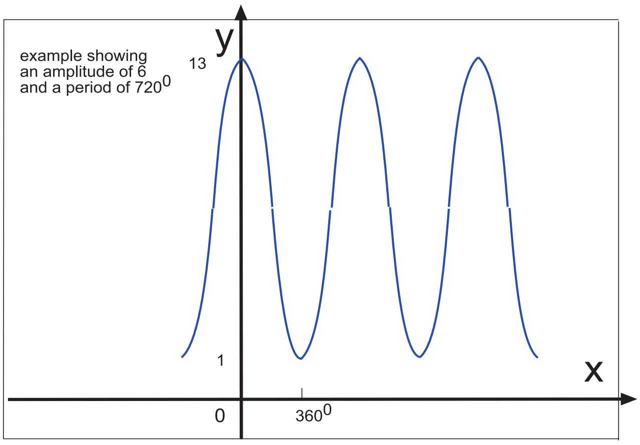
Using the two rules for k f (x) and for f (k x) gives

an amplitude of 2 and a period of 72° (360 ÷ 5)

To find the amplitude and period given the graph of such a function needs a little more care.

In this case it is very important to use the definitions of amplitude and period to find these values.

From a graph the maximum/minimum values can be found relatively easily if not already given and a visual check will identify the pattern and how it repeats.



Activity

At this stage it would be useful to check back on the section on related functions.

Use the information given in the rules for that section and experiment on a graphics calculator by graphing a variety of trig. functions.

Use the trace and calc. facilities to check the period and amplitude of these graphs.

The final step is to graph functions such as

a cos (bx + c) + d or a sin (bx + c) + d

Two simpler examples are shown.

Examples

1. Graph the function f (x) = cos (3x - 30)°

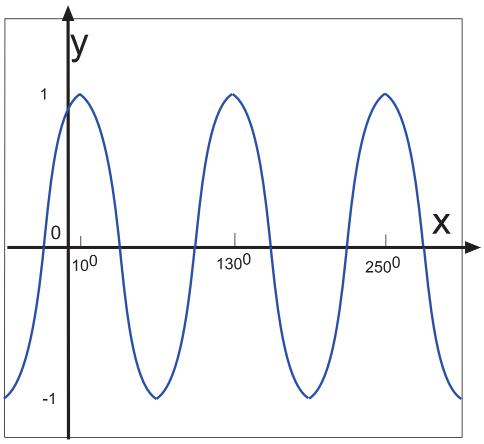
Answer:

This graph is of the form cos a(x + $^{b}/_{a}$)° where a is 3 and b = -30°

The value of $^{b}/_{a}$ (= -10) indicates that the graph moves 10° to the right of the basic cos graph.

The value of a indicates that the graph will also repeat every 120°
(It has a period of $^{360}/_{3}$ = 120°)

(Remember to move before scaling around the new origin.)



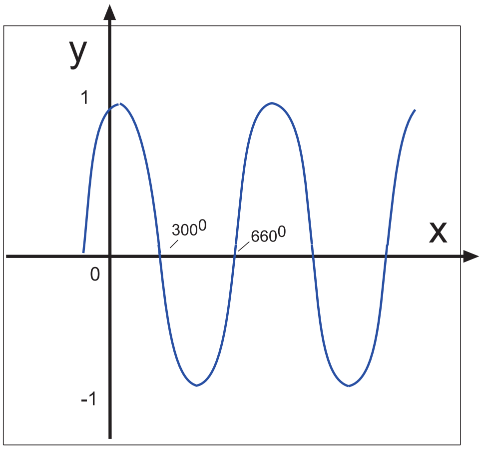
2. Graph the function f (x) = sin ($^{1}/_{2}$x + 30)°

Answer:

The graph is of the form sin (ax + b)° where a = $^{1}/_{2}$ and b = 30° and can be stated as y = sin $^{1}/_{2}$(x + 60)°

The value of $^{b}/_{a}$ indicates that the graph moves 60° to the left of the basic sin graph.

The value of a indicates that it repeats every 720°(the period is 360° divided by a half)



Trig. functions exercise

20 min

Q70: Graph the following functions:

a) sin (3x - 30)°
b) cos (x + 45)°
c) sin (2x + 120)°
d) sin ($^{1}/_{3}$x - 60)°
e) cos (2x + 90)°

It may be apparent from some of the examples and exercises in this section that it is possible to give a function in terms of sin or cos.
Since both graphs are essentially the same, this should not come as a surprise.

Activity

Using a graphics calculator, graph the functions shown in the last exercise.

For the sine functions, try to define these as functions of cosine and check that the graphs are identical by also graphing them on the calculator.

For the cosine functions, define these as sine functions and graph these to check.

2.9 Features of exponential and logarithmic graphs

Learning Objective

Recognise the graphs of logarithmic and exponential functions

2.9.1 Exponential functions and graphs

Exponential function calculator investigation

5 min

On graph paper plot the graph of y = 2^x by calculating the values of y for x = [-2, 4].

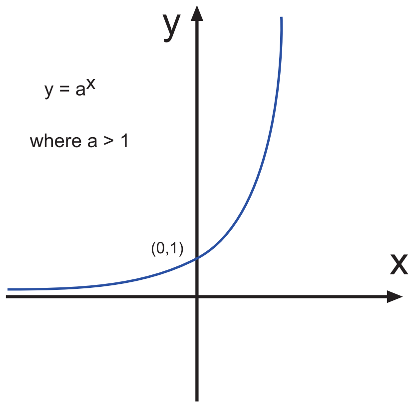
Use a graphics calculator and plot the graphs of 5^x and 6^x and note the relationship between the three graphs.

Exponential function

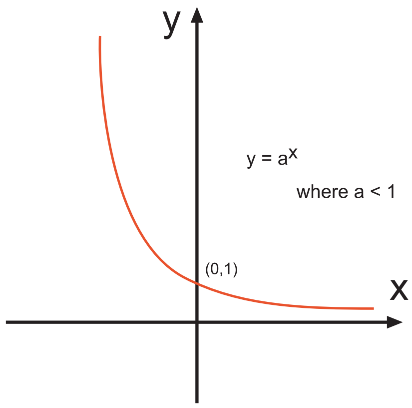
A function of the form $f(x) = a^x$ where $a > 0$ and $a \neq 1$ is called an exponential function

Features of $f(x) = a^x$

- The graph passes through the point (0, 1)
- If $a > 1$ the graph is increasing with the shape



- If $a < 1$ the graph is decreasing with the shape



Exponential investigation

10 min

Using a graphics calculator plot the graphs of $f(x) = a^x$ for values of a between 0 and 1 on the same screen. (use $^1/_2$, $^1/_3$, $^1/_4$ and $^1/_5$)

On a clear screen plot the graph of a^{-x} for $a \in [2, 5]$

What is the connection between $y = a^{-x}$ where $a > 1$ and $y = a^x$ where $0 < a < 1$?

The investigations demonstrate the relationship that

$(^1/_a)^x = a^{-x}$ for $a > 1$ and $0 < a < 1$

It is also possible to apply the rules given earlier for sketching related graphs to exponential functions. Here are two examples.

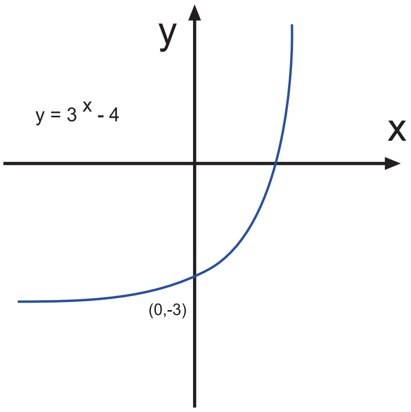
Examples

1. $y = a^x + k$

Sketch the graph of $3^x - 4$

Answer:

Using the rule for f (x) + k , the graph of $3^x - 4$ slides down the y-axis by 4 units.

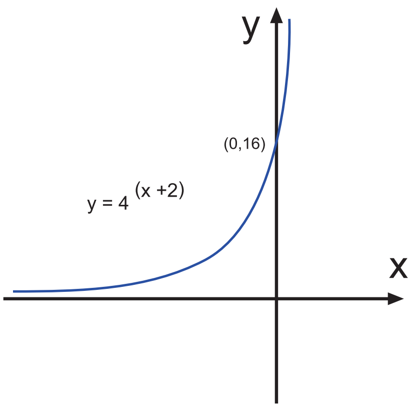


2. $y = a^{x + k}$

Sketch the graph of $y = 4^{x + 2}$

Answer:

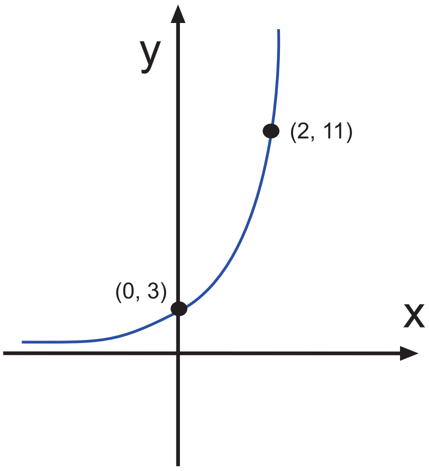
Using the rule for f (x + k) , the graph of $4^{x + 2}$ slides to the left by 2 units.



It is straightforward to find the equation of a function of the form f (x) = $a^x + k$ if two points on the graph are known.

Example : The equation of an exponential function from two points

Find the equation of the function which has the form f (x) = $a^x + k$ as represented on the graph:



Answer:

Using the point (0, 3) in the equation gives

$3 = a^0 + k \Rightarrow 3 = 1 + k \Rightarrow k = 2$

Using the point (2, 11) and k = 2 in the equation gives

$11 = a^2 + 2 \Rightarrow a^2 = 9 \Rightarrow a = 3$

(remember that a is positive)

The equation of the function is $f(x) = 3^x + 2$

10 min

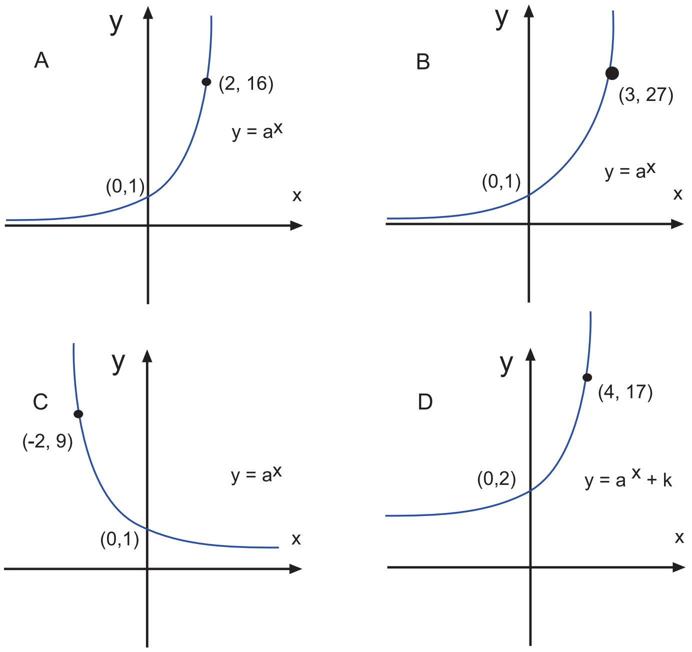
Exponential exercise

Q71: Sketch the graph of the function $f(x) = 2^x - 3$

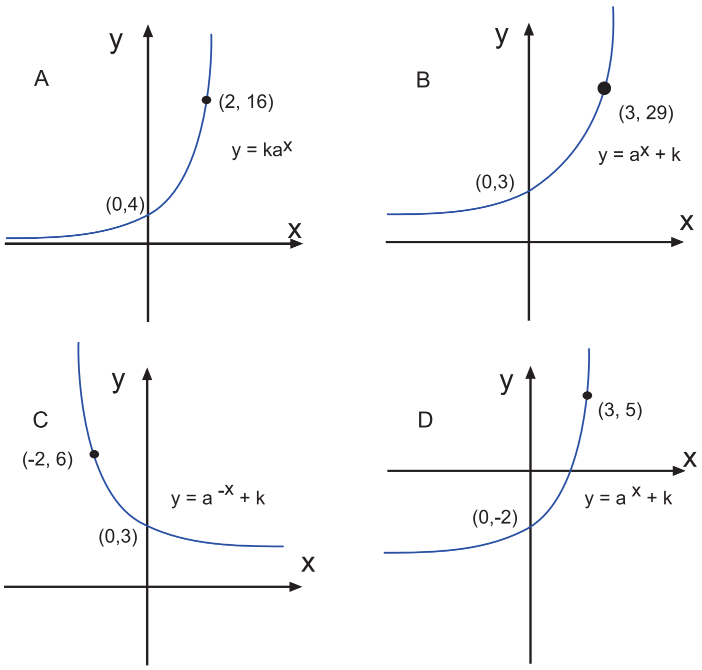
Q72: Sketch the graph of the function $f(x) = 2^{2x}$

Q73: Sketch the graph of the function $f(x) = 3^{-x}$

Q74: Find the equations of the following graphs in the form shown on each.



Q75: Find the equations of the following graphs in the form shown on each.



2.9.2 Logarithmic functions and graphs

In an earlier subsection of this topic, inverse functions were explored and a useful method of obtained them was introduced.

Recall, that, if a function is one-to-one and onto then an inverse exists and the graph of the inverse function $f^{-1}(x)$ is obtained from the graph of the original function $f(x)$ by reflecting it in the line $y = x$

Consider the exponential functions shown in the previous section. These are one-to-one functions and can be restricted to give onto functions. If this is the case then an inverse exists for each of these functions.

Logarithmic function calculator investigation

5 min

Using a graphics calculator plot the graph of y = x, e^x and $\log_e x$ on the same screen. Note that $\log_e x$ is the same as ln x

Now clear the screen and plot the graph of y = x, 10^x and $\log_{10} x$

on the same screen.

Determine the relationship between the graphs e^x and $\log_e x$
and between 10^x and $\log_{10} x$

On separate sheets of graph paper plot the following graphs and their reflection in the line y = x (remember that a point (x, y) on the graph will become the point (y, x) on the reflected graph:

a) 2^x

b) 3^x

c) 4^x

These activities should demonstrate that the inverse of an exponential function is a logarithmic function.

To be precise if f (x) = a^x then the inverse function f^{-1} (x) = $\log_a x$

A calculator will produce any values of a^x but it can only produce logarithmic function values for two functions: to the base 10 (the inverse of 10^x and shown on a calculator as 'log'): to the base e (the inverse of e^x and shown on the calculator as 'ln').

At this stage, the easiest way to plot a log function is to relate it to the exponential function and reflect the graph in the line y = x

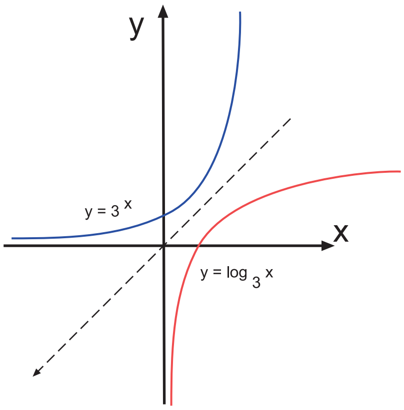
Example : Plotting a log function

Plot the graph of f (x) = $\log_3 x$

Answer:

Since exponential and logarithmic functions are inverses of each other, f (x) has an inverse $f^{-1}(x) = 3^x$

This can be plotted and then reflected in the line y = x to produce f (x) = $\log_3 x$



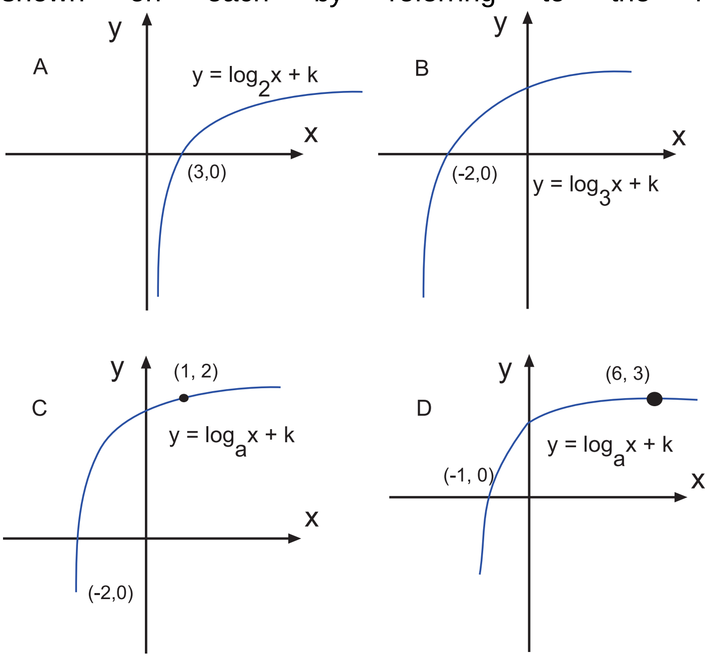
Logarithmic function exercise

10 min

Q76: Plot the following logarithmic functions by relating each to its exponential inverse:

a) $\log_2 x$
b) $\log_4 x$
c) $\log_{10} x$
d) $\log_e x$

Q77: Identify the functions in the following sketches in the form shown on each by referring to the related exponential graphs:



2.10 Identifying functions from graphs

Learning Objective

Identify the probable form of a function equation from the graph of the function

Identifying the probable form of a function from its graph can be useful.

Any straight line graph depicts a function of the form
$y = mx + c$ where m is the gradient and c is the y-intercept.

Conversely, with this information, an accurate sketch can be made of a straight line graph.

With quadratic functions, the relationships shown in this topic can be used to provide enough information to make a sketch or to identify a function.

Using the simplest of quadratic functions, $f(x) = x^2$ it is possible to identify the graphs of functions such as:

- $-x^2$ (x^2 reflected in the x-axis.)
- $(x - 2)^2$ (x^2 is moved to the right by 2 units.)
- $3x^2$ (x^2 is stretched vertically by a factor of 3)

and so on.

The other features such as crossing the x-axis and y-axis can be determined as shown in the subsection on completing the square.

Recall also that completing the square is useful in finding the turning points.

Example Sketch the function $(x + 3)^2$

The function has a graph $y = (x + 3)^2$

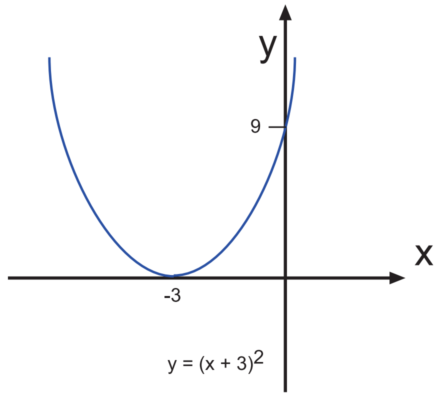
It crosses the x-axis at x = 0, that is when y = 9

This is the point (0, 9)

It crosses (or touches) the y-axis at y = 0, that is when x = -3. This is the point (-3, 0)

This is also the turning point.

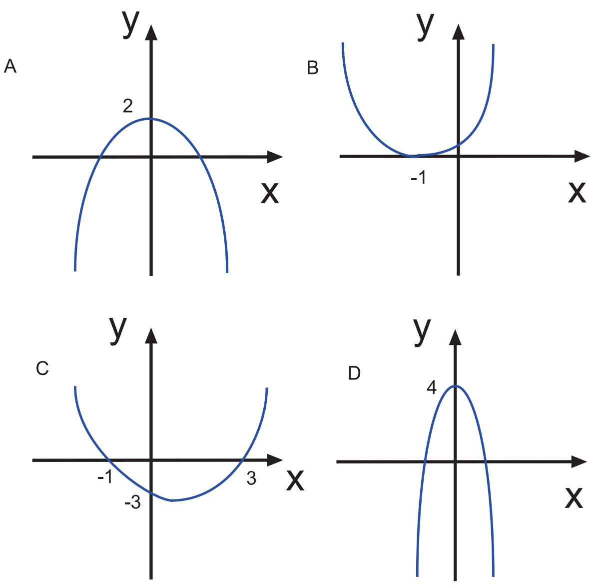
From the relationship to x^2 the graph moves to the left by 3 units.



Identify graphs exercise

10 min

Q78: Identify the following graphs:



Q79: Sketch the graphs of the following functions using the techniques shown in this topic:

a) $(x + 1)^2 - 2$

b) $x^2 - 4x + 2$

c) $(x - 3)^2$

2.11 Summary

The following points and techniques should be familiar after studying this topic:

- Identifying domain and range of a function.
- Sketching inverse functions from a given graph.
- Finding composite functions.
- Using the rules for related functions to identify graphs or sketch functions.
- Completing the square for quadratic functions.
- Recognising and sketching trig functions.
- Identifying the relevant features of trig graphs.
- Converting between radians and degrees.
- Stating the exact values of the trig ratios for common angles in degrees or radians.
- Recognising and sketching logarithmic and exponential functions.
- Identifying the features of logarithmic and exponential graphs.
- Identifying the possible form of a function from a graph.

2.12 Extended information

Learning Objective

Display a knowledge of the additional information available on this subject

There are links on the web which give a selection of interesting sites to visit. These sites can lead to an advanced study of the topic but there are many areas which will be of passing interest. Note in this topic there are many sites and a search is best done on a specific subject rather than on the topic in general.

Cauchy

In his early years Cauchy's work concentrated on geometry and functions. In later years he become involved in differential equations and ideas on continuity of functions. He is however, credited with introducing a rigorous approach to Mathematics.

Leibnitz

Leibnitz was a German mathematician whose work centred on calculus. He was the man who first defined a function as the slope of a curve.

Euler

It was Euler who defined a function in modern day terms. He also made major contributions to the work on number theory and sequences.

The work on trigonometric functions dates back a very long way. Try to find some mathematicians on this. (The Greek mathematicians are always a good place to start.)

2.13 Review exercise

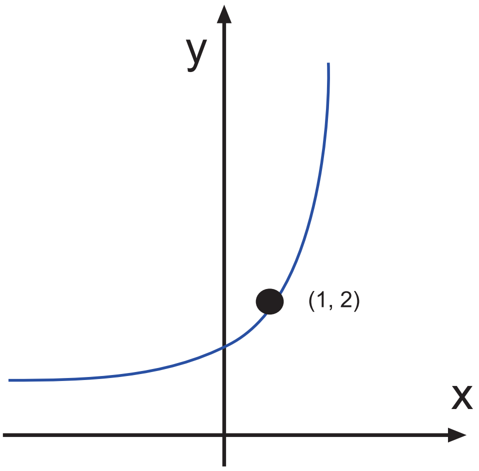
Review exercise

This is an exercise which reflects the work covered at 'C' level in this topic.

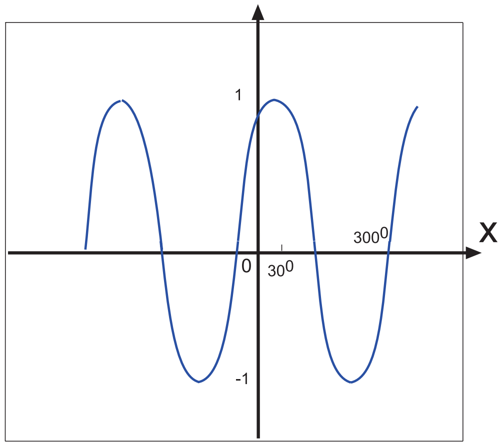
15 min

Q80: Identify the equation for f (x) for the following graphs:

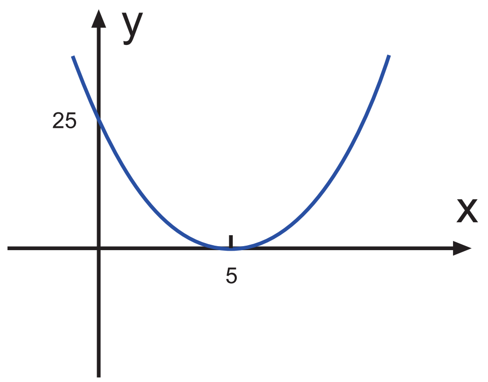
a)



b)



c)



Q81: Sketch the graphs of :

a) $2(x - 2)^2$

b) $\log_3 x - 2$

Q82: If f (x) = 3 sin x, g (x) = $(x - 1)^2$ and h (x) = 3x + 4 find the following composite functions:

a) g (f (x))

b) h (f (x))

c) f (h (x))

2.14 Advanced review exercise

Advanced review exercise

15 min

Q83: Find the minimum and/or maximum values of the following:

a) $4(x + 1)^2$

b) $3(\cos x - 1)^2 - 2$

c) $-4 - (x + 3)^2$

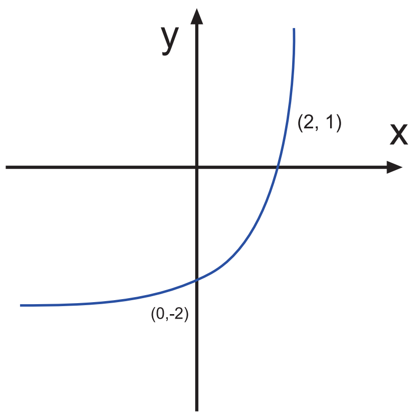
Q84: Complete the square for the following expressions:

a) $3x^2 - x + 5$

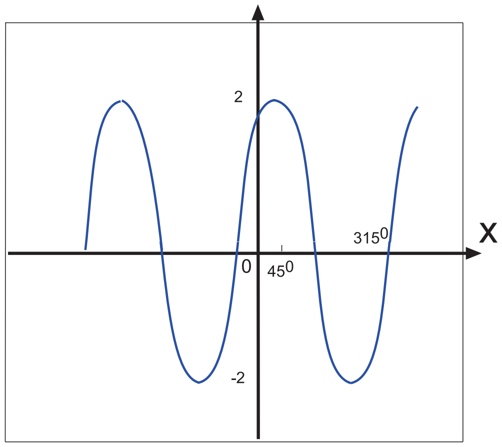
b) $2x^2 - 3x + 1$

Q85: Identify the equation for f (x) for the following graphs:

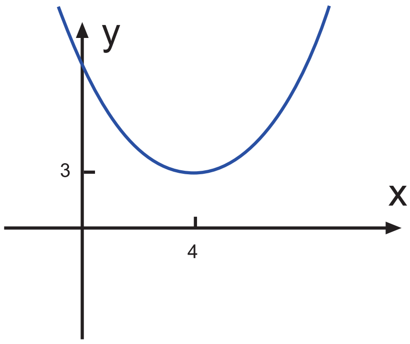
a)



b)



c)



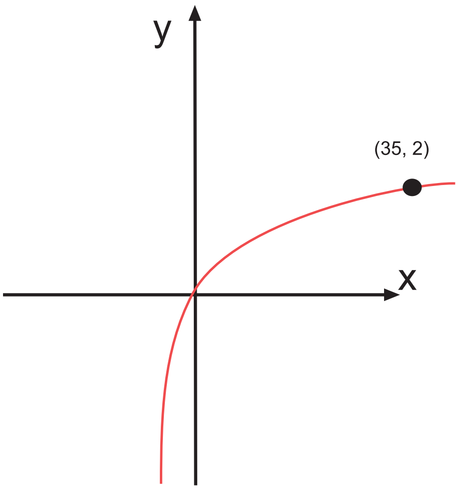
2.15 Set review exercise

15 min

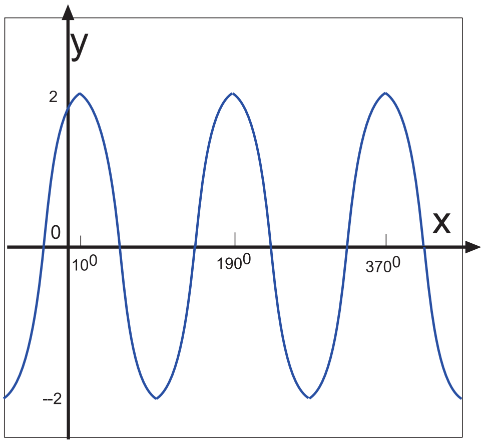
Set review exercise

Work out your answers and then access the web to input your answers in the online test called set review exercise.

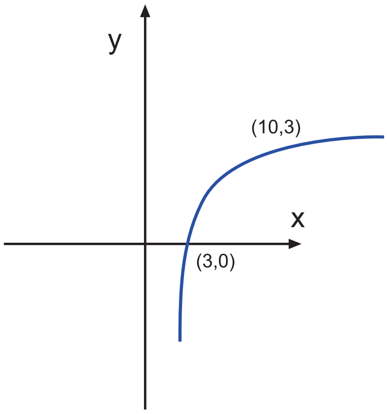
Q86: Identify the equation for f (x) for the following graph:



Q87: Identify the equation for f (x) for the following graph:



Q88: Identify the equation for f (x) for the following graph:



Q89: Find f (g (x)) and g (f (x)) for the following pairs of functions:

a) If f (x) = x^2 - 3 and g (x) = 4x + 3, what is f (g (x)) ?

b) If f (x) = x^2 - 3 and g (x) = 4x + 3, what is g (f (x)) ?

c) If f (x) = - 2 - x^4 and g (x) = $\frac{1}{2x - 1}$, what is f (g (x)) ?

d) If f (x) = - 2 - x^4 and g (x) = $\frac{1}{2x - 1}$, what is g (f (x)) ?

e) If f (x) = 2 - 3x and g (x) = cos x, what is f (g (x)) ?

f) If f (x) = 2 - 3x and g (x) = cos x, what is g (f (x)) ?

Topic 3

Basic Differentiation

Contents

Learning Objectives

Use basic differentiation

Minimum performance criteria:

- *Differentiate a function reducible to a sum of powers of x.*
- *Determine the gradient of a tangent to a curve by differentiation.*
- *Determine the coordinates of the stationary points on a curve and justify their nature using differentiation.*

Prerequisites

Before attempting this unit you should be familiar with algebraic operations on indices. It will be useful to remember the following rules

- $x^0 = 1$
- $x^1 = x$
- $\sqrt{x} = x^{1/2}$
- $\sqrt[n]{x^m} = x^{m/n}$
- $\frac{1}{x^n} = x^{-n}$
- $x^m \times x^n = x^{m+n}$
- $\frac{x^m}{x^n} = x^{m-n}$
- $(x^m)^n = x^{mn}$
- $(xy)^m = x^m y^m$

Examples

1. Write $\frac{2}{3\sqrt{x}}$ in the form ax^n

Solution

$$\frac{2}{3\sqrt{x}} = \frac{2}{3x^{1/2}} = \frac{2}{3}x^{-1/2}$$

2. Simplify $\frac{3x^2 + 1}{2\sqrt{x}}$

Solution

$$\begin{aligned}\frac{3x^2 + 1}{2\sqrt{x}} &= \frac{3x^2}{2\sqrt{x}} + \frac{1}{2\sqrt{x}} \\ &= \frac{3x^2}{2x^{1/2}} + \frac{1}{2x^{1/2}} \\ &= \frac{3}{2}x^{3/2} + \frac{1}{2}x^{-1/2}\end{aligned}$$

3.1 Revision exercise

Learning Objective

Identify areas which need revision

Revision - Exercise 1

30 min

These questions are intended to practice skills that you should already have. If you have difficulty you could consult a tutor or a classmate.

An on-line exercise is available at this point, which you might find helpful.

Try the following exercise to test your skills.

Some revision may be necessary if you find this difficult.

Q1: Simplify the following expressions (give your answers with positive powers)

a) $4x^2 \times x^5$

b) $\frac{y^5}{y^2}$

c) $\frac{a^7}{3a^4}$

d) $\left(b^3\right)^2$

e) $(2y)^3$

f) $\frac{2}{x^{-2}}$

g) $\left(m^{1/4}\right)^4$

h) $\left(x^{-2}\right)^3$

i) a^0

j) $\frac{3x}{\sqrt{x}}$

Q2: Write each of the following in the form ax^n

a) $\frac{1}{x^2}$

b) $\frac{5}{x}$

c) $\frac{1}{3x^4}$

d) $\frac{1}{2x}$

e) $5\sqrt{x}$

f) $\sqrt[3]{x}$

g) $\sqrt[3]{x^4}$

h) $\frac{1}{\sqrt{x}}$

i) $\frac{1}{3\sqrt{x}}$

j) $\frac{2}{\sqrt[3]{x}}$

Q3: Simplify the following, leaving your answer as a sum or difference of terms in the form ax^n

a) $\sqrt{x}\left(\sqrt{x} + 2x\right)$

b) $\frac{1}{\sqrt{x}}\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)$

c) $\left(\sqrt[3]{x} + 1\right)\left(\sqrt[3]{x} - 1\right)$

d) $\frac{(x - 3)^2}{2x}$

e) $\left(x + \frac{1}{x}\right)^2$

f) $\frac{3x^4 - x^5}{x^2}$

g) $\frac{6x^4 + x^2}{2x^2}$

h) $\frac{3x^2 + 5x}{\sqrt{x}}$

i) $\left(\frac{2}{\sqrt{x}} + \sqrt{x}\right)\left(\frac{2}{\sqrt{x}} - \sqrt{x}\right)$

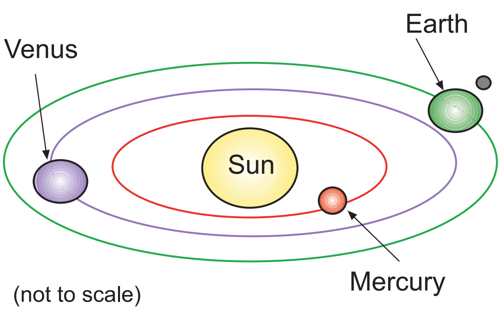
j) $\frac{x^5 - 3x^2}{x\sqrt{x}}$

3.2 Introduction

Learning Objective

Gain an appreciation of the relevance of Calculus.

Much of our understanding of the world we live in depends on our ability to describe how things change with time. This could range from the motion of a bouncing ball, the orbit of a planet, the flight path of a rocket, the spread of an epidemic, the growth of an economy or the decay of a radioactive substance.



Calculus was developed independently by two seventeenth century mathematicians Sir Isaac Newton (1642 - 1727) and Gottfried Leibniz (1646 - 1716) as a method for analyzing the motion of moving objects. Indeed calculus can be used to study any situation involving a rate of change. It introduces two new operations called **differentiation** and **integration**, which like addition and subtraction are opposites of each other.

differentiation

Differentiation is the method for calculating the derived function, f'(x), from f(x)

integration

Integration is the method for finding anti-derivatives.

Calculus

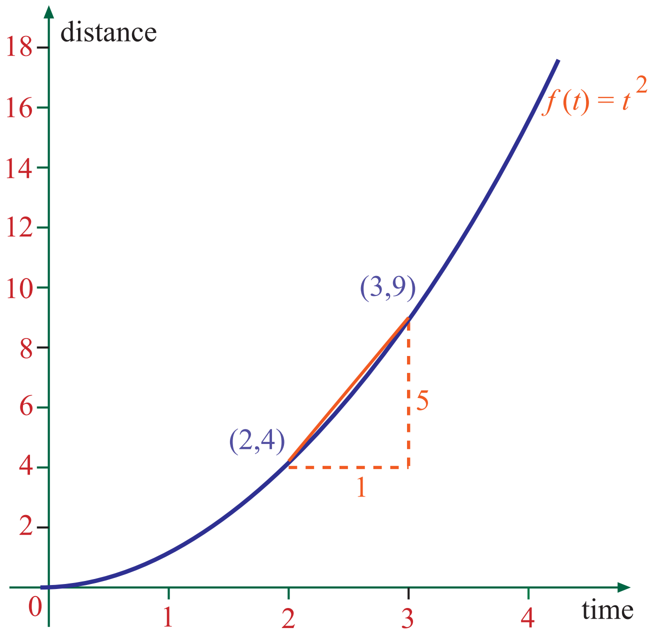
Calculus is the mathematics of motion and change.

3.3 Instantaneous speed

Learning Objective

Recognise that instantaneous speed can be obtained as the derivative of a function.

The following diagram shows a distance-time graph for a car travelling along a road on the first 4 seconds of its journey.



You should recall that we can calculate average speed as, average speed = $\frac{\text{distance}}{\text{time}}$

Thus the average speed for the car in the first three seconds of its journey is

$$\begin{aligned} \text{average speed} &= \frac{\text{distance}}{\text{time}} \\ &= \frac{9}{3} \\ &= 3 \text{ m/s} \end{aligned}$$

Look again at the distance-time graph and answer the following questions.

Q4: When does it seem that the car is travelling faster, at t = 1 or at t = 4?

Q5: How can you tell this from the graph?

Q6: Why can we only calculate the *average* speed for the first three seconds of the journey and not the exact speed?

Now consider this question.

What is the speed at exactly t = 2?

We can obtain a first estimate if we calculate the average speed between t = 2 and t = 3.

$$\begin{aligned}\text{average speed} &= \frac{\text{distance}}{\text{time}}\\ &= \frac{f(3) - f(2)}{3 - 2}\\ &= \frac{9 - 4}{3 - 2}\\ &= \frac{5}{1}\\ &= 5 \text{ m/s}\end{aligned}$$

To obtain a better estimate for the instantaneous speed at t = 2 we could choose shorter time intervals. See the following table of results,

time interval	distance	time	average speed
2 - 3	5	1	5
2 - 2.5	2.25	0.5	4.5
2 - 2.2	0.84	0.2	4.2
2 - 2.1	0.41	0.1	4.1
2 - 2.01	0.0401	0.01	4.01
2 - 2.001	0.004001	0.001	4.001

You should notice that as the time tends to zero the speed tends to a limit of 4 m/s. It seems reasonable to conclude that the instantaneous speed of the car at t = 2 is 4 m/s.

In a similar way it is also possible to calculate the instantaneous speed at t = 1,2,3,4,5... etc. The results are shown here.

time	1	2	3	4	5
instantaneous speed	2	4	6	8	10

You can check these results for yourself in the simulation 'Calculating instantaneous speed' in the on-line materials.

Perhaps you have already noticed that the instantaneous speed at time t can be calculated as 2t m/s.

derived function

The instantaneous speed, or rate of change of distance with respect to time, can be written as f'(t) and is known as the **derived function** of f(t).

In a similar way it is also possible to consider other functions apart from f (t) = t^2 and the results that are obtained for the derived function are as listed here.

distance $f(t)$	t	t^2	t^3	t^4	t^5	......	t^n
speed $f'(t)$	1	$2t$	$3t^2$	$4t^3$	$5t^4$		nt^{n-1}

If you wish, it is possible to check some of these results for yourself, using the same method as for $f(t) = t^2$

Notice that we have obtained a general rule for finding the derived function

When $f(t) = t^n$ then $f'(t) = nt^{n-1}$

Calculating instantaneous speed

A simulation is available in the on-line materials to complement this section.

20 min

3.4 Differentiation from first principles.

Learning Objective

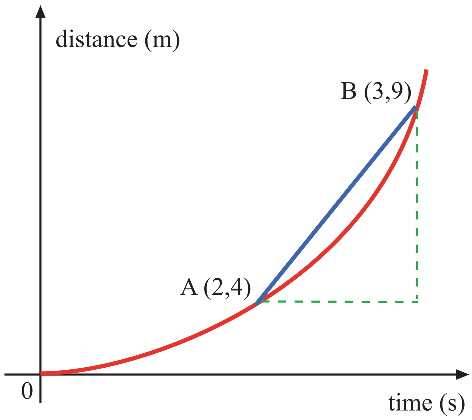
Know, from first principles, the definition for f'(x)

We can obtain the instantaneous speed of a car at time t = 2 seconds by calculating the average speed over smaller and smaller time intervals. It is important to relate this to the features of the graph for f(t).

The first estimate was the average speed between t = 2 and t = 3.

Notice that

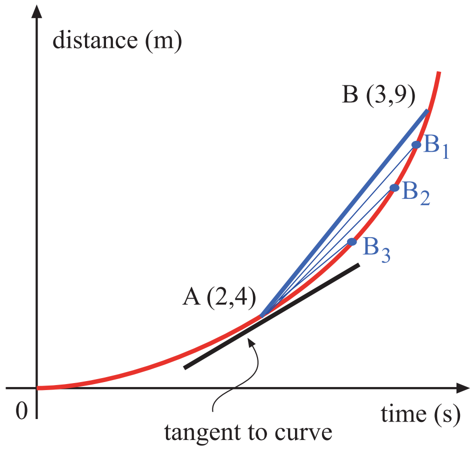
$$\text{average speed} = \frac{\text{distance}}{\text{time}}$$
$$= \text{gradient of chord AB}$$



Better estimates for the instantaneous speed at t = 2 are obtained by taking progressively shorter time intervals.

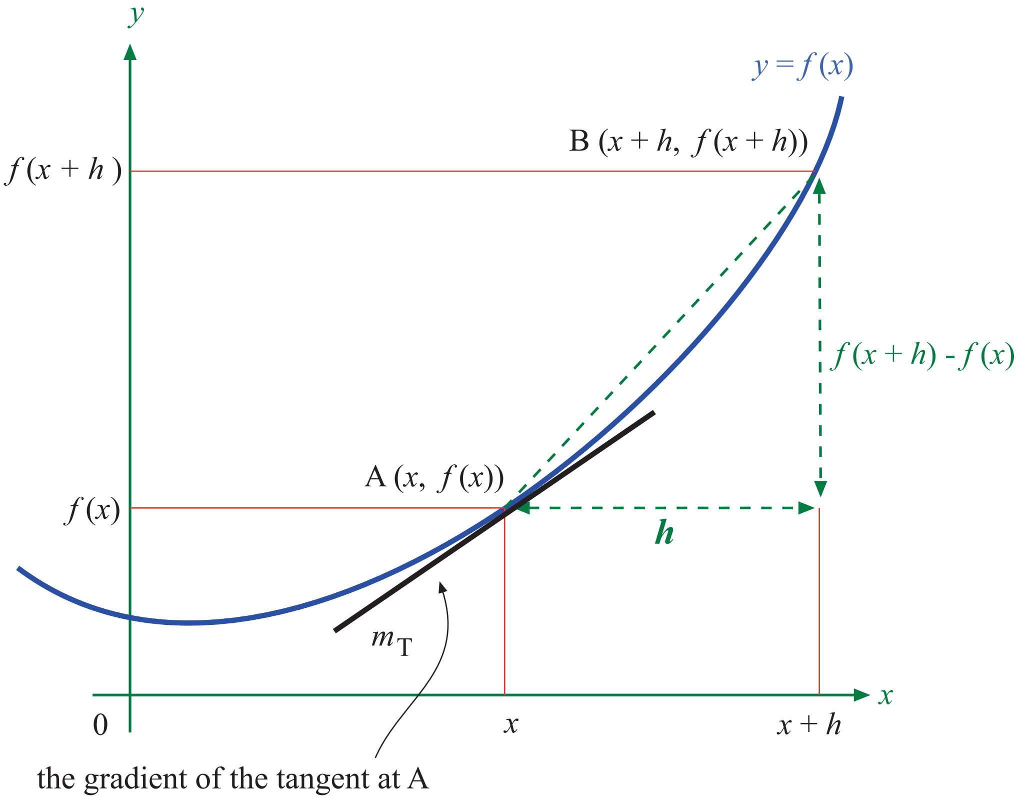
Notice that the gradients of the chords AB_1, AB_2 and AB_3 move closer to the gradient of the tangent to the curve at t = 2

Indeed the instantaneous speed at t = 2 is equal to the gradient of the tangent to the curve at that point.



In general the instantaneous speed, f'(t), is equal to the gradient of the tangent to the curve at that time.

The gradient of a tangent can now be used to calculate the instantaneous rate of change of any function as we see in the following explanation.



If we wish to find the gradient of the curve y = f(x) at the point A (x, f(x)) then we can take a second point B (x+h, f(x+h)) and obtain a first estimate which is the gradient of the chord AB.

$$m_{AB} = \frac{f(x + h) - f(x)}{(x + h) - x}$$

$$= \frac{f(x + h) - f(x)}{h}$$

The gradient of the curve at A which is equal to the gradient of the tangent to the curve at A is then derived in the same way as when we calculated an instantaneous speed, by reducing the interval h towards 0.

As $h \Rightarrow 0$, (h tends to 0), $\frac{f(x + h) - f(x)}{h}$ tends to a limit f'(x) and we can write the following.

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

f'(x) is the derived function or the derivative of f(x). It represents the gradient of the tangent to the graph of the function and the rate of change of the function.

When we use this formula to obtain f'(x) this is called differentiating from first principles.

Example

Find, 'from first principles', (i.e. from the definition) the derivative of f with respect to x, whenf(x) = x^2

Solution

Note that when $f(x) = x^2$

then $f(x + h) = (x + h)^2$

hence
$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$
$$= \lim_{h \to 0} \frac{(x + h)^2 - x^2}{h}$$
$$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h}$$
$$= \lim_{h \to 0} \frac{h(2x + h)}{h}$$
$$= \lim_{h \to 0} 2x + h$$
$$= 2x$$

Therefore when $f(x) = x^2$ then $f'(x) = 2x$ (as you knew already).

Now try the question in Exercise 2

Exercise 2

15 min

Note that differentiation from first principles is not included in this syllabus however, you might like to try the following questions for yourself.

Find, from first principles, the derivatives of the following functions.

Q7: $f(x) = x^3$

Q8: $f(x) = 5x^2$

Note that, in general, a function is differentiable at a point x = a if the curve y = f(x) has a tangent at that point.

Also, the function f(x) is differentiable over the interval $a \leqslant x \leqslant b$ if f'(x) exists for each x in the interval.

3.5 A rule for differentiation

Learning Objective

Learn the rule for differentiating $f(x) = x^n$

From first principles when f (x) = x^2 then f ′ (x) = 2x

It is also possible, in a similar way, to obtain the derivatives of other functions as shown in the following table.

$f(x)$	1	x	x^2	x^3	x^4	x^5	......	x^n
$f'(x)$	0	1	$2x$	$3x^2$	$4x^3$	$5x^4$		nx^{n-1}

You should notice that

When f (x) = x^n then f ′ (x) = nx^{n-1}

Note, in the following examples, that before differentiating the function must be expressed in the form x^n

You should also be aware that the variable may be denoted by other letters apart from x such as a,t,s,u,etc. and the function maybe denoted by another letter such as g(x) or h(t)

Examples

1.

Differentiate f (x) = x^{10}

Solution

f ′ (x) = $10x^9$

2.

Differentiate g(x) = $x^{4/3}$

Solution

g'(x) = $\frac{4}{3}x^{1/3}$

3.

Differentiate f (t) = t^{-3}

Solution

f ′ (t) = $-3t^{-4}$

4.

Differentiate $f(x) = \sqrt{x}$

Solution

We must first rewrite f(x) in terms of x^n

$f(x) = \sqrt{x} = x^{1/2}$

Therefore

$$\begin{aligned} f'(x) &= \tfrac{1}{2}x^{-1/2} \\ &= \frac{1}{2x^{1/2}} \\ &= \frac{1}{2\sqrt{x}} \end{aligned}$$

5.

Differentiate $f(u) = \dfrac{1}{u}$

Solution

Again we need to rewrite f(u) in terms of x^n

$f(u) = \dfrac{1}{u} = u^{-1}$

Therefore

$$\begin{aligned} f'(u) &= -u^{-2} \\ &= -\frac{1}{u^2} \end{aligned}$$

Now try the questions in the following exercise.

Exercise 3

30 min

There are randomised questions on-line which you can use for further practice.

Differentiate the following functions.

Q9:

a) $f(x) = x^8$
b) $f(x) = x^3$
c) $f(t) = t^7$
d) $f(a) = a^{20}$

Q10:

a) $f(x) = x^{-2}$
b) $g(x) = x^{-1}$
c) $h(t) = t^{-19}$
d) $f(s) = s^{-50}$

Q11:

a) $f(x) = x^{5/4}$

b) $f(t) = t^{3/4}$

c) $f(x) = x^{-2/3}$

d) $g(x) = x^{-3/7}$

e) $f(x) = x^{2/7}$

f) $f(t) = t^{-8/5}$

Q12: In these questions be careful to express the function in the form x^n before you attempt to differentiate.

a) $f(x) = \frac{1}{x^2}$

b) $f(x) = \frac{1}{x^{2/3}}$

c) $f(x) = \sqrt[3]{x}$

d) $f(x) = \frac{1}{x^{7/4}}$

e) $f(x) = \sqrt{x^5}$

f) $f(x) = \frac{1}{\sqrt[4]{x}}$

g) $f(x) = \sqrt[3]{x^2}$

h) $f(x) = \frac{1}{\sqrt[5]{x^2}}$

3.6 Rates of Change

Learning Objective

Know that f'(a) is the rate of change of f at a and know how to translate from words to symbols and vice versa.

In an example the motion of a car in the first few seconds of its journey was described by the function $f(t) = t^2$. We were asked to estimate the instantaneous speed of the car at time t = 2 and we found that this is simply the gradient of the tangent to the curve at that time, $f'(t) = 2t$.

So the speed at t = 2, which is a measure of the ***rate of change*** of distance with respect to time, is $f'(2) = 2 \times 2 = 4$ m/s

Similarly, the speed of the car at t = 3 is $f'(3) = 2 \times 3 = 6$ m/s.

Examples

1.

Water flowing in a stream travels $\sqrt{t^3}$ metres from its source in t seconds.

Calculate the speed of the water after 16 seconds.

Solution

The distance travelled by the water is

$d(t) = \sqrt{t^3}$
$= t^{3/2}$

Thus the speed of the water is given by

$d'(t) = \frac{3}{2}t^{1/2}$
$= \frac{3}{2}\sqrt{t}$

After 16 seconds the speed of the water is

$d'(16) = \frac{3}{2}\sqrt{16}$
$= 6$ m/s

2. Find the gradient of the curve $f(x) = \frac{1}{x^3}$ at x = 2

Solution

$f(x) = \frac{1}{x^3}$
$= x^{-3}$

Hence

$f'(x) = -3x^{-4}$
$= -\frac{3}{x^4}$

and the gradient of the function at x =2 is

$$f'(2) = -\frac{3}{2^4} = -\frac{3}{16}$$

Exercise 4

30 min

There is an on-line version of this exercise , which you might find helpful.

Q13: When $f(x) = x^{2/3}$ find $f'(8)$

Q14: When $f(u) = u^4$ find the derivative of f at u = 5

Q15: Find the gradient of the tangent to the curve, $f(x) = \sqrt{x}$ at x = 9

Q16: For $f(x) = \sqrt[3]{x^4}$ find the rate of change of f at x = 64

Q17: Find the gradient of the curve $h(t) = \frac{1}{t^2}$ when t =0.5

Q18:

For $s(t) = \frac{1}{t}$ find $s'(4)$

Q19:

The volume of liquid, in m^3, flowing into a storage tank is calculated as $V(t) = t^{-1/2}$ where t is time in seconds.

Calculate the rate of change in the volume of liquid in the tank when t = 4

Q20:

The number of bacteria in an experimental culture is $B(t) = t^{5/4}$ where t is the time in minutes from the start of the experiment.

Calculate the growth rate of the culture when t =16

3.7 Further rules for differentiating

Learning Objective

Learn and use some further rules for differentiating

So far we have concentrated on functions that can be written in the form $f(x) = x^n$

It is also possible to find derivatives of other similar functions. You will find the following rules for differentiating useful.

1. When $f(x) = ax^n$ then $f'(x) = nax^{n-1}$, where a is a constant.
2. When f(x) = a then f'(x) = 0, where a is a constant.
3. When f(x) = g(x) + h(x) then f'(x) = g'(x) + h'(x)

Examples

1.

Differentiate f (x) = $8x^3$

Solution

$$\begin{aligned} f'(x) &= 3 \times 8x^2 \\ &= 24x^2 \end{aligned}$$

2.

Find f'(x) when f(x) = $\frac{1}{3x^2}$

Solution

First we must rewrite f(x) in the form ax^n

$$\begin{aligned} f(x) &= \frac{1}{3x^2} \\ &= \frac{1}{3}x^{-2} \end{aligned}$$

You should notice that 3 remains in the denominator of the fraction. It is now straightforward to find f'(x)

$$\begin{aligned} f'(x) &= -\frac{2}{3}x^{-3} \\ &= -\frac{2}{3x^3} \end{aligned}$$

3.

Calculate the derivative of f(x) = 7

Solution

f'(x) = 0

4.

Differentiate f(x) = $3x^4 + 5x^2 + 9$

Solution

f'(x) = $12x^3 + 10x$

5.

Find the derivative of f(x) = $\frac{3}{\sqrt{x}} + \frac{\sqrt{x}}{5}$

Solution

First we need to rewrite f(x) in the correct form.

$$f(x) = 3x^{-1/2} + \frac{1}{5}x^{1/2}$$

Now we can differentiate as follows.

$$\begin{aligned} f'(x) &= -\frac{3}{2}x^{-3/2} + \frac{1}{10}x^{-1/2} \\ &= -\frac{3}{2x^{3/2}} + \frac{1}{10\sqrt{x}} \end{aligned}$$

50 min

Exercise 5

There are randomised questions on-line which you can use for further practice.

Differentiate each of the following functions.

Q21: $f(x) = 2x^7$

Q22: $f(x) = \frac{2}{3}x^6$

Q23: $f(y) = \dfrac{5}{y^2}$

Q24: $f(x) = \dfrac{1}{4x^3}$

Q25: $g(x) = \dfrac{3}{5x}$

Q26: $f(x) = 5\sqrt{x}$

Q27: $f(x) = \dfrac{4}{\sqrt{x}}$

Q28: $f(x) = 9\sqrt[3]{x^2}$

Q29: $f(t) = 5t^2 - 2t + 9$

Q30: $f(x) = 7x^2 + \dfrac{1}{3x^3}$

Q31: $f(x) = \dfrac{\sqrt{x}}{2} - \dfrac{2}{\sqrt{x}}$

Q32: $g(x) = 5 - \dfrac{8}{3\sqrt{x}} + 7x$

Q33: Find the gradient of the curve $f(x) = x^2 - 5x + 6$ at $x = 8$

Q34: Calculate h'(2) when $h(t) = 2t^3 + 1 + \dfrac{1}{t^2}$

Q35: Find the rate of change of $d(x) = \dfrac{6}{\sqrt{x}} - \dfrac{1}{2}$ at $x = 9$

3.8 Differentiating products and quotients

Learning Objective

Differentiate more complicated expressions

It is also possible to differentiate more complicated expressions. The following examples show how this is done.

Example Differentiate the function f(x) = (x + 3)(2x - 5)

Solution

This function is written as a product of terms. Before we can differentiate we need to multiply the terms.

$$\begin{aligned} f(x) &= (x + 3)(2x - 5) \\ &= 2x^2 + x - 15 \end{aligned}$$

Now we are able to differentiate term by term.

f'(x) = 4x + 1

Example Differentiate the function $f(x) = \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2$

Solution

Multipy the function so that it can be expressed as a sum of individual terms.

$$\begin{aligned} f(x) &= \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2 \\ &= \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right) \\ &= \left(\sqrt{x}^2 + 2\frac{\sqrt{x}}{\sqrt{x}} + \frac{1}{\sqrt{x}^2}\right) \\ &= x + 2 + \frac{1}{x} \\ &= x + 2 + x^{-1} \end{aligned}$$

Now we are able to differentiate term by term.

$$\begin{aligned} f'(x) &= 1 - x^{-2} \\ &= 1 - \frac{1}{x^2} \end{aligned}$$

Example Differentiate the function $f(x) = \frac{x^5 + 3x^2 - x}{x^2}$

Solution

This function is written as a quotient. We need to simplify the expression for f(x) before we are able to differentiate.

$$\begin{aligned} f(x) &= \frac{x^5 + 3x^2 - x}{x^2} \\ &= \frac{x^5}{x^2} + \frac{3x^2}{x^2} - \frac{x}{x^2} \\ &= x^3 + 3 - x^{-1} \end{aligned}$$

We are now able to differentiate.

$$f'(x) = 3x^2 + x^{-2}$$
$$= 3x^2 + \frac{1}{x^2}$$

Now try the questions in the following exercise.

Exercise 6

50 min

There are randomised questions on-line which you can use for further practice.

Find the derivative for each of the following. Remember that you will need to simplify the expression for f(x). It is good practice to try to give your answers with positive indices whenever appropriate.

Q36: $f(x) = (3x - 1)(x + 5)$

Q37: $f(x) = (\sqrt{x} - 3)(x + 2)$

Q38: $f(x) = \left(x - \frac{1}{x}\right)^2$

Q39: $f(x) = \frac{3x^4 - x^2 + 7}{x}$

Q40: $f(x) = \frac{x^3 + x - 1}{x^2}$

Q41: $f(x) = \frac{4x^4 + 3}{\sqrt{x}}$

Q42: $f(x) = \frac{1}{\sqrt{x}}(x - 6)$

Q43: $f(x) = \frac{(2x + 1)(x - 3)}{x^3}$

Q44: $f(x) = \left(2\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2$

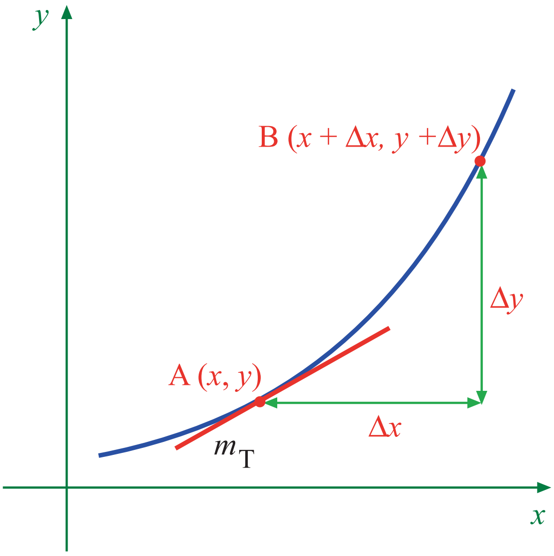
Q45: $f(x) = \frac{x^4 + 6x^3}{x\sqrt{x}}$

3.9 Leibniz notation

Learning Objective

Use and understand the $\frac{dy}{dx}$ notation for the derivative.

The gradient of the curve as the limit of the gradients of the chords can also be expressed using the alternative Leibniz notation.



Here Δx denotes a small change in x and Δy denotes a corresponding change in y.

Then $m_{AB} = \frac{\Delta y}{\Delta x}$

The gradient of the tangent at A is

$m_T = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$

This is usually written as $\frac{dy}{dx}$

Thus $\frac{dy}{dx}$ is equivalent tof'(x)

The following are all equivalent notations for the derivative
f'(x), y'(x) y', $\frac{dy}{dx}$, $\frac{d}{dx}$ (f(x)) , $\frac{df}{dx}$

Example For the function $y = (x^2 - 1)(x + 2)$ find $\frac{dy}{dx}$ at x = 3

Solution

First we must simplify the expression for y

$$y = \left(x^2 - 1\right)(x + 2)$$
$$= x^3 + 2x^2 - x - 2$$

Now we can differentiate to give

$$\frac{dy}{dx} = 3x^2 + 4x - 1$$

Therefore at x = 3

$$\frac{dy}{dx} = 3 \times 3^2 + 4 \times 3 - 1$$
$$= 38$$

Example Calculate $\frac{d}{dx}\left(x^5 + \frac{1}{x}\right)$

Solution

$$\frac{d}{dx}\left(x^5 + \frac{1}{x}\right) = \frac{d}{dx}\left(x^5 + x^{-1}\right)$$
$$= 5x^4 - x^{-2}$$
$$= 5x^4 - \frac{1}{x^2}$$

Now try the questions in the following exercise.

20 min

Exercise 7

There are randomised questions on-line, which you might like to attempt for further practice.

Q46: Find $\frac{dy}{dx}$ when $y = (3x + 2)^2$

Q47: Find the gradient of the tangent to the curve $y = 12 - 4x^2$ at the point $x = -\frac{1}{2}$

Q48: Find the rate of change of $y = 12\sqrt[3]{x}$ at $x = 8$

Q49: Calculate these derivatives

a) $\frac{d}{dx}\left(\frac{x - 1}{x}\right)$

b) $\frac{d}{dt}\left(9t^{4/3} + t\right)$

c) $\frac{d}{ds}\left(s\sqrt{s} + 7\right)$

Q50:

Find the coordinate of the point on the curve $y = x^2 - 3x + 5$ at which the tangent has gradient 1

Q51:

Find the coordinates of the two points on the curve $y = 2x^3 + 5$ at which the tangents have gradient 6

3.10 The equation of a tangent

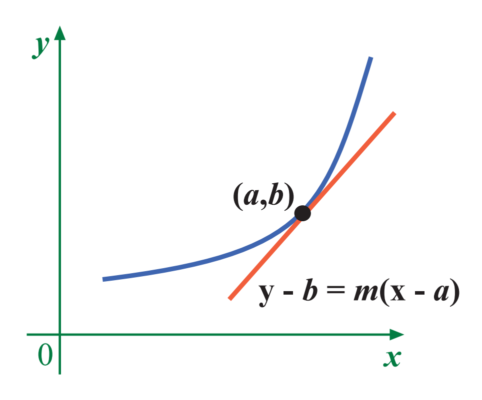
Learning Objective

Determine the equation of the tangent to a curve at a given point.

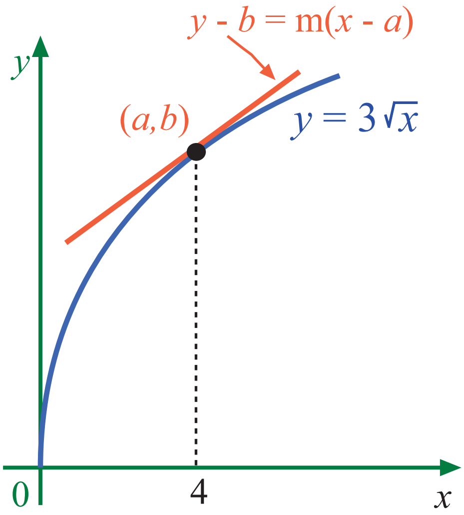
The tangent to a curve is a straight line and so we can write the equation for the tangent in the form $y - b = m(x - a)$

The following strategy is useful when finding the equation of the tangent.

1. Find the coordinates of (a,b), the point of contact of the tangent with the curve.
2. Calculate the gradient, m which is equal to the value of $\frac{dy}{dx}$ at x = a



Example



Find the equation of the tangent to the curve $y = 3\sqrt{x}$ at x = 4

Solution

1. We are given that x = 4, hence we are able to find the corresponding y coordinate by substituting x = 4 into the equation for the curve.

   When x = 4 then $y = 3\sqrt{x}$

   $= 3\sqrt{4}$

   $= 6$

   Thus the coordinates of the point of contact are (a,b) = (4,6)

2. When $y = 3\sqrt{x} = 3x^{1/2}$

   then

   $$\frac{dy}{dx} = \frac{3}{2}x^{-1/2} = \frac{3}{2\sqrt{x}}$$

   At x = 4

   $$\frac{dy}{dx} = \frac{3}{2\sqrt{4}} = \frac{3}{4}$$

   Thus the gradient of the tangent at x = 4 is $m = \frac{3}{4}$

We are now able to determine the equation of the tangent in the following way.

$$\begin{aligned} y - b &= m(x - a) \\ y - 6 &= \frac{3}{4}(x - 4) \\ 4y - 24 &= 3x - 12 \\ 3x - 4y + 12 &= 0 \end{aligned}$$

Thus the tangent to the curve $y = 3\sqrt{x}$ at $x = 4$ has equation $3x - 4y + 12 = 0$

Now try the questions in Exercise 8

Exercise 8

50 min

There are randomised questions on-line, which you might like to try for further practice.

Q52:

Find the equation of the tangent to the curve for each of the following.

a) $y = 2x^2$ at (2,8)

b) $y = x^2 + 4x$ at (1,5)

c) $y = \sqrt{x}$ at $x = 9$

d) $y = (x + 2)^2$ at $x = 1$

e) $y = \frac{2}{x^2}$ at $x = -1$

f) $y = \frac{x^2 - 1}{x}$ at $x = 2$

Q53:

a) Find the point on the parabola $y = 12 - 3x - x^2$ at which the gradient of the tangent is -1

b) Find the equation of this tangent.

Q54:

a) Find the equations of the tangents to $y = 3x - x^2$ at $x = 0$ and $x = 2$

b) Find the point of intersection of these tangents.

Q55:

The curve $y = (x - 1)(x^2 + 3)$ meets the x-axis at A and the y-axis at B.

Find

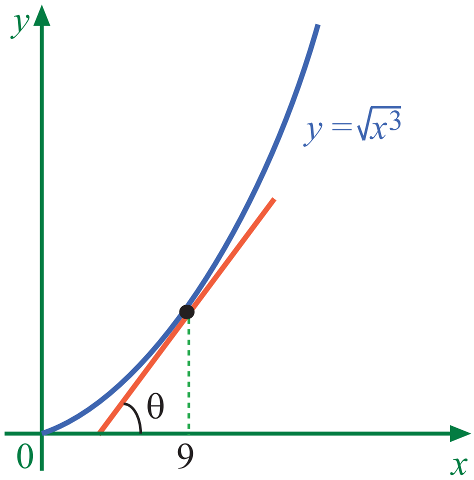
a) the equations of the tangents at A and B and

b) the coordinates of their point of intersection.

Q56:

The diagram shows the curve $y = \sqrt{x^3}$ with the tangent to the curve at x = 9

Calculate to the nearest degree, the angle that the tangent makes with the positive direction of the x-axis.



Q57:

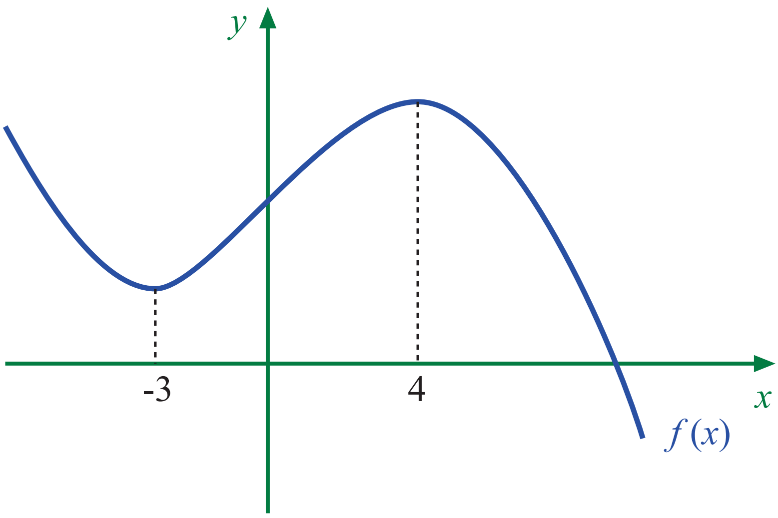
Find the equation of the tangent to the curve $y = x^2 + \frac{1}{4}$ which makes an angle of

3.11 Increasing and decreasing functions.

Learning Objective

Identify when a function is increasing or decreasing by considering the sign of f'(x)

Look at the graph shown here.



For the interval $-3 < x < 4$ as x increases the value of f(x) increases.

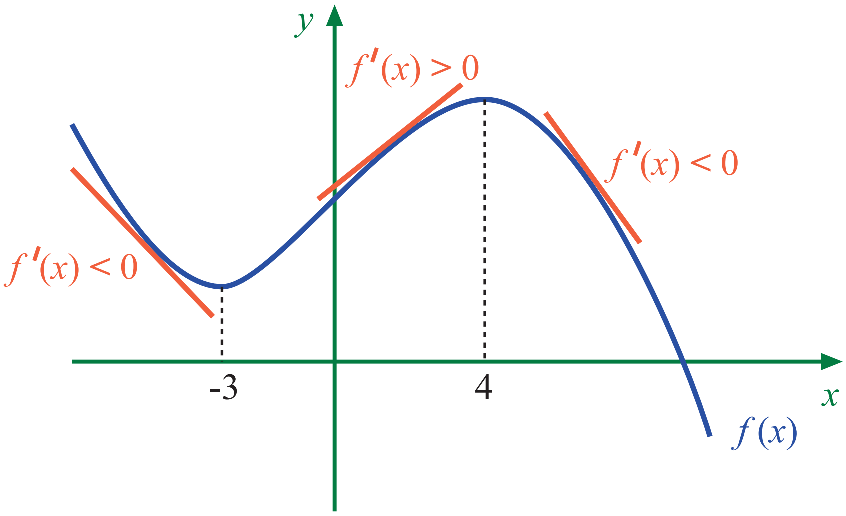
The graph is increasing for $-3 < x < 4$

For $x < -3$ and $x > 4$ as x increases the value of f(x) decreases.

The graph is decreasing for $x < -3$ and $x > 4$

Now study the same graph again as shown here.

Remember that a straight line that slopes up from left to right has positive gradient whereas a line that slopes down from left to right has negative gradient.

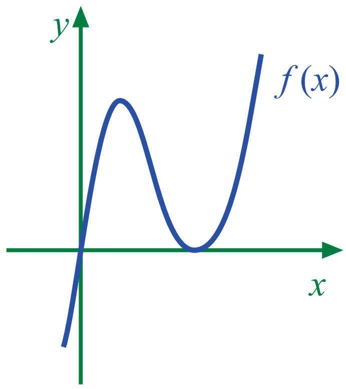


You should notice that

- when f(x) is increasing then $f'(x) > 0$
- when f(x) is decreasing then $f'(x) < 0$

Example

Find the interval in which the function f (x) = $x^3 - 6x^2 + 9x$ is increasing.



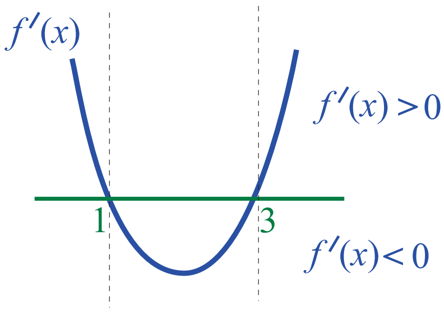
Solution

The function is increasing when $f'(x) > 0$

However, it is easier to identify when $f'(x) = 0$ first.

$$f'(x) = 3x^2 - 12x + 9 = 0$$
$$\Rightarrow 3\left(x^2 - 4x + 3\right) = 0$$
$$\Rightarrow 3(x - 1)(x - 3) = 0$$
$$\Rightarrow x = 1 \text{ or } x = 3$$

f'(x) is a parabola and has a curve as shown in the sketch here.



From the graph we can see that $f'(x) > 0$ when $x < 1$ or $x > 3$

Thus $f(x) = x^3 - 6x^2 + 9x$ is increasing when $x < 1$ or $x > 3$.

Check that this seems correct by looking at the sketch for f(x)

Example Show that the function $f(x) = \frac{1}{3}x^3 + 2x^2 + 4x$ is never decreasing.

Solution

$$f(x) = \frac{1}{3}x^3 + 2x^2 + 4x \Rightarrow f'(x) = x^2 + 4x + 4$$
$$= (x + 2)^2$$

Since $(x + 2)^2 \geq 0$ for all values of x then f(x) is never decreasing.

Exercise 9

There is an on-line exercise at this point, which you might find helpful.

30 min

Q58: Determine whether the following functions are increasing or decreasing at the given points.

a) $y = x^4 - 2x + 7$ at $x = 1$

b) $y = x^4 - 2x + 7$ at $x = -2$

c) $h(t) = 7t - 3t^2$ at $t = 2$

d) $f(x) = \frac{x^3 + 9}{x}$ at $x = 3$

Q59: For each of the following functions find the intervals in which they are increasing and decreasing.

a) $f(x) = x^2 + 7$

b) $f(x) = 2x^2 + 6x - 1$

c) $f(x) = x^3 + 3x^2 - 1$

d) $f(x) = 4x^2 - \frac{2}{3}x^3$

e) $f(x) = \frac{1}{3}x^3 - x^2 - 3x + 5$

f) $f(x) = x^3 + 12x^2 + 36x$

Q60: Show that the following functions are never decreasing.

a) $y = 2x^3 - 8$

b) $y = x^3 - 9x^2 + 27x - 18$

Q61: Show that the following functions are never increasing.

a) $y = \frac{1}{x}$

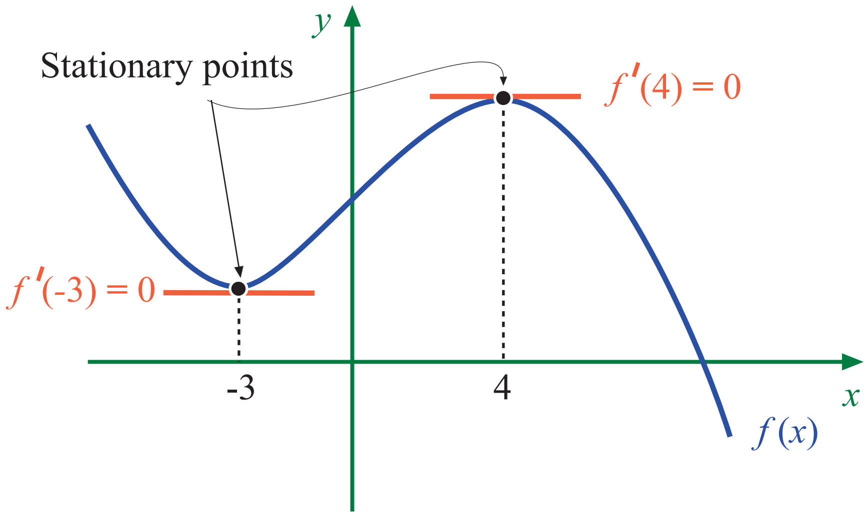
b) $y = 6x^2 + 15 - 6x - 2x^3$

3.12 Stationary points

Learning Objective

Determine the stationary points on a curve by identifying whenf'(x) = 0. Also, determine the nature of stationary points.

Look again at the graph we studied previously.



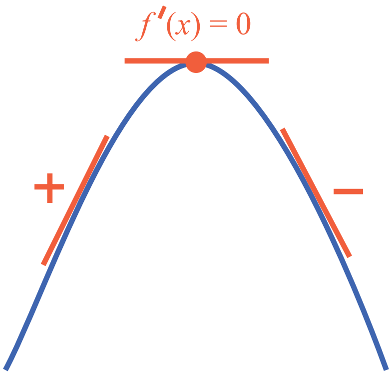
You can see that there are two points on this graph where the tangent to the curve is horizontal. These are called **stationary points** and f'(x) = 0 at these points.

stationary points

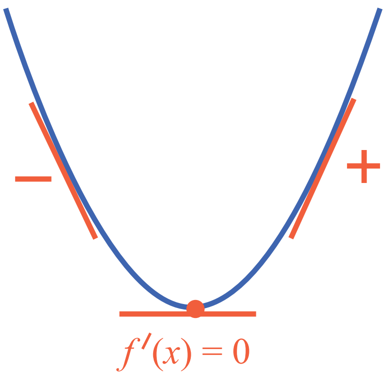
Stationary points are points on a curve where the function is neither increasing nor decreasing. At these points f'(x) = 0

The nature of a stationary point is determined by the sign of f'(x) on either side. Stationary points can be any of the following types.

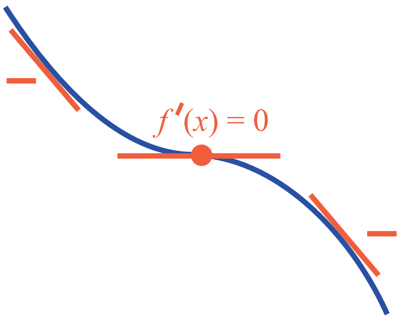
Maximum turning point



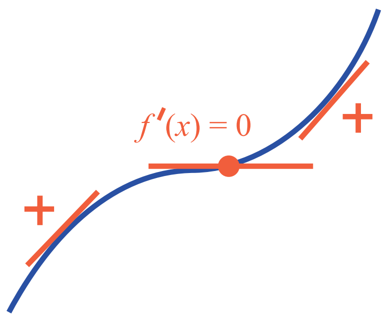
Minimum turning point



Falling point of inflexion



Rising point of inflexion



Example Find the stationary points on the curve $f(x) = 3x^4 + 8x^3 + 6$ and determine their nature.

Solution

$$f(x) = 3x^4 + 8x^3 + 6 \Rightarrow f'(x) = 12x^3 + 24x^2$$
$$= 12x^2(x + 2)$$

Stationary points occur when $f'(x) = 0$

Hence to find the stationary points we must solve

$$12x^2(x + 2) = 0$$
$$\Rightarrow 12x^2 = 0 \text{ or } x + 2 = 0$$
$$\Rightarrow x = 0 \text{ or } x = -2$$

When $x = 0$ then $f(0) = 6$

When $x = -2$ then $f(-2) = -10$

Thus the coordinates of the stationary points are (0,6) and (-2,-10)

To determine the nature of these stationary points we need to evaluate the gradient of the curve on either side. We can do this by completing a table of sign for $f'(x)$ as shown here. (Note that 0^- indicates a value just smaller than 0 whereas 0^+ indicates a value just bigger than 0).

x	-2^-	-2	-2^+
$f'(x)$	$-$	0	$+$
slope	↘	→	↗

There is a minimum turning point at (-2,-10)

x	0^-	0	0^+
$f'(x)$	$+$	0	$+$
slope	↗	→	↗

There is a rising point of inflection at (0,6)

Now try the questions in the following exercise.

50 min

Exercise 10

There is an on-line version of this exercise, which you might find helpful.

Find the stationary points on the following curves and determine their nature.

Q62: $f(x) = x^2 - 6x + 4$

Q63: $y = 3 + 8x - 2x^2$

Q64: $y = x^3 + 2$

Q65: $f(x) = 2x\left(3 - x^2\right)$

Q66: $f(x) = x^3 - 3x + 5$

Q67: $y = x^3 + 3x^2 - 9x$

Q68: $y = x^3 - 9x^2 + 15x - 7$

Q69: $f(x) = 4x^3 - x^4$

3.13 Curve sketching

Learning Objective

Sketch a curve with given equation by finding intersections with the axes, stationary point(s) and their nature and behaviour of y for large positive and negative values of x

In order to make a good sketch of a curve we need to find some information about the curve;

1. the points of intersection with the x and y axes,
2. the stationary points and their nature,
3. the behaviour of the curve for large positive and negative values of x

Example Sketch the curve $y = 3x^2 - x^3$

Solution

1.

Intersection with the axes.

x-axis

The curve cuts the x-axis when y = 0

$\Rightarrow 3x^2 - x^3 = 0$

$\Rightarrow x^2(3 - x) = 0$

$\Rightarrow x^2 = 0$ or $3 - x = 0$

$\Rightarrow x = 0$ or $x = 3$

Hence the x-intercepts are (0,0) and (3,0)

y-axis

The curve cuts the y-axis when x = 0

$\Rightarrow y = 3 \times 0^2 - 0^3$

$= 0$

Hence the y-intercept is (0,0)

2.

Stationary points and their nature.

$$y = 3x^2 - x^3 \Rightarrow \frac{dy}{dx} = 6x - 3x^2$$
$$= 3x(2 - x)$$

Stationary points occur when $\frac{dy}{dx} = 0$ hence we must solve

$3x(2 - x) = 0$

$\Rightarrow 3x = 0$ or $2 - x = 0$

$\Rightarrow x = 0$ or $x = 2$

When x = 0 then y = 0

When x = 2 then y = 4

Thus the stationary points are (0,0) and (2,4)

x	0^-	0	0^+
$f'(x)$	−	0	+
slope	↘	→	↗

There is a minimum turning point at (0,0)

x	2^-	2	2^+
$f'(x)$	+	0	−
slope	↗	→	↘

There is a maximum turning point at (2,4)

3.

Large positive and negative values of x

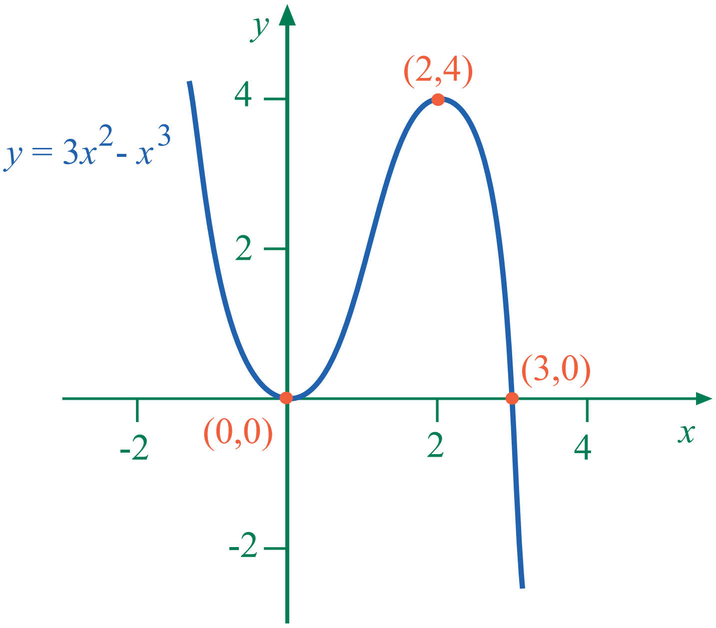
When x is very large the dominant term in the equation for the curve is the term with the largest positive power.

For $y = 3x^2 - x^3$ the dominant term is $-x^3$

As $x \Rightarrow +\infty$ then $y = 3x^2 - x^3 \Rightarrow -\infty$

As $x \Rightarrow -\infty$ then $y = 3x^2 - x^3 \Rightarrow -x^3 \Rightarrow +\infty$

We can now use all this information to make a good sketch of the curve.



Notice that important points on the curve are clearly labelled.

Now try the questions in the following exercise.

50 min

Exercise 11

There is an on-line exercise at this point, which you might find helpful.

Sketch and label the following curves.

It will help if you identify

1. the points of intersection with the x and y axes,
2. the stationary points and their nature,
3. the behaviour of the curve for large positive and negative values of x

Q70: $y = x^2 - 8x + 7$

Q71: $f(x) = 5 + 4x - x^2$

Q72: $f(x) = 8 - x^3$

Q73: $y = x^3 - 3x$

Q74: $y = x^4 - 4x^3$

Q75: $y = x(6 - 2x)^2$

Q76: $f(x) = 5x^3 - 3x^5$

Q77: $f(x) = (x^2 + 2)(4 - x^2)$

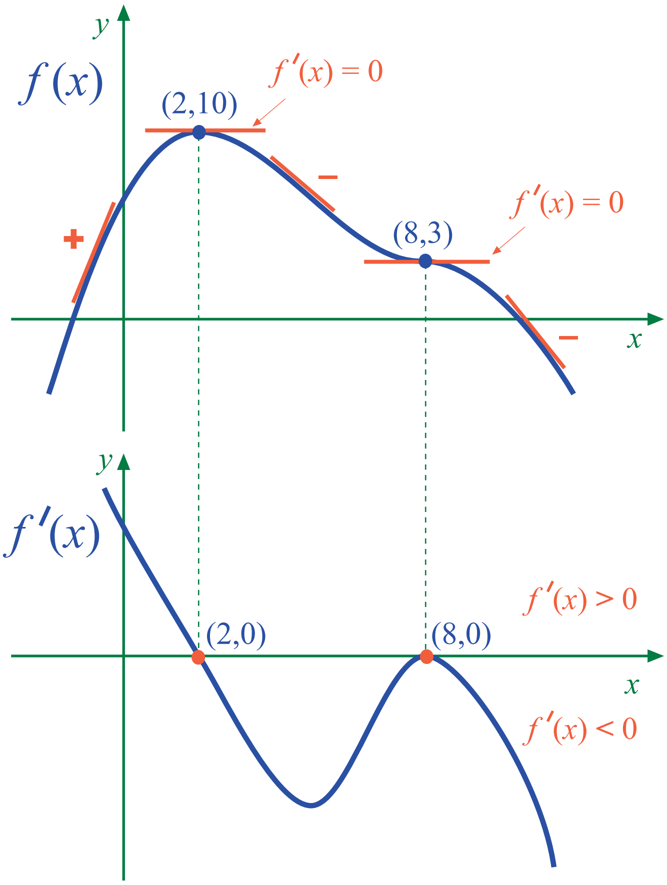
3.14 Sketching the derived function

Learning Objective

Make a sketch of the derived function from a sketch of a function.

When we are given the graph of a function, f(x) then it is usually possible to make a sketch of its derivative, f ′ (x)

You can see how this is done in the following example.



Notice the following features in these graphs.

1. At the stationary points, (2,10) and (8,3), f ′ (x) = 0, therefore the corresponding points on f ′ (x) are (2,0) and (8,0)
2. f(x) is increasing for $x < 2$, therefore $f'(x) > 0$ for $x < 2$
3. f(x) is decreasing for $x > 2$, therefore $f'(x) < 0$ for $x > 2$

These results can be summarised as follows

1. The stationary points on f(x) become x-intercepts for f ′ (x)
2. When f(x) is increasing the graph for f ′ (x) is above the x-axis
3. When f(x) is decreasing the graph for f ′ (x) is below the x-axis

You should be aware that

- if f(x) is a quadratic function then the derived function f ′ (x) will be a straight line
- if f(x) is a cubic function then the derived function f ′ (x) will be a quadratic function.

(Similar statements can also be made about functions of higher order).

Now try the questions in the following exercise as well as the derivative puzzle.

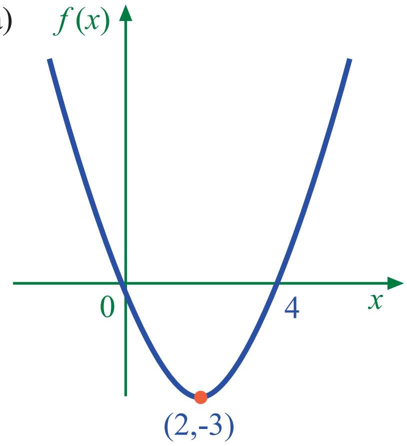
20 min

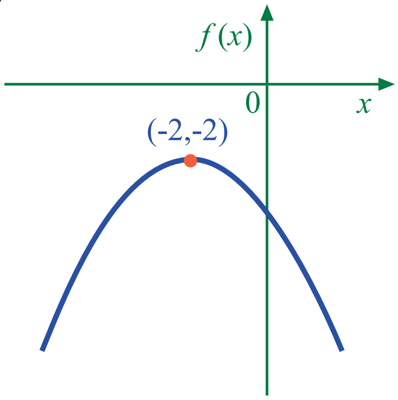
Exercise 12

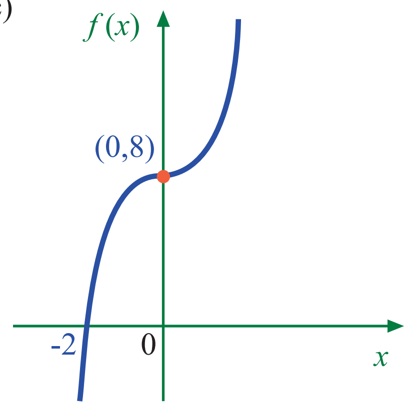
There is an on-line version of this exercise, which you might find helpful

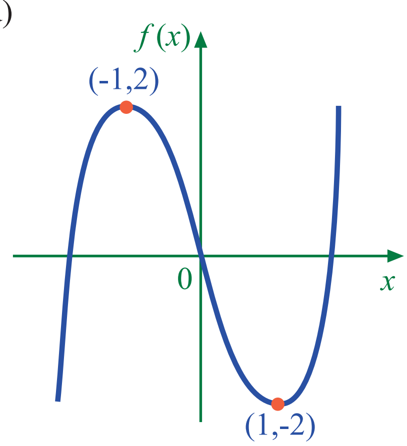
Q78:

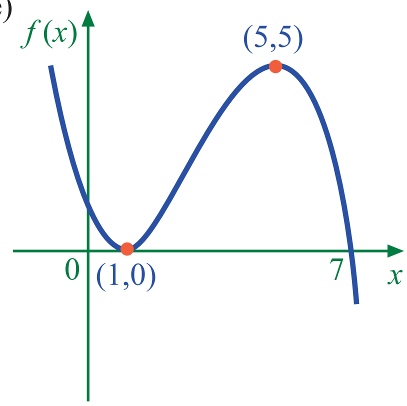
For each of the following functions make a sketch of the derived function.

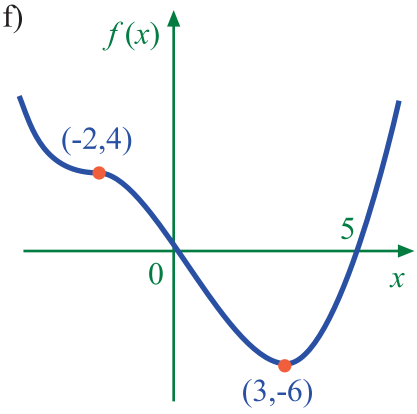
a)


b)


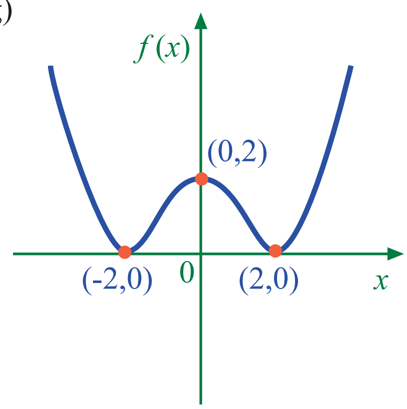
c)


d)


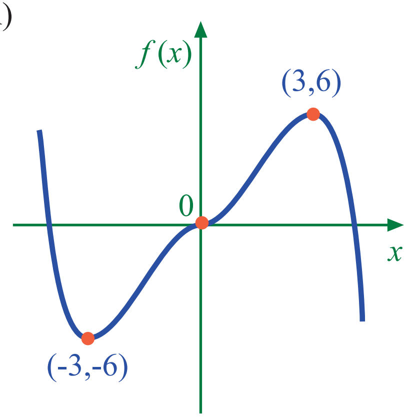
e)


f)


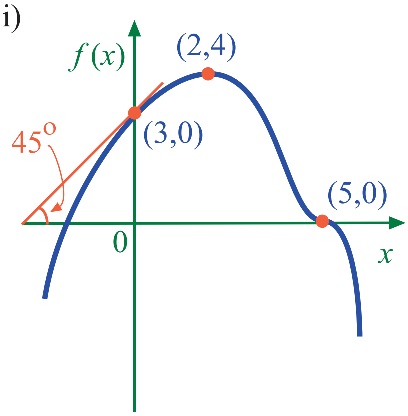
g)



h)



i)



Sketching the derivative puzzle

10 min

3.15 Maximum and minimum values on a closed interval

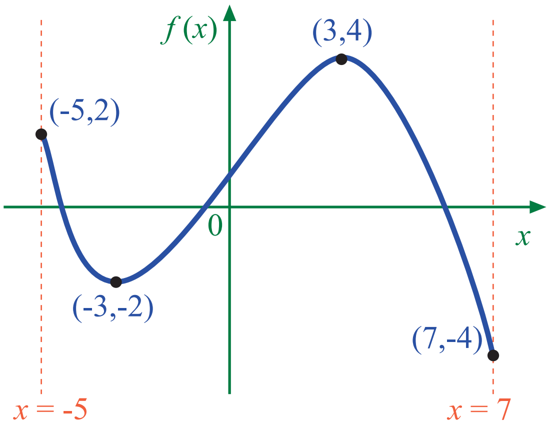
Learning Objective

Identify maximum and minimum values within a closed interval.

The graph shown here has been drawn for $-5 \leqslant x \leqslant 7$

$-5 \leqslant x \leqslant 7$ is a **closed interval** and can be denoted as [-5,7]

closed intervalA closed interval is a subset of the real number line which includes both end points. Thus, for example, [-2,5] is the closed interval $-2 \leqslant x \leqslant 5$



You should notice the following

- The maximum value of the function is 4 at the *stationary point* (3,4)
- The minimum value of the function is -4 at the *end point* (7,-4)

You can see that there is a minimum turning point at (-3,-2), but in this case the end point of the function gives a lower value for the function.

The maximum and minimum values in a closed interval occur at either

- stationary points or
- end points

Example Find the maximum and minimum values of $f(x) = 3x - x^3$ in the closed interval $-1.5 \geq x \geq 3$

Solution

To find the maximumm and minimum values we must check the stationary values and the end points of the function in the closed interval.

Stationary points

$$f(x) = 3x - x^3 \Rightarrow f'(x) = 3 - 3x^2$$
$$= 3\left(1 - x^2\right)$$
$$= 3(1 - x)(1 + x)$$

At stationary points

$$f'(x) = 0 \Rightarrow 3(1 - x)(1 + x) = 0$$
$$\Rightarrow (1 - x) = 0 \text{ or } (1 + x) = 0$$
$$\Rightarrow x = 1 \text{ or } x = -1$$

When x = -1 then f(-1) =-2

When x = 1 then f(1) = 2

Thus the stationary points occur at (-1,-2) and (1,2)

x	-1^-	-1	-1^+
$f'(x)$	−	0	+
slope	↘	→	↗

There is a minimum turning point at (-1,-2)

x	1^-	1	1^+
$f'(x)$	+	0	−
slope	↗	→	↘

There is a maximum turning point at (1,2)

End points

When x = -1.5 then f(-1.5) = -1.125

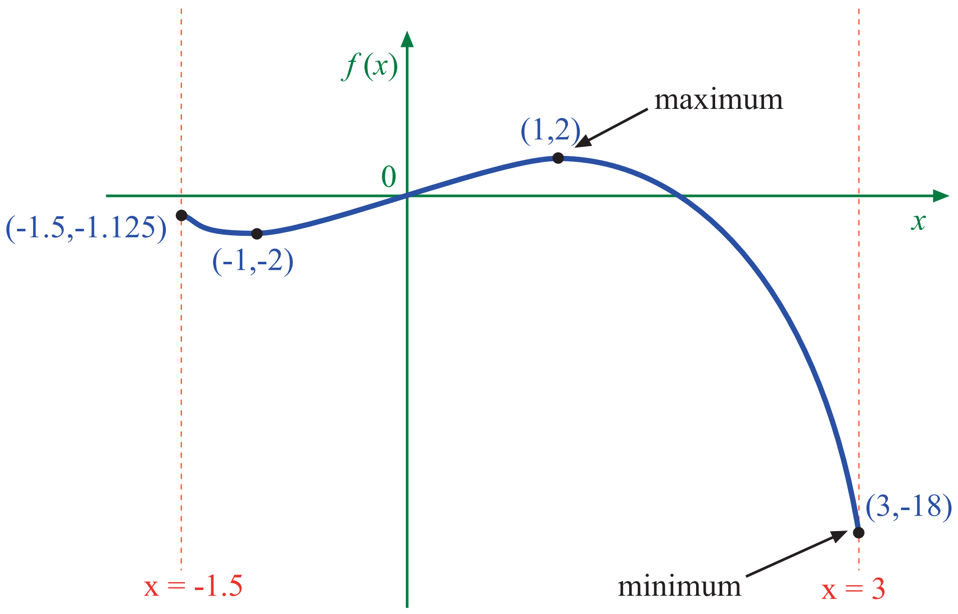
When x = 3 then f(3) = -18

From the above information we can conclude that in the closed interval

$-1.5 \geq x \geq 3$ the function $f(x) = 3x - x^3$ has

- maximum value 2 at the stationary point (1,2)
- minimum value -18 at the end point (3,-18)

You can see this more clearly in the following sketch.



Exercise 13

30 min

There is an on-line version of this exercise, which you might find helpful.

Q79: For each of the graphs shown here write down

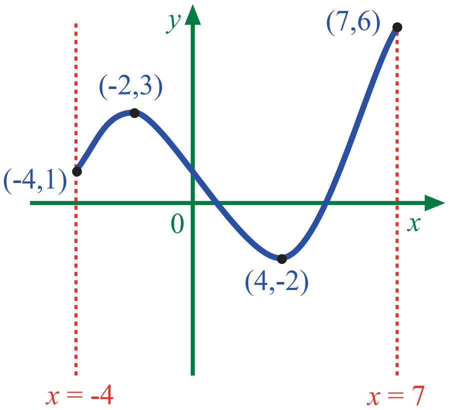
i the maximum value

ii the minimum value

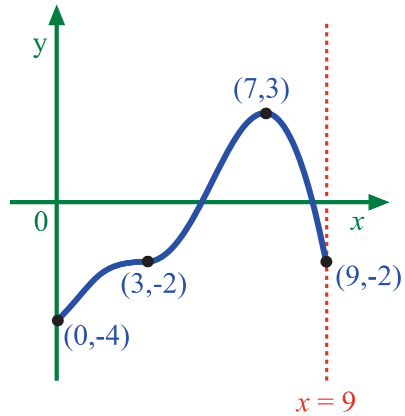
within the given closed intervals.

State whether these values occur at stationary points or end points and write down their coordinates.

a) Closed interval [-4,7]



b) Closed interval [0,9]



Q80: For each of the following functions calculate

i the maximum value

ii the minimum value

within the given closed intervals.

State whether these values occur at stationary points or end points and write down their coordinates.

a) $y = x^2 - 4x - 1$ for $-1 \leqslant x \leqslant 4$

b) $f(x) = x(x-3)^2$ for $0 \leqslant x \leqslant 5$

c) $y = 3x^2 + x^3$ for $-4 \leqslant x \leqslant 2$

d) $f(x) = 2x^2 - x^4 - 2$ for $-2 \leqslant x \leqslant 2$

3.16 Applications

Learning Objective

Translate relationships in science, which may be expressed as a rate of change, into mathematical language and vice versa.

Problems can often be solved by forming some kind of mathematical model such as a formula, an equation or a graph. When a rate of change is involved this is written as a derivative.

Example : Rate of change of volume

A sphere with radius r and volume V = $\frac{4}{3}\pi r^3$ is inflated.

Find

a) the rate of change of the volume of the sphere with respect to the radius

b) the rate of change of the volume when the radius is 2 cm

Solution

a) The rate of change of the volume of the sphere with respect to the radius is $\frac{dV}{dr}$

We are given that V = $\frac{4}{3}\pi r^3$

Notice that $\frac{4}{3}\pi$ is a constant which makes it simple to calculate $\frac{dV}{dr}$

Hence $\frac{dV}{dr} = 4\pi r^2$

b) The rate of change of the volume when the radius is 2 cm is

$\frac{dV}{dr} = 4\pi \times 2^2 = 16\pi$

Example : Distance(s) - Velocity(v) - Acceleration(a)

We have already seen that velocity is a rate of change of distance and we can write

$v = \frac{ds}{dt}$

In a similar way acceleration can be considered as a rate of change of velocity and thus

$a = \frac{dv}{dt}$

Problem

The displacement of a particle at time t, relative to its starting position, is given by the formula $s(t) = 2t^3 - 6t^2 + 10t + 5$

s(t) is the distance in centimetres and t is the time in seconds.

Calculate

a) the velocity of the particle when t = 4

b) the acceleration of the particle when t = 4

Solution

a)

Velocity, $v = \frac{ds}{dt} = 6t^2 - 12t + 10$

When t = 4 then velocity $= 6 \times 4^2 - 12 \times 4 + 10$
$= 58$ cm/s

b)

Acceleration, $a = \frac{dv}{dt} = 12t - 12$

When t = 4 then acceleration $= 12 \times 4 - 12$
$= 36$ cm/s^2

Now try the following questions

Exercise 14

There is an on-line exercise at this point, which you might find helpful.

30 min

Q81:

A stone is dropped from the roof of a tall building. After t seconds it has travelled a distance $5t^2$ metres. Calculate the speed of the stone 3 seconds after it was dropped.

Q82:

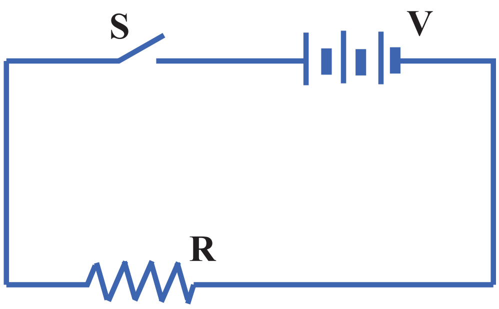
The number of bacteria, N(t), in a culture varies directly with time t seconds according to the formula $N(t) = 1000 + 200t + 5t^3$. Calculate the growth rate of the culture when t =4.

Q83:

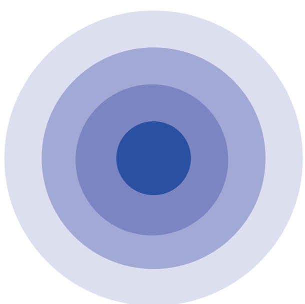
The resistance in an electrical circuit is given by

$R = \frac{300}{I}$

where I is the current in amps. Find the rate of change of R (ohms) with respect to I when the current is 10 amps.



Q84:



The radius(cm) of an ink blot t seconds after the start of an experiment is given by $R(t) = 3\sqrt{t}$

Calculate the rate of change of the radius when t = 4.

Q85:

The distance, s centimetres, that a toy car travels on a track is given by the formula $s(t) = 5 + 120t - t^3$

Calculate

a) the velocity and acceleration of the toy car at time t (seconds)

b) the velocity and acceleration of the toy car at time t = 4

(seconds)

Q86:

The volume of air, V cm^3, in an inflatable Bouncy Castle as it is inflated is given by the formula $V(t) = 10t^3 + 4t^2 + 16t$ where t is time in seconds.

Find the rate of change of the volume when t = 8

3.17 Optimisation

Learning Objective

Solve optimisation problems using calculus

Many problems involve finding a maximum or minimum value of a function. Using differentiation to find stationary points allows us to identify and determine these values. You can see this in the following examples.

Example : Maximum height

The height, h (metres), of a ball thrown upwards is given by the formula $h(t) = 20t - 5t^2$, where t is the time in seconds from when the ball is thrown.

a) When does the ball reach its maximum height?

b) Calculate the maximum height that the ball reaches?

Solution

a) The ball reaches its maximum height at a stationary point when $\frac{dh}{dt} = 0$

$$\frac{dh}{dt} = 0 \Rightarrow 20 - 10t = 0$$
$$\Rightarrow t = 2$$

We can check that the stationary point at t = 2 is a maximum by examining the gradient on either side.

x	2^-	2	2^+
$f'(x)$	+	0	−
slope	↗	→	↘

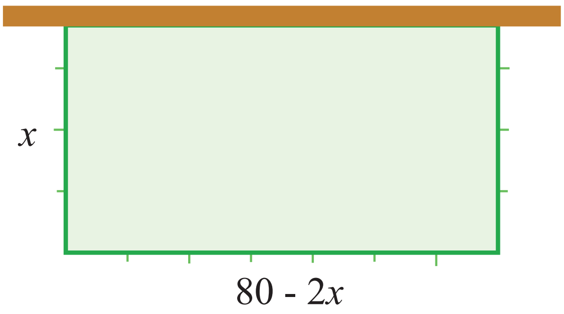
Thus there is a maximum at t = 2 and the ball reaches its maximum height at this time.

b) The ball reaches its maximum height at t = 2 and

$$\begin{aligned} h(2) &= 20 \times 2 - 5 \times 2^2 \\ &= 40 - 20 \\ &= 20 \end{aligned}$$

Thus the maximum height it reaches is 20 metres

Example : Maximum Area



A farmer has 80 metres of fencing to make a rectangular sheep pen against an existing wall. Find the greatest area that he can enclose.

Solution

First we need to assign variables for the length and breadth of the rectangle.

Let x be the breadth of the rectangle then, since there is 80 metres of fencing in total, the length of the rectangle must be 80 - 2x metres.

Thus the area of the rectangle is

$$\begin{aligned} A(x) &= \text{length} \times \text{breadth} \\ &= (80 - 2x)\,x \\ &= 80x - 2x^2 \end{aligned}$$

At a stationary point A'(x) = 0

$$\begin{aligned} A'(x) = 0 &\Rightarrow 80 - 4x = 0 \\ &\Rightarrow x = 20 \end{aligned}$$

We can check that the stationary point at x = 20 is a maximum by examining the gradient on either side.

x	20^-	20	20^+
$A'(x)$	+	0	–
slope	↗	→	↘

Thus for maximum area, breadth x =20 metres and length 80 - 2x = 40 metres

Maximum area = 40 × 20 = 800 metres^2

Now try the questions in the following exercise.

50 min

Exercise 15

There is an on-line exercise at this point, which you might find helpful.

Q87: The sum of two numbers x and y is 14

a) An expression for the product of the two numbers is P = xy
Rewrite this expression in terms of x alone.

b) Find the values for x and y which maximise P

c) Find the maximum value for P

Q88:

The speed of a skier on part of a slope is given by the formula

$v(t) = 8\sqrt{t} - 0.8t$

where t is time in seconds from the start and v(t) is speed in metres/second.

a) When does the skier reach her maximum speed?

b) What is the maximium speed that she reaches (give your answer in km/h)

Q89: A rectangle has breadth x and perimeter 48 cm.

a) Find, in terms of x, an expression for the area of the rectangle.

b) Calculate the dimensions of the rectangle that give the maximum area.

c) Find the maximum area.

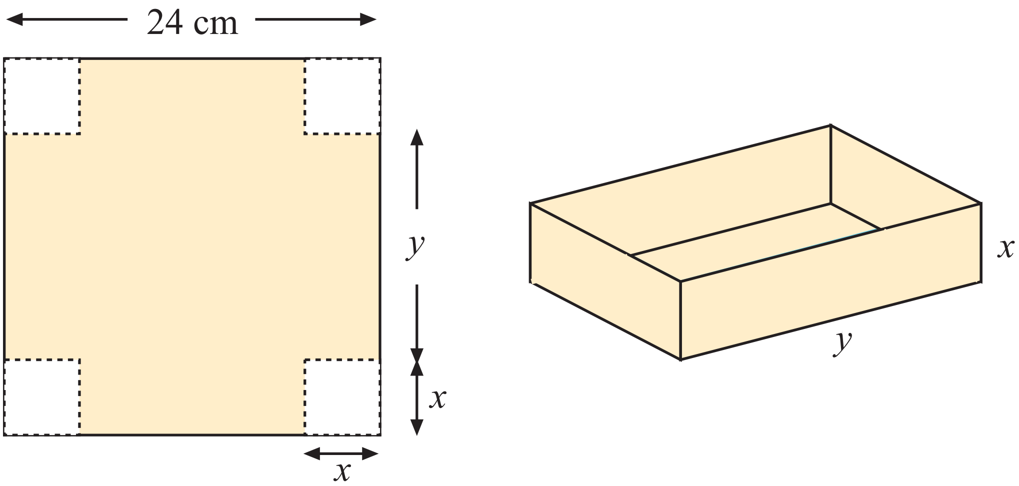
Q90: A gardener has enough roses to fill a rectangular area of 36 m^2.

a) Letting x represent the breadth of the rose bed show that the perimeter can be written as $P = 2x + \frac{72}{x}$

b) What dimensions should he make the rose bed so that he minimises the perimeter of the rose bed?

c) Find the minimum perimeter.

Q91: A cardboard square of side 24 cm has squares of side x cm cut from each corner. The cardboard is then folded to make an open box as shown in the diagram.



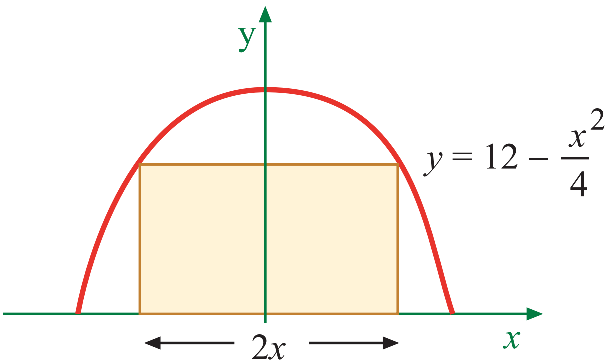
a) Find an expression for y in terms of x cm

b) Show that the volume of the open box can be written as V = $576x - 96x^2 + 4x^3$

c) Calculate the value of x for maximum volume.

d) Find the maximum volume.

Q92: A workshop is constructed underneath an old railway arch as shown in the following cross-section.



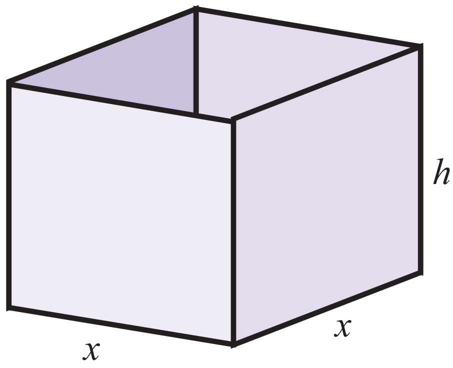
a) The shaded area in the diagram represents the cross-section of the workshop. Find an expression in terms of x(metres) for this shaded area.

b) Calculate the value of x for maximum area.

c) Find the maximum cross-sectional area of the workshop.

Q93:

A box is made in the shape of a cuboid with square base and no lid. The volume of the box is 4000 cm^3



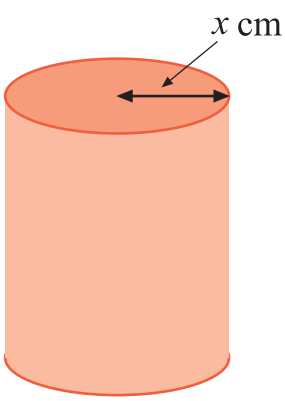
a) Find an expression for the height, h, of the box in terms of x

b) Show that the total surface area of the box may be expressed as

$$A = x^2 + \frac{16000}{x}$$

c) Calculate the dimensions of the box that minimise the surface area.

Q94:



A soup can is manufactured so that it can contain 400 cm^3 of soup.

a) Find an expression for the height of the can in terms of x

b) Show that the surface area, S, of the can is given by

$$S = \frac{800}{x} + 2\pi x^2$$

c) Find the value of x that will minimise the surface area of the can.

3.18 Summary

Learning Objective

Recall the main learning points from this topic

1. The instantaneous speed, or rate of change of distance with respect to time, can be written as f'(t) and is known as the derived function of f(t).

2. $f'(x) = \lim_{x \to 0} \frac{f(x + h) - f(x)}{h}$

3. Some rules for differentiating;

   - When $f(x) = x^n$ then $f'(x) = nx^{n-1}$, $n \in \mathbb{Q}$
   - When $f(x) = ax^n$ then $f'(x) = nax^{n-1}$, $n \in \mathbb{Q}$ and a is a constant
   - When f(x) = a then f'(x) = 0, a is a constant
   - When f(x) = g(x) + h(x) then f'(x) = g'(x) + h'(x)

4. The following are all equivalent notations for the derivative

   $f'(x),\quad y'(x),\quad y',\quad \frac{dy}{dx},\quad \frac{d}{dx}(f(x)),\quad \frac{df}{dx}$

5. The tangent to a curve is a straight line and can be written in the form y - b = m(x - a).

   The following strategy is useful when finding the equation of the tangent.

   - Find the coordinates of (a,b), the point of contact of the tangent with the curve.
   - Calculate the gradient, m, which is equal to the value of $\frac{dy}{dx}$ at x = a

6. The function f(x) is strictly increasing in a given interval if $f'(x) > 0$ for that interval.

   The function f(x) is strictly decreasing in a given interval if $f'(x) < 0$ for that interval.

7. At stationary points f'(x) = 0

   When f'(a) = 0 then f(a) is a stationary *value* and (a, f(a)) is a stationary *point*.

The *nature* of a stationary point is determined by the gradient of the curve on either side of it.

Stationary points can be any of the following types

8.
- maximum turning point
- minimum turning point
- falling point of inflection
- rising point of inflection

To make a good sketch for the graph of a function we need to find the following information about the curve;

9.
- the points of intersection with the x and y axes,
- the stationary points and their nature,
- the behaviour of the curve for large positive and negative values of x

10. In order to sketch the derived function, f'(x), from f(x) note that;

- the stationary points on f(x) will become the x-intercepts on f'(x)
- when f(x) is increasing the graph for f'(x) will be above the x-axis
- when f(x) is decreasing the graph for f'(x) will be below the x-axis

11. The maximum and minimum values in a closed interval occur at either a stationary point or an end point of the interval.

12. Velocity, v, is a rate of change of displacement, s, and thus

$v = \frac{ds}{dt}$

Acceleration, a, is a rate of change of velocity, v, and thus

$a = \frac{dv}{dt}$

13. Many problems involve finding a maximum or minimum value of a function. Using differentiation to find stationary points allows us to identify and determine these values.

3.19 Extended information

There are links on the web which give a variety of web sites related to this topic.

A Maths / Physics Connection

If you study Physics you will probably have come across the equation for displacement

$$s = ut + \frac{1}{2}at^2$$

Velocity is the rate of change of displacement hence we can write

$$v = \frac{ds}{dt} = \frac{d}{dt}\left(ut + \frac{1}{2}at^2\right)$$
$$= u + at$$

Thus we have the equation v = u + at. Physics scolars do you recognise this equation?

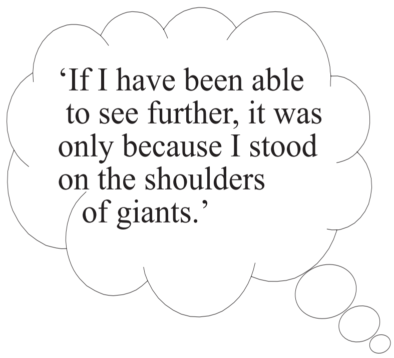
In a similar way, acceleration is the rate of change of velocity hence

$$a = \frac{dv}{dt} = \frac{d}{dt}(u + at)$$
$$= a$$

Which is as we would have hoped.

Sir Isaac Newton

- Born on 4th January 1643 in Woolsthorpe, England
- Died on 31st March 1727 in London.



Newton came from a family of farmers but never knew his father who died three months before he was born. Although a wealthy man, Newtons father was uneducated and could not sign his own name. His mother, Hannah Ayscough remarried when Newton was two years old. Newton was then left in the care of his grandmother and he had a rather unhappy childhood.

In 1653 he attended the Free Grammar School in Grantham. However, his school reports described him as idle and inattentive and he was taken away from school to manage his mothers estate. He showed little interest for this and, due to the influence of an uncle, he was allowed to return to the Free Grammar School in 1660. This time he was able to demonstrate his academic promise and passion for learning and on 5th June 1661 he entered Trinity College,Cambridge.

His ambition at Cambridge was to obtain a law degree but he also studied philosophy, mechanics and optics. His interest in mathematics began in 1663 when he bought an astrology book at a fair and found that he could not understand the mathematics in it. This spurred him on to read several mathematical texts and to make further deep mathematical studies.

Newton was elected a scholar at Cambridge on 28th April 1664 and received his bachelors degree in April 1665. In the summer of 1665 the University was closed due to the plague and Newton had to return to Lincolnshire. There, while still less than 25 years old, he made revolutionary advances in mathematics, physics, astronomy and optics. While at home, Newton established the foundations for differential and integral calculus, several years before the independent discovery by Leibniz. The method of fluxions as he named it was based on his crucial insight that integration is merely the inverse procedure to differentiating a function.

In 1672 he was elected a fellow of the Royal Society after donating a reflecting telescope. In that year he also published his first scientific paper on light and colour. However, he came in for some criticism from other academics who objected with some of his methods of proof and from then on Newton was torn between wanting fame and recognition and the fear of criticism. He found the easiest way to avoid this was to publish nothing.

Newton's greatest achievement was his work in physics and celestial mechanics that lead to his theory of universal gravitation. He was persuaded to write a full account of his new physics and its application to astronomy. In 1687 he published the Philosophiae naturalis principia mathematica or Principia as it is always known. This is recognised as the greatest scientific book ever written. It made him an international leader in scientific research.

On 15th January Newton was elected by the University of Cambridge as one of their two members to the Convention Parliament in London. This may have led him to see that there was a life in London which might appeal more to him than that of the academic world in Cambridge.

After suffering a nervous breakdown in 1693, Newton retired from research and decided to leave Cambridge to take up a government position in London as Warden and then later as Master of the Royal Mint. He made an effective contribution to the work of the Mint particularly on measures to prevent counterfeiting of the coinage.

In 1703 he was elected as president of the Royal Society, a position he retained until his death. He was knighted by Queen Anne in 1705, the first scientist to be honoured in this way for his work.

However, his last years were not easy, dominated in many ways over the controversy with Leibniz as to who had first invented calculus.

> ‘Truth is ever to be found in the simplicity, and not in the multiplicity and confusion of things.’

> ‘I know not what I appear to the world, but to myself I seem to have been only like a boy playing on the sea-shore, and diverting myself in now and then finding a smoother pebble or a prettier shell, whilst the great ocean of truth lay all undiscovered before me.’

Gottfried Leibniz

- Born on 1st July 1646 in Leipzig,Germany
- Died on 14th November1716 in Hannover.

> ‘The soul is the mirror of an indestructible universe.’

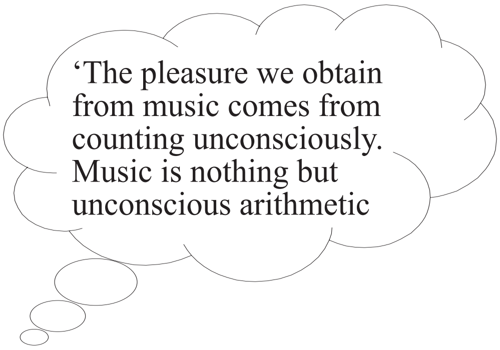
His father Friedrich was a professor of moral philosophy and his mother Catharina Schmuck was Friedrichs third wife. Friedrich died when Leibniz was only six, so he was brought up by his mother and it was her influence that played an important role in his life and philosophy.

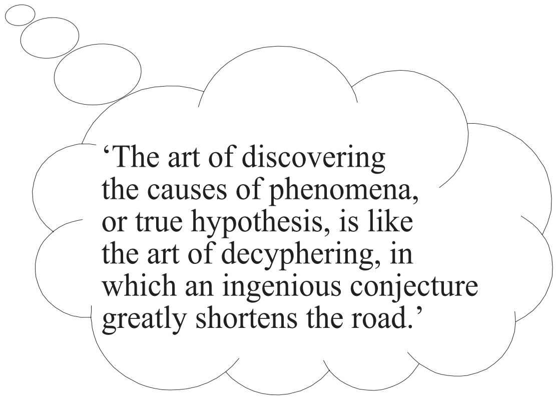
In 1661 Leibniz entered the University of Leipzig. He was only fourteen, which nowadays would be considered highly unusual, however at that time there would be others of a similar age. He studied philosophy and mathematics and graduated with a bachelors degree in 1663. Further studies took him on to a Masters Degree in philosophy and a bachelors degree and doctorate in Law.

By November 1667 Leibniz was living in Frankfurt where he investigated various different projects, scientific, literary and political. He also continued his law career.

In 1672 Leibniz went to Paris with the aim of contacting the French government and dissuading them from attacking German land. While there he made contact with mathematicians and philosophers and began construction of a calculating machine. On the January of the following year he went to England to try the same peace mission, the French one having failed and while there he visited The Royal Society of London and presented his incomplete calculating machine. The Royal Society elected him as a fellow on 19th April 1673 but by 1674 he had not kept his promise to finish his mechanical calculating machine and so he fell out of favour.

It was during his time in Paris that Leibniz developed his version of calculus. However, the English mathematician Sir Isaac Newton, several years before Leibniz had already laid the foundations for differential and integral calculus. This lead to much controversy over who had invented calculus and caused Newton to fly into an irrational temper directed against Leibniz. Neither Leibniz nor Newton thought in terms of functions, both always worked in terms of graphs. Leibniz concentrated on finding a good notation for calculus and spent a lot of time thinking about it, whereas Newton wrote more for himself and tended to use whatever notation he thought of on the day.





Amongst Leibniz’s other achievements in mathematics were his development of the binary system of arithmetic and his work on determinants which arose from his developing methods to solve systems of linear equations. He also produced an important piece of work on dynamics.

Leibniz is descibed as:

"a man of medium height with a stoop, broad shouldered but bandy-legged, as capable of thinking for several days sitting in the same chair as of travelling the roads of Europe summer and winter. He was an indefatigable worker, a universal letter writer (he had more than 600 correspondents), a patriot and cosmopolitan, a great scientist, and one of the most powerful spirits of Western civilisation."

It was also said about him

"It is rare to find learned men who are clean, do not stink and have a sense of humour." !!

3.20 Review exercise

Review exercise on basic differentiation

20 min

There is an on-line exercise at this point, which you might find helpful.

These questions are intended to test basic competency in this topic.

Q95:

Differentiate the following with respect to x,

a) $f(x) = (2x - 3)(x + 4)$

b) $f(x) = \dfrac{3x - 5}{\sqrt{x}}$

Q96:

Find the gradient of the tangent to the curve $f(x) = (x - 3)^2$ at $x = 2$

Q97:

a) Find the coordinates of the stationary points on the curve $y = x^3 - 6x^2 + 9x - 4$
b) Determine the nature of these stationary points and justify your answer.

3.21 Advanced review exercise

Advanced review exercise in basic differentiation

30 min

There is an on-line exercise at this point, which you might find helpful.

These questions are intended to reflect the standard of question that might be posed in a final exam.

Q98:

Given that $f(x) = \dfrac{x^4 - 3x + 1}{2\sqrt{x}}$ find $f'(x)$

Q99:

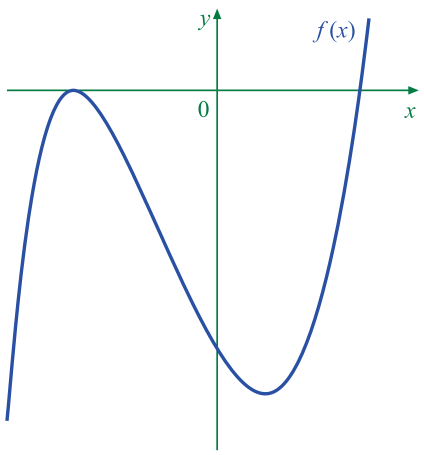
Find $f'(2)$ when $f(x) = (3x + 2)(2x - 5)$

Q100:

Find the equation of the tangent to the curve $y = 5 + x - 2x^3$ at the point where $x = -1$

Q101:

The diagram shows part of the graph of the function $f(x) = (x^2 - 9)(x + 3)$



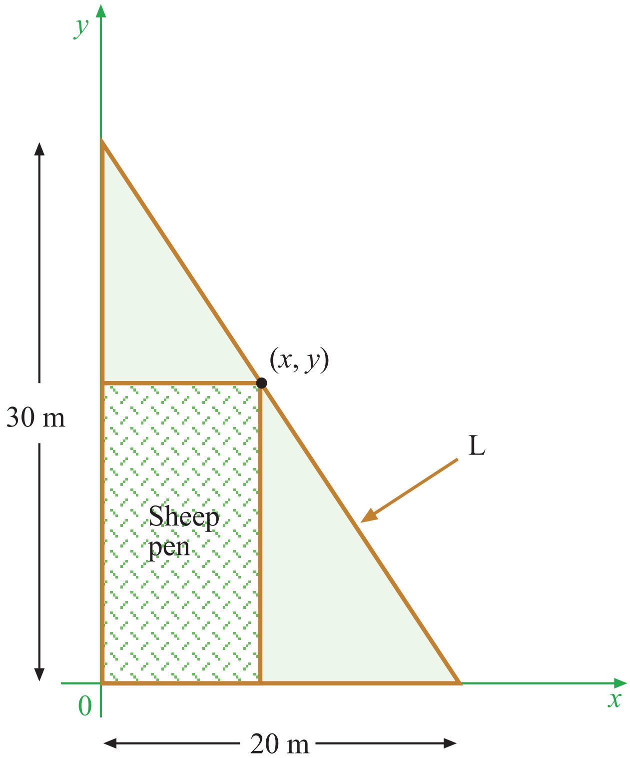
a) Write down the coordinates of the points where the curve cuts the axes.
b) Find the coordinates of the stationary points and justify their nature.

Q102:

Find the maximum and minimum values of the function $y = x^3 - 3x^2 - 9x + 5$ on the closed interval $0 \leqslant x \leqslant 4$

Q103:

A farmer wants to build a rectangular sheep pen in a triangular shaped field as shown in the diagram.



a) Find the equation of the line L

b) Show that the area, A, of the sheep pen can be written as $A = 30x - \frac{3}{2}x^2$

c) Find the dimensions of the sheep pen for maximum area.

3.22 Set review exercise

Set review exercise in basic differentiation

20 min

An on-line assessment is available at this point, which you will need to attempt to have these answers marked. These questions are not randomised on the web site. The questions on the web may be posed in a different manner but you should have the required answers in your working. These questions are intended to test basic competency in this topic.

Q104:

Differentiate the following with respect to x

a) $y = \sqrt{x}\left(x + \sqrt{x}\right)$

b) $f(x) = \frac{x^3 - 2}{3x}$

Q105:

Calculate the gradient of the tangent to the curve $y = \frac{3}{x}$ at $x = 2$

Q106:

a) Find the coordinates of the stationary points on the curve $f(x) = x^3(x + 4)$
b) Determine the nature of these stationary points and justify your answer.

Topic 4

Recurrence relations

Contents

Learning Objectives

Design and interpret mathematical models of situations involving recurrence relations.

- *Define and interpret a recurrence relation in the form* $u_{n+1} = mu_n + c$ *(m, c constants) in a mathematical model.*
- *Find and interpret the limit of the sequence generated by a recurrence relation in a mathematical model (where the limit exists).*

Prerequisites

You should be competent in the use of formulae and be able to solve a pair of simultaneous equations.

4.1 Revision exercise

Learning Objective

Identify areas that need revision

Revision exercise

10 min

There is an on-line exercise at this point, which you might find helpful.

Q1:

Use the formula u = ab + c to calculate u when a = 10, b = -5 and c = 30

Q2:

Evaluate $L = \frac{b}{1 - a}$ when a = 0.8 and b = 20

Q3:

Use the formula $u_{n+1} = au_n + b$ to calculate u_{n+1} when a = 2, $u_n = 6$ and b = 100

Q4:

Solve the simultaneous equations $\begin{cases} 4x + y = 8 \\ 3x + y = 5 \end{cases}$

Q5:

Solve the simultaneous equations $\begin{cases} y = 4x + 9 \\ 2y - 6x = 17 \end{cases}$

4.2 Sequences

Learning Objective

Calculate specified terms in a sequence from a given formula. Calculate the limit for a convergent sequence.

sequenceA **sequence** is an ordered list of terms.

Consider the following sequence of numbers

5, 8, 11, 14, 17, 20, ...

We can label the terms in this sequence in the following way

$$\begin{array}{cccccccc} u_1, & u_2, & u_3, & u_4, & u_5, & u_6, & \ldots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ 5, & 8, & 11, & 14, & 17, & 20, & \ldots \end{array}$$

One way to describe this pattern of numbers is by a formula for the n^{th} term. In this case the formula is $u_n = 3n + 2$

You can check that the formula gives the correct terms in the sequence by substituting for n

When n = 1 then $u_1 = 3 \times 1 + 2 = 5$

When n = 2 then $u_2 = 3 \times 2 + 2 = 8$

When n = 3 then $u_3 = 3 \times 3 + 2 = 11$

When n = 4 then $u_4 = 3 \times 4 + 2 = 14$

Thus the formula generates the required sequence of numbers.

Given a formula for the n^{th} term it is easy to calculate the value of any term in the sequence.

For example, the 20^{th} term in the sequence is $u_{20} = 3 \times 20 + 2 = 62$ and in a similar way the 100^{th} term is $u_{100} = 302$

Example A sequence is defined by the formula $u_n = 3 - \frac{1}{n}$

a) Calculate the first five terms in the sequence.

b) Calculate u_{10}, u_{100} and u_{1000}, giving your answer in decimal form.

c) Write down the value that u_n approaches as n gets larger. This is the limit as $n \Rightarrow \infty$ (n tends to infinity)

Solution

a)

$$u_1 = 3 - \frac{1}{1} = 2$$
$$u_2 = 3 - \tfrac{1}{2} = 2\tfrac{1}{2}$$
$$u_3 = 3 - \tfrac{1}{3} = 2\tfrac{2}{3}$$
$$u_4 = 3 - \tfrac{1}{4} = 2\tfrac{3}{4}$$
$$u_5 = 3 - \tfrac{1}{5} = 2\tfrac{4}{5}$$

b)

$$u_{10} = 3 - \tfrac{1}{10} = 2\tfrac{9}{10} = 2.9$$
$$u_{100} = 3 - \tfrac{1}{100} = 2\tfrac{99}{100} = 2.99$$
$$u_{1000} = 3 - \tfrac{1}{1000} = 2\tfrac{999}{1000} = 2.999$$

c) $u_n \Rightarrow 3$ as $n \Rightarrow \infty$

convergent sequence

A sequence is **convergent** if it has a limit as $n \Rightarrow \infty$

divergent sequence

A sequence is **divergent** if it does not have a limit as $n \Rightarrow \infty$

Now try the questions in the following exercise.

30 min

Exercise 1

Q6: Calculate the first five terms and u_{10} for each of the following sequences.

a) $u_n = 3n$

b) $u_n = 4n - 1$

c) $u_n = 2n^2 - 3$

d) $u_n = 2^n$

e) $u_n = \frac{1}{2n}$

f) $u_n = \frac{1}{2}n(n + 1)$

g) $u_n = \frac{n - 1}{n}$

h) $u_n = 1 + \left(\frac{1}{2}\right)^n$

i) Which of the above sequences are convergent?

Q7: The n^{th} term in a sequence is given by the formula $u_n = \frac{n}{n + 1}$

a) Calculate the first five terms in this sequence, giving your answer as a fraction.

b) Calculate u_9, u_{99} and u_{999} giving your answer as an exact decimal.

c) Calculate the value of u_n as $n \Rightarrow \infty$

Q8: Each of the following formulae are for convergent sequences. Calculate the limit for each.

a) $u_n = \frac{1}{n}$

b) $u_n = \left(\frac{1}{2}\right)^n$

c) $u_n = \frac{3n - 1}{n}$

d) $u_n = 5 + 0.2^n$

4.3 Recurrence relations

Learning Objective

Calculate specified terms in sequences that are defined as recurrence relations

Consider again the sequence of numbers

$$\begin{array}{ccccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & \dots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ 5 & 8 & 11 & 14 & 17 & 20 & \dots \end{array}$$

Notice that given $u_1 = 5$ we can calculate u_2, u_3, u_4, u_5, u_6, ... by repeatedly adding 3.

$u_1 = 5$

$u_2 = u_1 + 3 = 8$

$u_3 = u_2 + 3 = 11$

$u_4 = u_3 + 3 = 14$

$u_5 = u_4 + 3 = 17$

$u_6 = u_5 + 3 = 20$

This allows us to define this sequence in another way as $u_{n+1} = u_n + 3$

A sequence defined in this way is known as a recurrence relation because the pattern +3 recurrs.

Example List the first five terms in the sequence with recurrence relation $u_{n+1} = 2u_n + 1$ and $u_1 = 3$

Solution

Notice that the pattern that recurrs this time is "times 2, add 1"

$u_1 = 3$

$u_2 = 2 \times 3 + 1 = 7$

$u_3 = 2 \times 7 + 1 = 15$

$u_4 = 2 \times 15 + 1 = 31$

$u_5 = 2 \times 31 + 1 = 63$

(A graphic calculator can do these calculations very efficiently. For example, we can achieve the above results using the following steps

- key in $u_1 = 3$ and press ENTER
- key in $2 \times$ ANS + 1
- now pressing ENTER repeatedly will give u_2, u_3, u_4, u_5, ...

This technique will be very useful throughout the rest of this topic)

Now try the following questions.

30 min

Exercise 2

There is an on-line exercise at this point, which you might find helpful.

Q9: Use the following recurrence relations to write down the first five terms in each sequence. Also calculate u_{10}

a) $u_{n+1} = u_n + 5$, $u_1 = 2$

b) $u_{n+1} = 3u_n$, $u_1 = 1$

c) $u_{n+1} = -2u_n + 3$, $u_1 = 5$

d) $u_{n+1} = 0.8u_n + 4$, $u_1 = 200$

e) $u_{n+1} = -u_n$, $u_1 = 5$

f) $u_{n+1} = {}^1/_5 u_n + 5$, $u_1 = 10$

Q10: Write down a recurrence relation for each of the following sequences. Remember to include the first term.

a) 2, 7, 12, 17, 22, ...

b) 4, 12, 36, 108, 324, ...

c) 3, 7, 15, 31, 63, ...

d) 8, -14, 30, -58, 118, ...

Q11:

a) Use the recurrence relation $u_{n+2} = u_{n+1} + u_n$ with $u_1 = 1$ and $u_2 = 1$ to write down the first ten terms in the sequence.
(This type of recurrence relation is called the Fibonacci sequence after the Italian mathematician who first discovered it. You can read more about Fibonacci in the extended information section at the end of this topic. There are also web links in this section to some sites that give interesting facts about the Fibonacci numbers and their relevance in nature).

b) From the sequence you have generated calculate the following sequence
$\frac{u_2}{u_1}, \frac{u_3}{u_2}, \frac{u_4}{u_3}, \frac{u_5}{u_4}, \frac{u_6}{u_5}, \frac{u_7}{u_6}, \frac{u_8}{u_7}, \frac{u_9}{u_8}, \frac{u_{10}}{u_9}, \ldots$
giving your answer as a decimal. What do you notice?

4.4 Two special recurrence relations

Learning Objective

Solve problems involving arithmetic and geometric sequences

Geometric sequence

Example Under certain laboratory conditions a bamboo plant grows at a rate of 20% per day. At the start of the experiment the height of the bamboo plant is $B_0 = 30$ cm.

a) Write down a recurrence relation that describes the growth of the plant.

b) Calculate how tall the plant is after 5 days.

c) Find a formula for the height of the plant after n days giving your answer in terms of B_0

(Notice that the notation is slightly different in this question as we have used B_0 for the initial height of the plant. Indeed, it makes sense that B_1 represents the height of the plant after 1 day and similarily for $B_2, B_3, B_4, \ldots$)

Solution

a) We are given that $B_0 = 30$.

Since the plant grows by 20% each day, the height will be 120% = 1.2 times its height from the previous day.

The recurrence relation is therefore $B_{n+1} = 1.2B_n$ with $B_0 = 30$

b)

$$B_0 = 30$$
$$B_1 = 1.2 \times B_0 = 1.2 \times 30 = 36$$
$$B_2 = 1.2 \times B_1 = 1.2 \times 36 = 43.2$$
$$B_3 = 1.2 \times B_2 = 1.2 \times 43.2 = 51.84$$
$$B_4 = 1.2 \times B_3 = 1.2 \times 51.84 = 62.208$$
$$B_5 = 1.2 \times B_4 = 1.2 \times 62.208 = 74.6496$$

Thus after five days the plant will be 75 cm tall (to the nearest cm).

c) We can rewrite the preceeding working in another way as shown here

$$B_0 = 30$$
$$B_1 = 1.2 \times B_0$$
$$B_2 = 1.2 \times B_1 = 1.2 \times (1.2 \times B_0) = 1.2^2 \times B_0$$
$$B_3 = 1.2 \times B_2 = 1.2 \times \left(1.2^2 \times B_0\right) = 1.2^3 \times B_0$$
$$B_4 = 1.2 \times B_3 = 1.2 \times \left(1.2^3 \times B_0\right) = 1.2^4 \times B_0$$
$$B_5 = 1.2 \times B_4 = 1.2 \times \left(1.2^4 \times B_0\right) = 1.2^5 \times B_0$$

You can see that a pattern develops and that in general

$B_n = 1.2^n B_0$

geometric sequence

A **geometric sequence** is a special type of recurrence relation that takes the form u_{n+1} = au_n

The n^{th} term is given by $u_n = a^n u_0 u_n = a^n u_0$

The following are further examples of geometric sequences

1, 3, 9, 27, 81, ...

40, 20, 10, 5, 2.5, ...

3, 6, 12, 24, 48, ...

Notice that for a geometric sequence the ratio of successive terms is constant.

Thus, for example, the sequence 3, 6, 12, 24„ 48, ... has common ratio $\frac{6}{3} = \frac{12}{6} = \frac{24}{12} = \frac{48}{24} = 2$ and the n^{th} term is given by $u_n = 2^n u_0$

Arithmetic sequence

Example A lorry driver travels 80 miles per day on his regular delivery round. At the start of the week his mileometer reads M_0 = 14200 miles

a) Write down a recurrence relation that describes the reading on the mileometer.

b) Calculate the reading on the mileometer after 5 delivery rounds.

c) Find a formula for the reading on the mileometer after n delivery rounds.

Solution

a) Each delivery round the driver travels a further 80 miles thus the recurrence relation is $M_{n+1} = M_n + 80$ with M_0 = 14200 miles

b)

$$M_0 = 14200$$
$$M_1 = M_0 + 80 = 14200 + 80 = 14280$$
$$M_2 = M_1 + 80 = 14280 + 80 = 14360$$
$$M_3 = M_2 + 80 = 14360 + 80 = 14440$$
$$M_4 = M_3 + 80 = 14440 + 80 = 14520$$
$$M_5 = M_4 + 80 = 14520 + 80 = 14600$$

c) We can rewrite the preceeding working in another way as shown here.

$$M_0 = 14200$$
$$M_1 = M_0 + 80$$
$$M_2 = M_1 + 80 = (M_0 + 80) + 80 = M_0 + 2 \times 80$$
$$M_3 = M_2 + 80 = (M_0 + 2 \times 80) + 80 = M_0 + 3 \times 80$$
$$M_4 = M_3 + 80 = (M_0 + 3 \times 80) + 80 = M_0 + 4 \times 80$$
$$M_5 = M_4 + 80 = (M_0 + 4 \times 80) + 80 = M_0 + 5 \times 80$$

You can see that a pattern develops and that in general

$M_n = M_0 + 80n$

arithmetic sequence

An **arithmetic sequence** is a special type of recurrence relation that takes the form $u_{n+1} = u_n + b$

The n^{th} term is given by $u_n = u_0 + bn$

> The following are further examples of arithmetic sequences.
>
> 1, 3, 5, 7, 9, ...
>
> 40, 35, 30, 25, 20, ...
>
> -10, -7, -4, -1, 2, ...
>
> Notice that for an arithmetic sequence the difference between successive terms is constant.
>
> Thus, for example, the sequence -10, -7, -4, -1, 2, ... has common difference +3 and the n^{th} term can be written as $u_n = u_0 + 3n$

Now try the following questions.

Exercise 3

There is an on-line exercise at this point, which you might find helpful.

30 min

Q12: The value of a new car depreciates by 20% each year. The value of the car was initially $C_0 = £18000$.

a) Write down a recurrence relation that describes the value of the car.
b) Calculate how much the car is worth after 4 years.
c) Find a formula for C_n, giving your answer in terms of C_0

Q13:

At the start of a week a farmer has 15 apple trees in a small orchard. He plants twelve more trees in the orchard each day for the next 5 days.

a) Write down a recurrence relation that describes the number of trees, T_{n+1}, in the orchard.
b) How many trees does the farmer have in his orchard after 5 days?

Q14: Joe bought a compact dvd player on hire purchase and paid a deposit of $P_0 = £20$. He makes a payment of £19 at the end of each month to repay the balance.

a) Write down a recurrence relation to describe the payments that Joe makes.

b) He has completed the payments after 12 months. How much did the dvd player cost him ?

c) Find a formula for P_n, giving your answer in terms of P_0

Q15: Claire deposits £1000 in a bank account which pays 6% compound interest each year. Let M_n represent the amount of money in her account after n years.

a) Write down a recurrence relation to describe the amount of money that she has in her account.

b) Calculate how much money she has in her account after 5 years.

c) Find a formula for M_n, giving your answer in terms of M_0

Q16:

Due to global warming, it is estimated that an iceberg is melting at a rate of 6% per year. At the start of the year 2000 the volume of the iceberg was approximately V_0 = 2100 km^3

a) Write down a recurrence relation that describes the volume of the iceberg.

b) Calculate the predicted volume of the iceberg afer 6 years.

c) Find a formula for V_n, giving your answer in terms of V_0

d) If the iceberg continues to melt at the same rate, calculate how many years it will take until the volume of the iceberg is less than 1000 km^3

Q17: In an experiment a certain population of bacteria grows at a rate of 13% per minute. At the start of the experiment there are $B_0 = 120$ bacteria.

a) Write down a recurrence relation to describe the growth of the bacteria.

b) Calculate how many bacteria there are present after 4 minutes.

c) How long does it take until the number of bacteria doubles?

4.5 Linear recurrence relations

Learning Objective

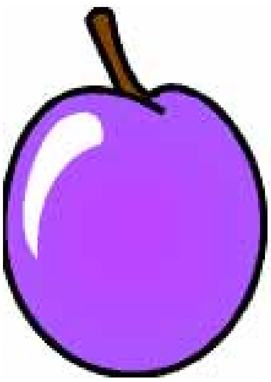
Define and interpret linear recurrence relations in a mathematical model.

linear recurrence relationA **linear recurrence relation** is a sequence defined by $u_{n+1} = au_n + b, a \neq 0$

As you can see this has the same form as the equation of the straight line y = mx + c

This type of recurrence relation occurs in many problems.

Example



A farmer grows a variety of plum tree which ripens during the months of July and August. On the last day in July there was 2000 kg of ripe fruit ready to be picked. At the beginning of August the farmer hires some fruit pickers who manage to pick 75% of the ripe fruit each day. Also, each day 60 kg more of the plums become ripe.

a) Find a recurrence relation for the weight of ripe plums left in the orchard.

b) What is the estimated weight of ripe plums left in the orchard at the end of the day on the 7th August ?

Solution

a) Let P_0 represent the weight of ripe plums available at the start then $p_0 = 2000$

   P_1 represents the amount of ripe plums left in the orchard at the end of the day on the 1st August and similarly for P_2, P_3, P_4, ...

   Since 75% of the ripe fruit is picked each day then 25% of the ripe fruit is left in the orchard for the next day. Also, each day 60 kg more of the plums ripen. Thus the recurrence relation is

   $P_{n+1} = 0.25 P_n + 60$

b) With $P_0 = 2000$ and $P_{n+1} = 0.25P_n + 60$ then, using a calculator, you can check that

   $P_1 = 560$

   $P_2 = 200$

   $P_3 = 110$

   $P_4 = 87.5$

   $P_5 = 81.875$

   $P_6 = 80.46875$

   $P_7 = 80.1171875$

Thus after the 7th of August there is approximately 80kg of ripe fruit in the orchard.

Now try the following questions

Exercise 4

30 min

There is an on-line exercise at this point, which you might find helpful

Q18: Brian is given a gift of £50 by a rich relative on his 12th birthday and on each birthday after that. He saves this money in a savings account which pays 6% annual interest.

a) Find a recurrence relation for the amount of money that Brian has in his account.

b) Calculate how much money he should expect to have in his account on his eighteenth birthday.

Q19:

A large shoal of 300 fish are observed and it is noticed that every minute 30% of the fish leave the shoal and 25 return.

a) Find a recurrence relation for the size of the shoal of fish.

b) How many fish are in the shoal after 5 minutes and after 10 minutes?

Q20: A pharmaceutical company is given permission to discharge a maximum of 60 kg of chemical waste into a section of river each day. Due to the natural tidal nature of the river 80% of the chemical is washed away each day.

a) Find a recurrence relation for the amount of chemical waste in the river.

b) How many kg of chemical waste are there in the river after 7 days and after 15 days?

c) Due to environmental concerns the chemical waste in the section of river must not exceed 80 kg. Is it safe for the pharmaceutical company to continue discharging waste at this rate? Give a reason.

Q21: 120 rabbits are bred for sale. Each month the rabbit population increases by 15% and each month 30 rabbits are sold to customers.

a) Write down a recurrence relation that describes the population of rabbits.

b) Calculate how many rabbits there are after 6 months.

c) What will happen to the rabbit population if it continues in this way?

Q22: A patient is injected with 90 units of a drug in hospital. Every eight hours 30% of the drug passes out of the bloodstream. The patient is therefore given a further dose of 20 units of the drug at eight hourly intervals.

a) Find a recurrence relation for the amount of the drug in the bloodstream.
b) Calculate how many units of the drug you would expect there to be in the patient's bloodstream after 48 hours.

Q23:

A family have taken out a £80000 mortgage to buy a cottage. The building society charge interest on this sum at 6% per annum. The family pay back £6000 each year.

a) Find a recurrence relation for the amount of money that they owe the building society.
b) How much money do they owe after 5 years, after 10 years?
c) During what year will the mortgage be paid off?

4.6 Further linear recurrence relations

Learning Objective

Identify particular features of linear recurrence relations

Occasionally we may be interested in specific terms in a recurrence relation.

Example A sequence is defined by the recurrence relation $u_{n+1} = 0.8u_n + 500, \quad u_0 = 10$

Calculate the value of u_4 and find the smallest value of n for which $u_n > 2000$

Solution

From the recurrence relation we can calculate (to two decimal places)

$u_0 = 10$
$u_1 = 0.8 \times 10 + 500 = 508$
$u_2 = 0.8 \times 508 + 500 = 906.4$
$u_3 = 0.8 \times 906.4 + 500 = 1225.12$
$u_4 = 0.8 \times 1225.12 + 500 = 1480.10$
$u_5 = 0.8 \times 1480.10 + 500 = 1684.08$
$u_6 = 0.8 \times 1684.08 + 500 = 1847.26$
$u_7 = 0.8 \times 1847.26 + 500 = 1977.81$
$u_8 = 0.8 \times 1977.81 + 500 = 2082.25$

From this we can see that $u_4 = 1480.10$ (to 2 d.p.) and that n = 8 for $u_n > 2000$

Now try the following questions.

20 min

Exercise 5

There is an on-line exercise at this point, which you might find helpful.

Q24:

A sequence is defined by the recurrence relation $u_{n+1} = 0.6u_n - 5, \ \ u_0 = 80$

a) Calculate the value of u_2

b) Find the smallest value of n for which $u_n < 0$

Q25:

A sequence is defined by the recurrence relation $u_{n+1} = 1.2u_n + 70, \ \ u_0 = 100$

a) Calculate the values of u_2 and u_4

b) How many terms in the sequence have value less than 1000?

Q26:

A sequence is defined by the recurrence relation $u_n = 1.05u_{n-1} + 200, \ \ u_1 = 0$

a) Calculate u_2 and u_5

b) How many terms in the sequence are between 1000 and 2000?

Q27:

A sequence is defined by the recurrence relation $u_{n+1} = 0.1u_n + 18, \ \ u_0 = 22$

a) Calculate u_2, u_3, and u_4

b) What value does u_n approach as $n \to \infty$?

4.7 Investigating recurrence relations

Learning Objective

Know the condition for the limit of the sequence resulting from a recurrence relation to exist

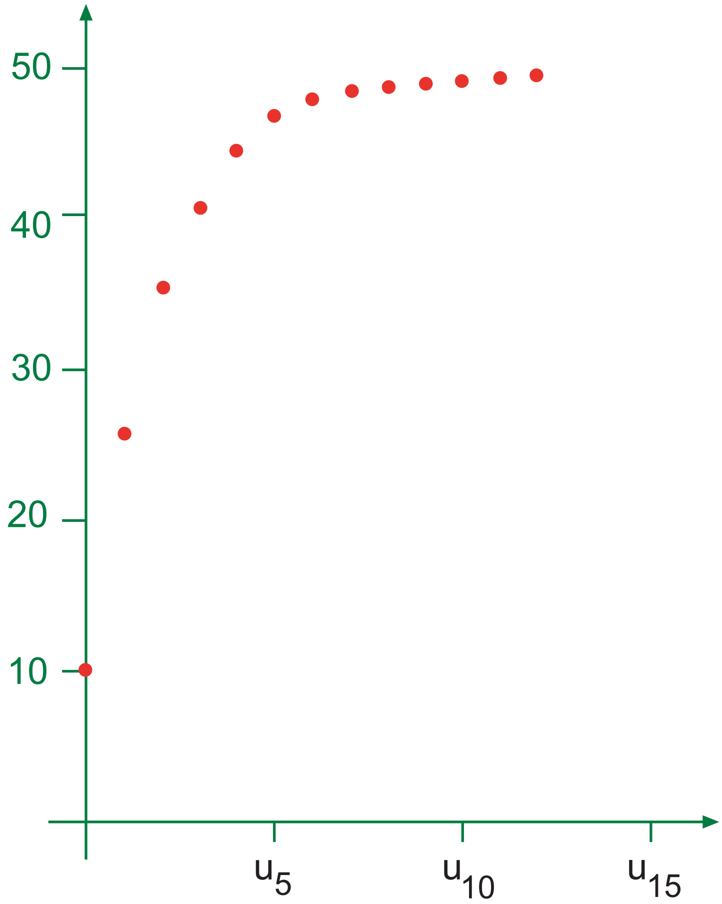
Examples

1.

$u_{n+1} = 0.6u_n + 20, \quad u_0 = 10$ generates the following sequence (to 2 decimal places)

We can represent this in a graph as shown here.

$u_0 = 10$
$u_1 = 26$
$u_2 = 35.6$
$u_3 = 41.36$
$u_4 = 44.82,$
$u_5 = 46.89$
$u_6 = 48.13$
$u_7 = 48.88$
$u_8 = 49.33$
$u_9 = 49.60$
$u_{10} = 49.76$
$u_{11} = 49.85$
$u_{12} = 49.91$



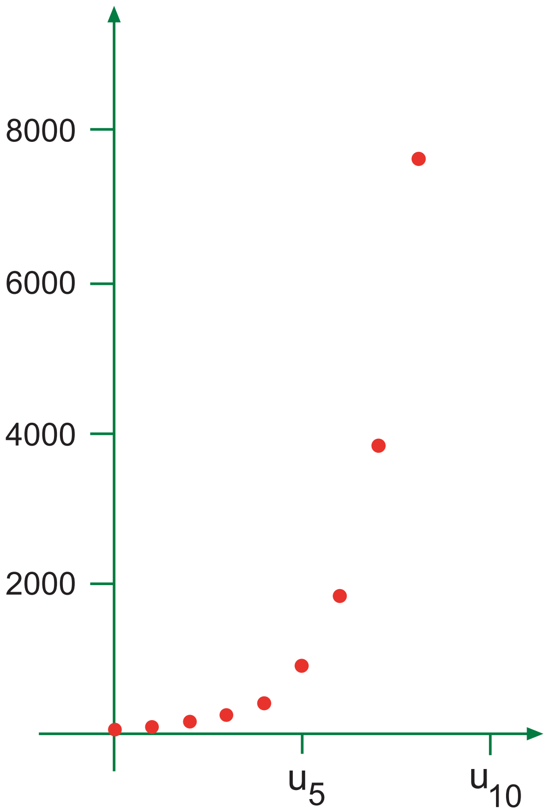
You can see that $u_n \to 50$ as $n \to \infty$. This is a convergent sequence.

2.

$u_{n+1} = 2u_n + 20, \quad u_0 = 10$ generates the following sequence (to 2 decimal places)

We can represent this in a graph as shown here.

$u_0 = 10$
$u_1 = 40$
$u_2 = 100$
$u_3 = 220$
$u_4 = 460$
$u_5 = 940$
$u_6 = 1900$
$u_7 = 3820$
$u_8 = 7660$
$u_9 = 15340$



Notice that this time the sequence is divergent. Subsequent terms continue getting bigger and bigger.

Compare the following two recurrence relations.

It is useful to be able to predict when these two situations arise. The following exercise should help you reach some conclusions.

30 min

Exercise 6

Q28:

Investigate the effect of altering a in the recurrence relation $u_{n+1} = au_n + 12$ with $u_0 = 4$

Try the following values of a

a)
- i 2
- ii 2
- iii 0.6
- iv 0.2
- v -0.2
- vi -0.5
- vii -2
- viii -3

b) Which of the above recurrence relations converges to a limit as $n \Rightarrow \infty$?

c) In general, which values of a for $a \in \mathbb{R}$, will give a convergent sequence?

Q29:

Investigate the effect of altering b in the recurrence relation $u_{n+1} = 0.25u_n + b$ with $u_0 = 4$

Try the following values of b

a)

- i 12
- ii 6
- iii 3
- iv 0.3
- v -0.9
- vi -6

b) Which of the above recurrence relations converges to a limit as $n \Rightarrow \infty$?

c) What effect does changing b have on the recurrence relation?

Q30:

Investigate the effect of changing u_0 in the recurrence relation $u_{n+1} = 0.1u_n + 18$

Try the following values of u_0

a)

- i 1000
- ii 10
- iii 1
- iv -100

b) Does the value of $u_0$effect the value of the limit?

4.8 The limit of a recurrence relation

Learning Objective

Find and interpret the limit of the sequence resulting from a recurrence relation in a mathematical model

In the previous section we were able to confirm that

For the linear recurrence relation $u_{n+1} = au_n + b$

If $-1 < a < 1$ then u_n tends to a limit, L, and L is given by the formula

$$L = \frac{b}{1 - a}$$

Proof

We can derive the above formula for L from the following.
In general we can say that if $u_{n+1} \to L$ as $n \to \infty$ then we also have that $u_n \to L$ as $n \to \infty$

Thus, as $n \to \infty$, the formula $u_{n+1} = au_n + b$ tends to the following

$$\begin{aligned} L &= aL + b \\ L - aL &= b \\ L(1 - a) &= b \\ L &= \frac{b}{1 - a} \end{aligned}$$

Examples

1.

a) Find the first six terms of the recurrence relation $u_{n+1} = 0.8u_n + 4$ with $u_0 = 2$

b) Give a reason why this recurrence relation generates a sequence which has a limit.

c) Calculate the value of the limit.

Solution

a) The first six terms in the sequence are

$u_0 = 2$

$u_1 = 5.6$

$u_2 = 8.48$

$u_3 = 10.784$

$u_4 = 12.6272$

$u_5 = 14.10176$

b) The sequence has a limit because $-1 < 0.8 < 1$

c) As $n \Rightarrow \infty$ the recurrence relation $u_{n+1} = 0.8u_n + 4$ tends to

$$\begin{aligned} L &= 0.8L + 4 \\ 0.2L &= 4 \\ L &= \frac{4}{0.2} \\ &= 20 \end{aligned}$$

Thus the sequence has a limit of 20

2.

Fish, like all animals need oxygen to survive. The fish in a certain tank use up 15% of the oxygen in the water each hour. However, due to the action of a pump, oxygen is added to the water at a rate of 1 part per metre^3 each hour.

The oxygen level in the tank should be between 5 and 7 parts per metre^3 for the survival of the fish. Initially the concentration of oxygen in the tank is 6ppm^3

1. Write down a recurrence relation to describe the oxygen level in the water.
2. Say whether or not a limit exists, giving a reason.
3. Determine, in the long term, whether the fish will survive.

Solution

a) Let F_n represent the oxygen level in the water after n hours then $F_{n+1} = 0.85F_n + 1$ with $F_0 = 6$

b) A limit does exist since $-1 < 0.85 < 1$

c) The fish will survive because when $n \Rightarrow \infty$ then $u_n \Rightarrow L$ and $u_{n+1} \Rightarrow L$thus

$$\begin{aligned} L &= 0.85L + 1 \\ 0.15L &= 1 \\ L &= \frac{1}{0.15} \\ &= 6.667 \end{aligned}$$

Since $5 < 6.667 < 7$ the fish will survive.

Now try the questions in the following exercise.

Exercise 7

40 min

There is an on-line exercise at this point, which you might find helpful.

Q31:

For each of the following recurrence relations

a) find the first six terms,
b) state whether or not a limit exists and give a reason,
c) calculate the limit of the sequence, if it exists.

i $u_{n+1} = 0.5u_n + 10, \quad u_0 = 3$

ii $u_{n+1} = 0.2u_n + 4, \quad u_0 = 100$

iii $u_{n+1} = u_n - 20, \quad u_0 = 200$

iv $u_{n+1} = -0.2u_n + 6, \quad u_1 = -4$

v $u_{n+1} = -0.25u_n + 10, \quad u_0 = 30$

vi $u_{n+1} = -1.5u_n + 25, \quad u_0 = 0$

Q32:

A sequence is defined by the recurrence relation $u_{n+1} = 0.3u_n + 5$ with first term u_1

a) Explain why this sequence has a limit as n tends to infinity.
b) Find the **exact** value of this limit.

S.C.E Mathematics Higher Mathematics 1996

Q33:

A farmer has 80 kg of strawberries in a field ready to be picked. Each day 75% of the crop are picked and during the day another 15 kg ripen.
Let W_n be the weight of strawberries ready for picking after n days.

a) Write down a recurrence relation to describe the weight of strawberries that are ready to be picked.
b) What weight of strawberries are ready to be picked after 4 days? (to 1 decimal place)
c) How do you know that the number of ripe strawberries ready to be picked will become constant?

d) Calculate the limit of the sequence.

Q34:

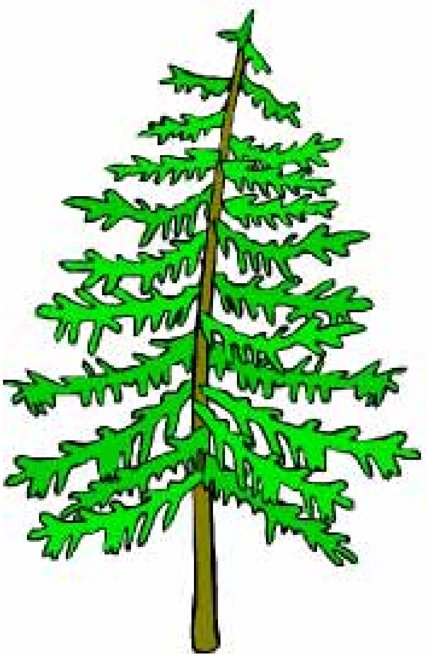
An office worker has 70 folders on his desk ready to be filed. Each hour he manages to file 62% of the folders. However, another 20 are also added to his pile each hour.

a) Letting F_n represent the number of folders on his desk after n hours, write down a recurrence relation to model this situation.

b) How many folders are ready to be filed after 5 hours?

c) In the long term how many folders should he expect to have on his desk ?

Q35:



Trees are sprayed weekly with the pesticide, 'Killpest', whose manufacturers claim it will destroy 65% of all pests. Between the weekly sprayings, it is estmated that 500 new pests invade the trees.

A new pesticide, 'Pestkill', comes onto the market. The manufacturers claim it will destroy 85% of existing pests but it is estimated that 650 new pests per week will invade the trees.

Which pesticide will be more effective in the long term?

S.C.E Higher Mathematics 1995

Q36:

a) At 12 noon a hospital patient is given a pill containing 50 units of antibiotic.
By 1 p.m. the number of units in the patient's body has dropped by 12%
By 2 p.m. a further 12% of the units remaining in the body at 1 p.m. is lost.
If this fall-off rate is maintained, find the number of units of antibiotic remaining at 6 p.m.

b) A doctor considers prescribing a course of treatment which involves a patient taking one of these pills every 6 hours over a long period of time.
The doctor knows that more than 100 units of this antibiotic in the body is regarded as too dangerous.
Should the doctor prescribe this course of treatment or not?
Give reasons for your answer.

S.C.E Higher Mathematics 1991

4.9 Solving recurrence relations

Learning Objective

Calculate variables in a recurrence relation

Example

Given the recurrence relation $u_{n+1} = au_n + b$ with $u_1 = 2$, $u_2 = 7$ and $u_3 = 17$

a) find the values of a and b

b) calculate u_5

Solution

a)

From the recurrence relation $u_{n+1} = au_n + b$ we can write down

$u_2 = au_1 + b$ and $u_3 = au_1 + b$

$7 = 2a + b$ and $17 = 7a + b$

We now have the simultaneous equations

$$\begin{cases} 7 = 2a + b \\ 17 = 7a + b \end{cases}$$

These can be solved to give a = 2 and b = 3.

Thus the recurrence relation is $u_{n+1} = 2u_n + 3$

b) Given the recurrence relation $u_{n+1} = 2u_n + 3$ and $u_3 = 17$ then

$u_4 = 2u_3 + 3 = 2 \times 17 + 3 = 37$

$u_5 = 2u_4 + 3 = 2 \times 37 + 3 = 77$

Thus $u_5 = 77$

Now try the following questions

Exercise 8

There is an on-line exercise at this point, which you might find helpful.

25 min

Q37: Given the recurrence relation $u_{n+1} = au_n + b$ find a and b in each of the following cases.

a) $u_1 = 5,\ u_2 = 13$ and $u_3 = 37$

b) $u_5 = 20,\ u_6 = 40$ and $u_7 = 90$

c) $u_0 = 80,\ u_1 = 35$ and $u_2 = 12.5$

d) $u_1 = 100,\ u_2 = -60$ and $u_3 = 68$

Q38: Given the recurrence relation $u_{n+1} = au_n + b$

a) find a and b when $u_1 = 8000,\ u_2 = 6500$ and $u_3 = 5375$

b) hence find u_0

c) also find the limit, L, of the sequence as $n \to \infty$

Q39: A recurrence relation is given as $u_{n+1} = 5u_n + b$ with $u_0 = 1$ and $u_2 = 97$. Find the value of b.

Q40:

The same sum of money is placed in a special investment acount on the 1st January and each year for five years. No money is withdrawn and the interest rate over the period of investment remains constant. The amount of money in the account at the start of the next three years is £1050, £1655 and £2320.50. Calculate the interest rate and the amount invested each year.

Q41:

A mushroom bed contains 4000 mushrooms on the first day of harvesting. Each day x% of the mushrooms are picked and each night another y mushrooms are ready for picking. On the second and third days there are 3700 then 3460 mushrooms ready to be picked. Calculate the values of x and y

Q42: Two sequences are defined by the recurrence relations

$u_{n+1} = 0.4u_n + x, \quad u_0 = 20$ and

$v_{n+1} = 0.9v_n + y, \quad v_0 = 10$

If both sequences have the same limit, express x in terms of y.

4.10 Summary

Learning Objective

Recall the main learning points from this topic

1. A sequence is a pattern of numbers that can be defined by a rule or formula.
2. A recurrence relation describes a sequence where each term is a function of the previous term.
3. A geometric sequence takes the form $u_{n+1} = au_n$
   The n^{th} term can be written as $u_n = a^n u_0$
4. An arithmetic sequence takes the form $u_{n+1} = u_n + b$.
   The n^{th} term can be written as $u_n = u_0 + nb$
5. A linear recurrence relation is a sequence defined by $u_{n+1} = au_n + b$, $a \neq 0$
6. For the linear recurrence relation $u_{n+1} = au_n + b$ if $-1 < a < 1$ then u_n tends to a limit, L, and the limit is given by the formula
   $L = \frac{b}{1 - a}$

4.11 Extended information

There are links on the web which give a variety of web sites related to this topic.

Leonardo Pisano Fibonacci

Fibonacci was born in 1170 and died in 1250 in Italy. Fibonacci is actually a nickname, his real name being Leonardo Pisano. He also sometimes called himself Bigollo, which may mean either a traveller or a good-for-nothing.

Although born in Italy he was educated in North Africa where his father held a diplomatic post. He travelled extensively with his father and grew to appreciate the different mathematical counting systems in the countries they visited.

When Fibonacci was about 30 he returned to Pisa and began writing a number of important texts. These include *Liber abbaci*(1202), *Practica geometriae*(1220), *Flos*(1225) and *Liber quadratorium*. It is remarkable that copies of these texts still exist, as Fibonacci lived in the days before printing, so his books were handwritten and the only way to have a copy was to have another handwritten version made.

Liber abbaci was based on the arithmetic and algebra that Fibonacci had observed during his travels. This book introduced the decimal system and the use of Arabic numerals into Europe. Linear simultaneous equations were presented in this text. It contains problems related to the price of goods and how to calculate profit. It detailed how to convert between various Mediterranean currencies. It also introduced the Fibonacci sequence which arises from the following problem and for which Fibonacci is probably best remembered today.

"A certain man put a pair of rabbits in a place surrounded on all sides by a wall. How many pairs of rabbits can be produced from that pair if it is supposed that every month each pair begets a new pair which from the second month on becomes productive?"

Practica geometriae contains a large collection of geometry problems. It also includes practical information for surveyors and demonstrates how to calculate the height of tall objects using similar triangles.

Flos provides an account of Fibonacci's solution to the root of the equation $10x + 2x^2 + x^3 = 20$, which comes from a book by Omar Khayyam on algebra. Fibonacci gives an approximate solution in sexagesimal notation as 1.22.7.42.33.4.40 (this is $1 + \frac{22}{60} + \frac{7}{60^2} + \frac{42}{60^3} + \ldots$) which converts to the decimal 1.3688081075. This is correct to nine decimal places and was a remarkable achievement at that time.

Liber quadratorium means the book of squares and is considered as Fibonacci's most impressive work. It is a number theory book and includes methods for finding Pythagorean triples.

4.12 Review exercise

Review exercise in recurrence relations

10 min

There is an on-line exercise at this point which you might find helpful.

These questions are intended to test basic competency in this topic.

Q43:

A gardener feeds her geraniums with "Growmore" plant food. Each week the amount of plant food in the geraniums drops by about 40% and each week the gardener applies another 2g of "Growmore" to each plant.

a) Letting G_n represent the amount of Growmore plant food present in the plant after n weeks write down a recurrence relation for G_{n+1}

b) Find the limit of this recurrence relation and explain what it means in the context of this question.

Q44:

At a university students form a queue to buy food at the canteen. Every 10 minutes 75% of the queue are served. Also, every 10 minutes another 15 students join the queue.

a) Letting S_n denote the number of students in the queue after 10n minutes write down a recurrence relation for S_{n+1}

b) Find the limit of this recurrence relation and explain what it means in the context of this question.

4.13 Advanced review exercise

Advanced review exercise in recurrence relations

20 min

There is an on-line exercise at this point , which you might find helpful.

These questions are intended to reflect the standard of question that might be posed in a final exam.

Q45:

Two sequences are defined by the recurrence relations

$G_{n+1} = 2G_n + 0.5$ with $G_0 = 0$

$K_{n+1} = 0.1K_n + 2$ with $K_0 = 1$

a) Explain why only one of these recurrence relations approaches a limit as $n \to \infty$

b) Find the ***exact*** value of the limit.

Q46:

The level of silt in a harbour just after high tide is 0.5 metres at present. A dredger manages to clear 85% of the silt in between high tides. However, each high tide another 1.2 metres of silt is deposited.

a) Using D_n for the level of silt in the harbour after n high tides write down a recurrence relation for D_{n+1}

b) How much silt is there in the harbour after 4 high tides?

c) To keep the harbour open to fishing boats the level of silt must not exceed 1.5 metres. Will the harbour remain open? Justify your answer.

Q47:

Biologists calculate that, when the concentration of a particular chemical in a sea loch reaches 5 milligrams per litre (mg/l), the level of pollution endangers the life of the fish.

A factory wishes to release waste containing this chemical into the loch. It is claimed that the discharge will not endanger the fish.

The Local Authority is supplied with the following information:

1. The loch contains none of this chemical at present.
2. The factory manager has applied to discharge waste once per week which will result in an increase in concenration of 2.5 mg/l of the chemical in the loch.
3. The natural tidal action will remove 40% of the chemical from the loch every week.

a) Show that this level of discharge would result in the fish being endangered.

When this result is announced, the company agrees to install a cleaning process that reduces the concentration of chemical released into the loch by 30%

b) Show the calculations you would use to check this revised application.
Should the Local Authority grant permission?

(S.C.E. Higher Mathematics 1992)

Q48:

Secret Agent 004 has been captured and his captors are giving him a 25 milligram dose of a truth serum every 4 hours. 15% of the truth serum present in his body is lost every hour.

a) Calculate how many milligrams of serum remain in his body after 4 hours (that is, immediately ***before*** the second dose is given).

b) It is known that the level of serum in the body has to be continuously above 20 milligrams before the victim starts to confess. Find how many doses are needed before the captors should begin their interrogation.

c) Let u_n be the amount of serum (in milligrams) in his body just ***after*** his n^{th} dose. Show that $u_{n+1} = 0.522u_n + 25$

d) It is also known that 55 milligrams of this serum in the body will prove fatal, and the captors wish to keep Agent 004 alive. Is there any maximum length of time for which they can continue to administer this serum and still keep him alive?

(S.C.E. Higher Mathematics 1993)

4.14 Set review exercise

Set review exercise in recurrence relations

10 min

An on-line assessment is available at this point, which you will need to attempt to have these answers marked. These questions are not randomised on the web site. The questions on the web may be posed in a different manner but you should have the required answers in your working. These questions are intended to test basic competency in this topic.

Q49:

The sand on a beach is combed each night for rubbish which results in $\frac{3}{4}$ of the rubbish being removed. However, each day sunbathers drop around 60 kg. more rubbish. There are u_n kilograms of rubbish on the beach at the start of a particular day.

a) Write down a recurrence relation for u_{n+1}, the amount of rubbish on the beach at the start of the next day.

b) Find the limit of the sequence generated by this recurrence relation and explain what the limit means in the context of this question.

Q50:

A car driver uses $\frac{4}{5}$ of the screen wash in the tank in her car every month. However, at the end of each month she adds 1 pint of screen wash to top up the tank. There are u_n pints of screen wash in her car at the start of a particular month.

a) Write down a recurrence relation for u_{n+1}, the amount of screen wash in her car at the start of the next month.

b) Find the limit of the sequence generated by this recurrence relation and explain what the limit means in the context of this question.

Glossary

Amplitude of a graph

The amplitude of a graph is half of the distance along the y-axis between the maximum and minimum values of the graph.

arithmetic sequence

An **arithmetic sequence** is a special type of recurrence relation that takes the form $u_{n+1} = u_n + b$

The n^{th} term is given by $u_n = u_0 + bn$

Calculus

Calculus is the mathematics of motion and change.

closed interval

A closed interval is a subset of the real number line which includes both end points. Thus, for example, [-2,5] is the closed interval $-2 \leqslant x \leqslant 5$

Codomain of a function

For a function $f : A \rightarrow B$, B is called the codomain of the function f.

Completing the square

After completing the square,

a quadratic function has the form $(x + k)^2 + m$

Concurrency

Lines which meet at a point are said to be concurrent.

convergent sequence

A sequence is **convergent** if it has a limit as $n \Rightarrow \infty$

derived function

The instantaneous speed, or rate of change of distance with respect to time, can be written as f'(t) and is known as the **derived function** of f(t).

differentiation

Differentiation is the method for calculating the derived function, f'(x), from f(x)

divergent sequence

A sequence is **divergent** if it does not have a limit as $n \Rightarrow \infty$

Domain of a function

For a function $f : A \rightarrow B$, A is called the domain of the function f.

Equation of a straight line - alternative form

Given one point and the gradient, the equation of the straight line can be found by using the formula $y - y_1 = m(x - x_1)$ where (x_1, y_1) is the known point and m is the gradient.

Equation of a straight line - general form

The general form of the equation of a straight line is

$Ax + By + C = 0$ (A, B not both zero).

Equation of a straight line - standard form

The equation of a straight line takes the form $y = mx + c$ where m is the gradient of the line and c is the y-intercept.

Exponential function

A function of the form $f(x) = a^x$ where $a > 0$ and $a \neq 1$ is called an exponential function

Function f

A function f from set A to set B is a rule which assigns to each element in A exactly one element in B. This is often written as $f : A \to B$.

geometric sequence

A **geometric sequence** is a special type of recurrence relation that takes the form $u_{n+1} = au_n$

The n^{th} term is given by $u_n = a^n u_0 u_n = a^n u_0$

Gradient

The slope of a straight line from the point A (x_1, y_1) to B (x_2, y_2) is called the gradient.

It is denoted m_{AB} and defined as $m_{AB} = \frac{y_2 - y_1}{x_2 - x_1}$ where $x_1 \neq x_2$

Gradient as a tangent

The gradient of a straight line is the tangent of the angle made by the line and the positive direction of the x-axis. (assuming that the scales are equal).

If A is the point (x_1, y_1) and B is the point (x_2, y_2) then $m_{AB} = \tan\theta$ where θ is the angle made by the line and the positive direction of the x-axis.

The image set or range of a function

For a function $f : A \to B$, the set C of elements in B which are images of the elements in A under the function f is called the image set or range of the function f. C is always contained in or equal to B. This is written $C \subseteq B$

integration

Integration is the method for finding anti-derivatives.

Intersection point of the medians

Let the points M (a, b), N (c, d) and P (e, f) be the vertices of a triangle. The point of intersection of the medians has coordinates

$$\left(\frac{a + c + e}{3}, \frac{b + d + f}{3}\right)$$

Inverse function

Suppose that f is a one-to-one and onto function. For each $y \in B$ (codomain) there is exactly one element $x \in A$ (domain) such that f (x) = y

The inverse function is denoted $f^{-1}(y) = x$

linear recurrence relation

A **linear recurrence relation** is a sequence defined by $u_{n+1} = au_n + b$, $a \neq 0$

Many-to-one function

A function which maps more than one element in the domain to the same element in the range or image set is called a many-to-one or a many-one function.

The function is said be in many-to-one correspondence.

It is also common to say that such a function is not one-to-one.

The modulus function

For $x \in \mathbb{R}$ the modulus function of f (x), denoted by $|f(x)|$ is defined by

$$|f(x)| = \begin{cases} f(x) & \text{if } f(x) \geqslant 0 \\ -f(x) & \text{if } f(x) < 0 \end{cases}$$

One-to-one function

A function $f: A \to B$ is a one-to-one function if whenever f (s) = f (t)

then s = t where $s \in A$ and $t \in A$

The function is said to be in one-to-one correspondence.

Onto function

An onto function is one in which the range is equal to the codomain.

Period of a graph

The length along the x-axis over which the graph traces one full pattern is called the period of the graph.

Radian

An angle subtended at the centre of a circle by an arc of length equal to the radius of the circle is called a radian.

Rule for graphing y = $|$ f (x) $|$

To sketch the graph of a modulus function $|f(x)|$, first sketch the graph of the function y = f (x)

Take any part of it that lies below the x-axis and reflect it in the x-axis.

The modulus function $y = |f(x)|$ is the combined effect of the positive part of the original function and the new reflected part.

Rule for graphing y = f (k x)

To obtain y = f (k x) scale the graph of y = f (x) horizontally by a factor of $^1/_k$

This type of transformation is called a horizontal scaling.

Rule for graphing y = f (-x)

To obtain the graph of y = f (-x) take the graph of y = f (x) and reflect it in the y-axis.

Rule for graphing y = - f (x)

To obtain the graph of y = - f (x) take the graph of y = f (x) and reflect it in the x-axis.

Rule for graphing y = f (x) + k

To obtain the graph of y = f (x) + k take the graph of y = f (x).

- For k $>$ 0 slide the graph UP the y axis by k units.
- For k $<$ 0 slide the graph DOWN the y axis by k units.

This type of transformation is known as a vertical translation.

Rule for graphing y = f (x + k)

To obtain y = f (x + k) take y = f (x)

- For k $>$ 0 slide the graph to the *left* by k units.
- For k $<$ 0 slide the graph to the *right* by k units.

This type of transformation is called a horizontal translation or in trigonometric terms it is also called a phase shift.

Rule for graphing y = k f (x)

To obtain the graph of y = k f (x) scale the graph of y = f (x) vertically by a factor of k.

This type of transformation is known as a vertical scaling.

sequence

A **sequence** is an ordered list of terms.

Standard number sets

The standard number sets are:

- $\mathbb{N}$ = {1, 2, 3, 4, 5, ...} the set of natural numbers.
- $\mathbb{W}$ = {0, 1, 2, 3, 4, 5, ...} the set of whole numbers.
- $\mathbb{Z}$ = {..., -3, -2, -1, 0, 1, 2, 3, ...} the set of integers.
- $\mathbb{Q}$ = the set of all numbers which can be written as fractions, called the set of rational numbers.
- $\mathbb{R}$ = the set of rational and irrational numbers (such as $\sqrt{2}$), called the set of real numbers.

stationary points

Stationary points are points on a curve where the function is neither increasing nor decreasing. At these points f'(x) = 0

Turning point coordinates from completing the square

For a quadratic function in the form $(x + k)^2 + m$, the coordinates of the turning point are given by (-k, m)

Answers to questions and activities

1 Properties of the straight line

Revision exercise (page 3)

Q1:

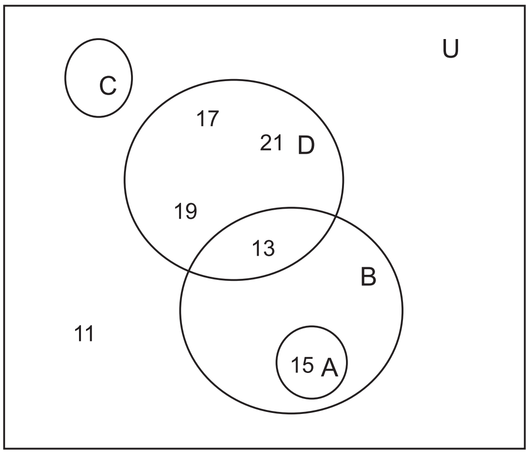
a) U = {1, 2, 3, 4, 5, 6, 7, 8, 9}
b) D
c) 4
d) A
e) C = {1, 4, 5, 6, 7}
f) yes

Q2: The *smallest* sets are as follows:

a) The set $\mathbb{R}$
b) The set $\mathbb{W}$
c) The set $\mathbb{N}$
d) The set $\mathbb{Z}$
e) The set $\mathbb{Q}$
f) The set $\mathbb{R}$

Remember that these answers identify the smallest set to which the particular number belongs.

Q3:



Gradients of straight lines exercise (page 9)

Q4:

a) $m_{AB} = \frac{10 - 2}{2 + 10} = \frac{8}{12} = \frac{2}{3}$

b) $m_{PQ} = \frac{-4-6}{-2-2} = \frac{-10}{-4} = \frac{5}{2}$

c) $m_{RS} = \frac{-8-2}{-2+8} = \frac{-10}{6} = -\frac{5}{3}$

d) $m_{CD} = 0$ (parallel to x-axis)

Q5:

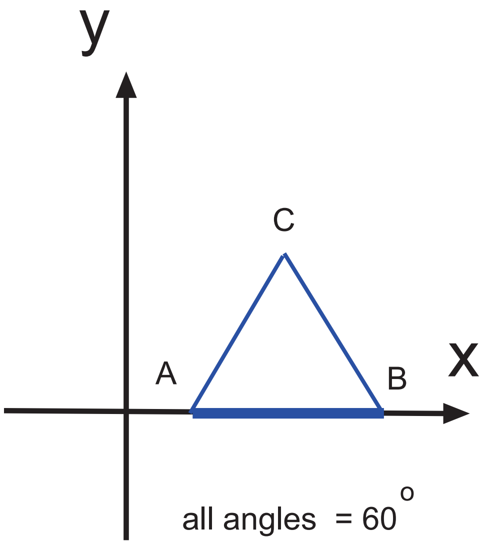
a) $m_{MN} = \frac{2-6}{3-(-4)} = -\frac{4}{7}$

b) m_{LK} undefined or infinite(The line is parallel to the y-axis).

c) $m_{PQ} = 0$ (The line is parallel to the x-axis).

d) $m_{TO} = \frac{-4}{-(-1)} = -4$

Q6: Draw a diagram:

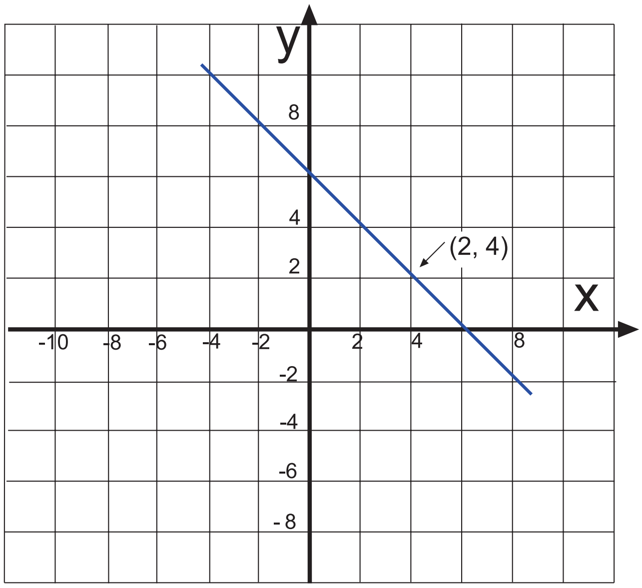


m_{AC} = tan 60° = 1.7 to 1 decimal place.

m_{CB} = tan (-60) = -1.7 to 1 decimal place.

m_{AB} = 0 (parallel to the x-axis).

Q7: If the gradient is -1 then tan θ = -1 where θ is the angle between the line and the positive x-axis. So θ = -45°



The line crosses the x-axis at (6, 0) and the y-axis at (0, 6)

Q8: AB has zero gradient

CD has a negative gradient

EF has an infinite gradient

GH has a positive gradient

Distance between two points exercise (page 11)

Q9:

a) $\sqrt{(3-2)^2 + (-5+1)^2} =$
$\sqrt{17} = 4.1$ to 1d.p.

b) $\sqrt{(1+2)^2 + (1+3)^2} =$
$\sqrt{25} = 5.0$ to 1d.p.

c) $\sqrt{(1+2)^2 + (1-0)^2} =$
$\sqrt{10} = 3.2$ to 1d.p.

Equation of a line as y = mx + c exercise (page 13)

Q10:

a) AB has equation y = -4x - 2
b) CD has equation y = 4
c) EF has equation x = 1
d) GH has equation y = -2x - 3

Q11:

a) The gradient is -2 and the y-intercept is -1
b) The gradient is 3 and the y-intercept is 0
c) The gradient is -5 and the y-intercept is -5
d) The gradient is undefined or infinite and there is no y-intercept
e) The gradient is 1 and the y-intercept is -1

Conversion between equation forms exercise (page 14)

Q12:

a) y + 2x + 3 = 0 and A = 1, B = 2, C = 3
b) y - 3x = 0 and A = 1, B = -3, C = 0
c) y - 2x + 1 = 0 and A = 1, B = -2, C = 1
d) y - 4 = 0 and A = 1, B = 0, C = -4
e) x + 5 = 0 and A = 0, B = 1, C = 5

Q13:

a) This equation is x = 4 and the line is parallel to the y-axis. It does not have a y-intercept.
b) y = -3. This line has gradient zero and y-intercept of -3
c) y = -2x + 3 with gradient = -2 and y-intercept of 3
d) y = 2x + 3 with gradient of 2 and y-intercept of 3

Further equations exercise (page 15)

Q14:

a) The point (0, -2) is on the y-axis and so the y-intercept is -2
The gradient is 4 and so the equation of the line AB is y = 4x - 2
b) $m_{GH} = \frac{-2-4}{1-3} = \frac{-6}{-2} = 3$
The point (3, 4) is on the line and so the equation takes the form
$y - 4 = 3(x - 3) \Rightarrow y = 3x - 5$
The equation of the line GH is y = 3x - 5
c) RS has an equation of the form $y - y_1 = -4(x - x_1)$
The point (2, -3) lies on the line $\Rightarrow y + 3 = -4(x - 2)$
The equation of RS is y = -4x + 5
d) CD is parallel to the y-axis and so the equation takes the form x = k
Here k = -1 since (-1, 0) is on the x-axis and the equation of CD is x = -1
e) EF has gradient of zero. Notice that the y co-ordinates are equal. This means that the line is parallel to the x-axis and the equation takes the form y = c
The point (0, 5) is on the y-axis and so the y-intercept is y = 5
The equation of EF is y = 5

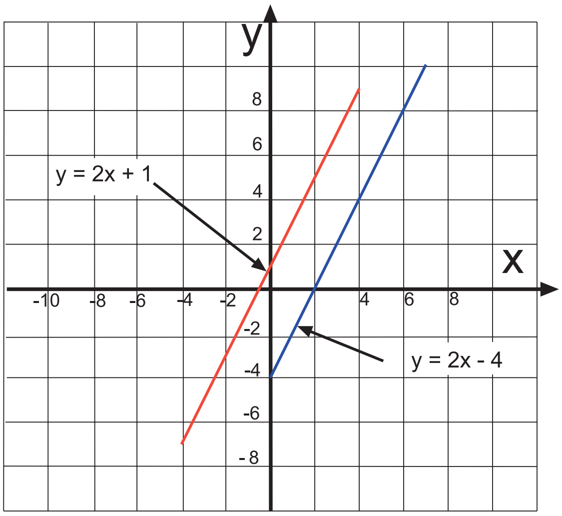
Answers from page 16.

Q15: The gradient of y = 2x - 4 is 2

The gradient of y = 2x + 1 is 2

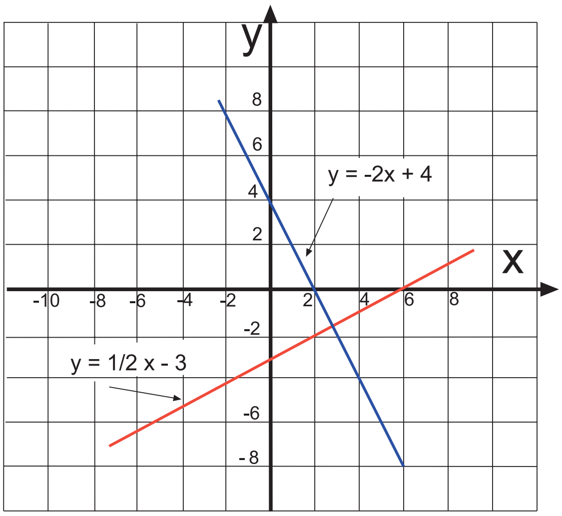
So the lines have the same gradient.

A graph shows that the lines are also parallel.



Answers from page 18.

Q16:



The two lines are perpendicular; that is they are at right angles.

Perpendicular and parallel lines exercise (page 19)

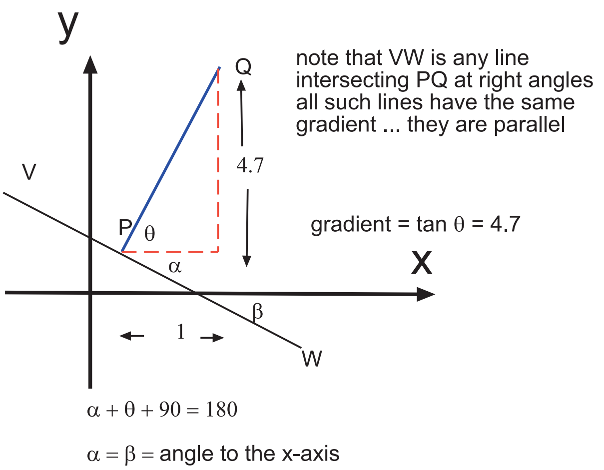
Q17: $m_{AB} = 4$

If lines are perpendicular then the product of the gradients is -1

Let the gradient of the line perpendicular to AB be m_p

then $m_p m_{AB} = -1 \Rightarrow m_p = {}^{-1}/_4$

Q18:

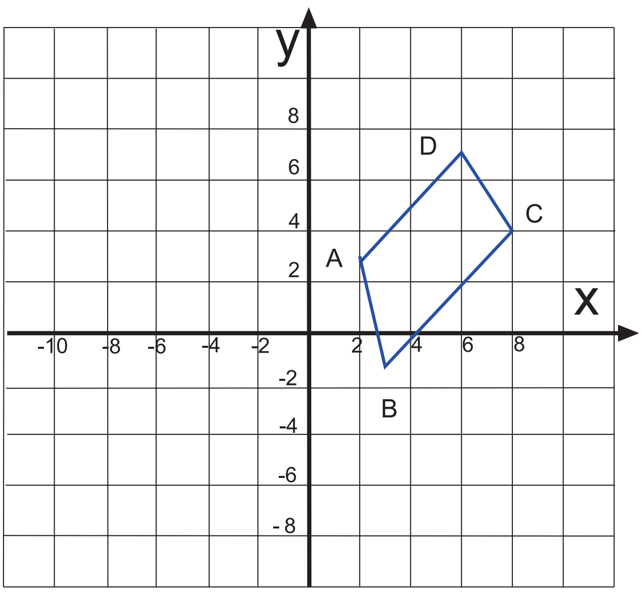


$m_{PQ} = 4.7 \Rightarrow$ PQ is at an angle of $\tan^{-1}(4.7)°$ to the positive x-axis

That is, at an angle of 78° to the nearest degree.

Thus if VW is perpendicular to PQ then VW is at an angle of 90 + 78 = 168° or - 12° to the x-axis.

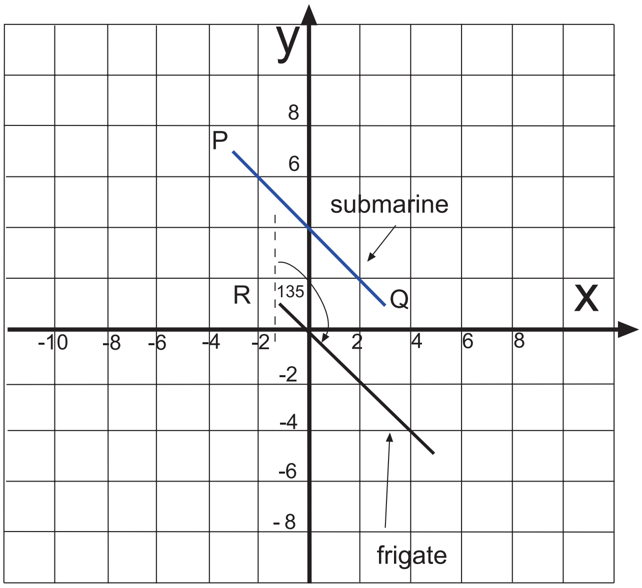
Q19:



$m_{AB} = -4$, $m_{AD} = 1$, $m_{CD} = {}^{-3}/_2$, $m_{BC} = 1$

The lines AD and BC are parallel. The shape is a trapezium.

Q20:



The gradient of the submarine path is -1

The submarine path makes an angle of -45° or 135° with the positive x-axis ($\tan^{-1}(-1)$)

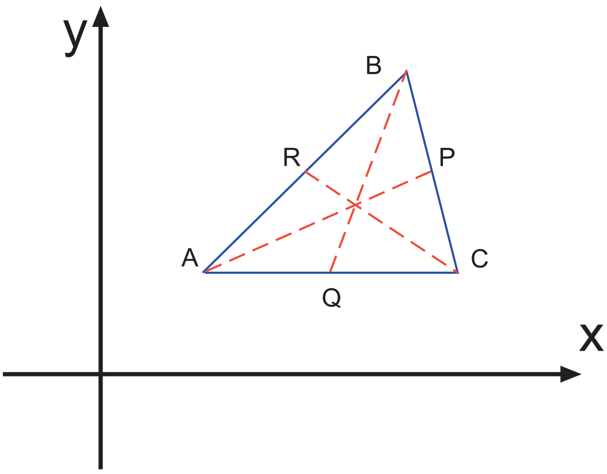
The frigate is on a bearing of 135°. This path makes an angle of 135° or -45° also with the x-axis.

The ship and the submarine are on parallel paths and the frigate will not cross the path of the submarine.

Medians exercise (page 21)

Q21: A median construction is from the vertex to the midpoint of the opposite side.

The first step is therefore to find the midpoints of each side.



The mid point R of AB is $\left(\frac{6+2}{2}, \frac{6+2}{2}\right) = (4, 4)$

The mid point P of BC is $\left(\frac{6+7}{2}, \frac{6+2}{2}\right) = (\frac{13}{2}, 4)$

The mid point Q of AC is $\left(\frac{2+7}{2}, \frac{2+2}{2}\right) = (\frac{9}{2}, 2)$

$m_{AP} = \frac{4 - 2}{\frac{13}{2} - 2} = \frac{4}{9}$

$m_{BQ} = \frac{2 - 6}{\frac{9}{2} - 6} = \frac{8}{3}$

$m_{CR} = \frac{4 - 2}{4 - 7} = -\frac{2}{3}$

The equation of AP using the gradient just found and the point A is

$(y - 2) = \frac{4}{9}(x - 2)$

$y = \frac{4}{9}x + \frac{10}{9}$

The equation of BQ using the gradient just found and the point B is

$(y - 6) = \frac{8}{3}(x - 6)$

$y = \frac{8}{3}x - 10$

The equation of CR using the gradient just found and the point C is

$(y - 2) = -\frac{2}{3}(x - 7)$

$y = -\frac{2}{3}x + \frac{20}{3}$

Q22:

a) Let V be the midpoint of AB, W the midpoint of BC and X the midpoint of AC then the coordinates of these points are
V (0, 4), W (-1, 5) and X (2, 5)
Using the gradient formula gives

$m_{AW} = -\frac{1}{4}$

$m_{BX} = \frac{1}{5}$

$m_{CV} = 2$

Using the alternative equation of a straight line $y - y_1 = m(x - x_1)$ for each median in turn gives the equations:

AW : $y = -\frac{1}{4}x + \frac{19}{4}$

BX : $y = \frac{1}{5}x + \frac{23}{5}$

CV : $y = 2x + 4$

b) Let the midpoints be I for PQ, J for line QR and K for line PR
line IR has equation $y = -{}^{4}/_{7}x + {}^{10}/_{7}$
line JP has equation $y = {}^{1}/_{5}x + {}^{2}/_{5}$
line KQ has equation $y = -{}^{5}/_{2}x + 4$

Q23:

a) Let V be the midpoint of CD, W the midpoint of DE and X the midpoint of CE then the coordinates of these points are:

V = (-1, -1), W = (1, -4) and X = (3, -2)

The equations of the medians are:

CW: x = 1 (note that the gradient is undefined and so the median is parallel to the y-axis. The two points indicate that x = 1)

DX: $y = {}^{1}/_{6}x - {}^{5}/_{2}$

CW: $y = -{}^{2}/_{3}x - {}^{5}/_{3}$

The point of intersection is

$$\left(\frac{1 - 3 + 5}{3}, \frac{1 - 3 - 5}{3}\right) = \left(1, -\frac{7}{3}\right)$$

b) Let X be the midpoint of KM, Y be the midpoint of KL and Z be the midpoint of LM

These points have coordinates: X = (-1, 1), Y = (-1, -1) and Z = (1, -1)

The equations of the medians are:

XL: y = -2x - 1

YM: y = x

KZ: $y = -{}^{1}/_{2}x - {}^{1}/_{2}$

The point of intersection of the medians is $\left(-\frac{1}{3}, -\frac{1}{3}\right)$

Altitude exercise (page 23)

Q24:

a) Let the points of intersection of the altitudes with AB, AC and BC be X, Y and Z respectively then

AZ has gradient ${}^{1}/_{3}$, BY has gradient -1, CX has gradient 3

The equations of the altitudes are:

AZ: $y = {}^{1}/_{3}x + {}^{11}/_{3}$

BY: y = -x + 3

CX: y = 3x + 5

b) Let the points of intersection of the altitudes with LM, KM and LK be X, Y and Z respectively then

KX has gradient -4, LY has gradient ${}^{1}/_{4}$, MZ has gradient $-{}^{3}/_{5}$

Note that this is a right angled triangle and some work can be saved by noticing that the altitude LY is in fact the side LM and similarly the altitude KY is the side KM

The equations of the altitudes are:

KX: y = -4x + 12

LY: $y = {}^{1}/_{4}x - {}^{3}/_{4}$

MZ: $y = -{}^{3}/_{5}x + {}^{9}/_{5}$

Bisector exercise (page 25)

Q25: The mid point of BC is $(-^5/_2, -2)$

The gradient of BC is $^8/_5$ and so the perpendicular bisector has gradient $-^5/_8$

The equation of the perpendicular bisector is $y + 2 = -^5/_8(x + ^5/_2)$

That is, $16y + 10x + 57 = 0$

Q26: The side AB has gradient 2 since the product of the gradients of perpendicular lines is -1.

The equation of the side AB is $y + 2 = 2 (x - 3)$ since (3, -2) is on the line.

That is the equation of AB is $y = 2x - 8$

Review exercise (page 26)

Q27: The mid point of PR is $(-^1/_2, -^7/_2)$

The gradient of PR is $^1/_3$ and so the gradient of the perpendicular bisector is -3

Thus the equation is $y + ^7/_2 = -3 (x + ^1/_2)$

That is, $y = -3x - 5$

Q28: $m_{EF} = \tan 35° = 0.7$

The equation is $y = 0.7(x - 2)$

That is, $y = 0.7x - 1.4$ or $10y - 7x + 14 = 0$

Q29: The gradient of the line through the two points is $-^5/_3$

The line parallel to this line through the origin has a y-intercept of 0 and so has the equation $y = -^5/_3x$

Advanced review exercise (page 26)

Q30: C is the point (6, 3)

The gradient of AB is $^5/_4$ and so the gradient of the altitude to AB is $-^4/_5$

The equation of the altitude is $y - 3 = -^4/_5(x - 6)$

That is, $5y - 15 = -4(x - 6)$

Thus $5y + 4x - 39 = 0$

Q31:

a) The equation of AD is $y - 2 = 5(x - 1)$ that is $y = 5x -3$

b) The gradient of AB is $^5/_2$
Thus the equation of AB is $y - 2 = ^5/_2(x - 1)$
That is $2y = 5x - 1$

c) BC is parallel to AD and so has gradient 5
The equation of BC is $y + 3 = 5(x + 1)$
That is $y = 5x + 2$

Since AB is parallel to CD then CD has gradient $^{5}/_{2}$ and has an equation of the form
2y = 5x + c

but y = -8 when x = 0 and so c = -16

Thus the equation is 2y = 5x - 16

Set review exercise (page 27)

Q32: The answers are only available on the web.

Q33: The answer is only available on the web.

Q34: The answers are only available on the web.

Q35: The answers are only available on the web.

2 Functions and graphs

Revision exercise (page 31)

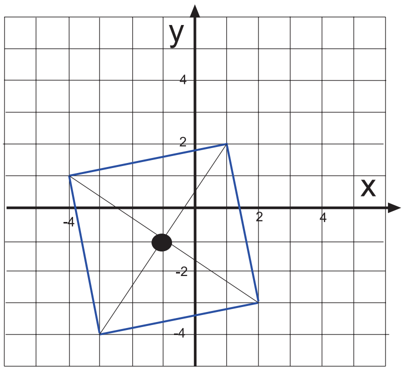
Q1: (x - 9)(x - 2)

Q2: $3x^2 - 12x$

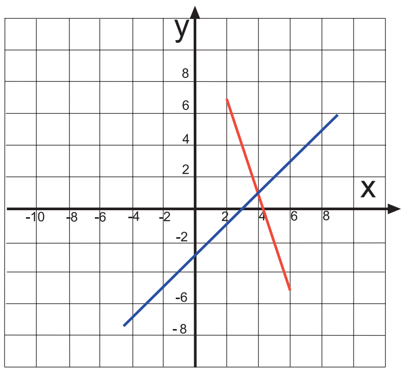
Q3: $(x - {}^1/_2)(x + {}^1/_2)$

Q4: The other two corners are (-4, 1) and (2, -3)

The centre of the square is at (-1, -1)



Q5: The intersection is at the point (4, 1)



Is it a function exercise (page 33)

Q6:

a) No: if x = 4 then y can be either - 2 or + 2 but the definition of a function states that it can only map each element of the domain to one and only one element in the codomain.

b) No: $y^2 = x^2$ solves to give x = ±y and for a value of x there is two values for y

c) Yes: For each value of x there is only one value for k

d) No: $t = \pm\sqrt{x}$ so for any value of x there is two values of t

e) No: There is no rule assigning a unique value to y for x = 3. The number of values for y is infinite.

Domain and range exercise (page 36)

Q7: $x \in \{-1, 0, 3, 8\}$

Note that when x = 1 then f (x) = -1 and this illustrates the importance of not purely looking at the end points of the interval.

Q8: $\{x \in \mathbb{R} : x \geq 2\}$

Note that if x was less than two f (x) is the square root of a negative number.

Q9: $\{y \in \mathbb{R} : y \neq 0\}$ The condition $y \neq 0$ is needed since $\frac{1}{0}$ is not defined.

Q10: $\{z \in \mathbb{R}: z \neq 0\}$

Q11: $\{ f(x) \in \mathbb{Q} : 0 < f(x) \leq 1\}$

Q12: $\{ g(y) \in \mathbb{N}\}$

or this can also be stated as $\{ g(y) \in \mathbb{Z}^+\}$

or $\{ g(y) \in \mathbb{Z} : g(y) > 0\}$

Q13: $\{h(w) \in \mathbb{Z}^+\}$

One-to-one and onto function exercise (page 41)

Q14: Yes.

Q15: Yes.

Q16: No. (For example y = -2 and y = 2 both give g (y) = -4)

Q17: No. (w = 0 and w = 4 both give k (w) = 0)

Q18: Yes.

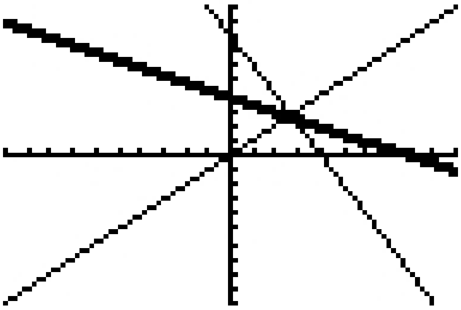
Q19: Yes.

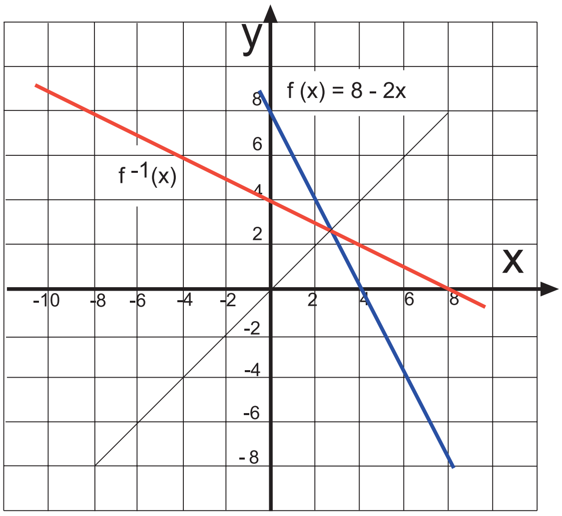
Q20: No. The range is {-4, -3, -2, -1, 0}. This does not equal the codomain.

Q21: No. The range is [0,1]. This does not equal the codomain.

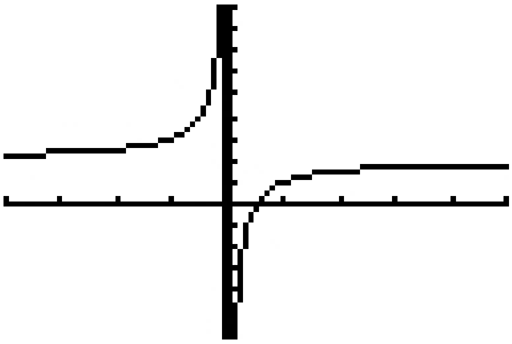
Answers from page 45.

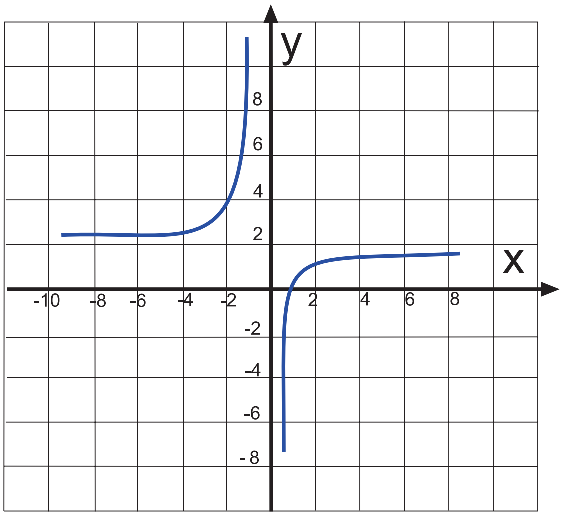
Q22: $f^{-1}(x) = \frac{8 - x}{2}$ where $x \in \mathbb{R}$. The first drawing is the screen dump from a graphics calculator showing the line y = x, the original function and a bold line denoting the inverse function.



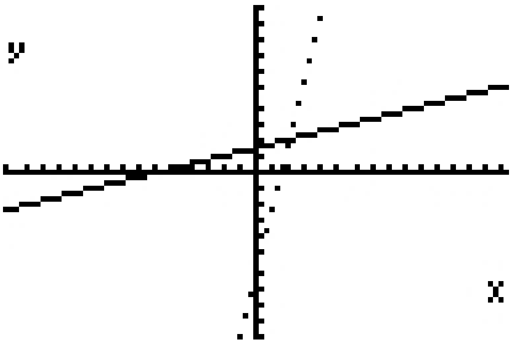


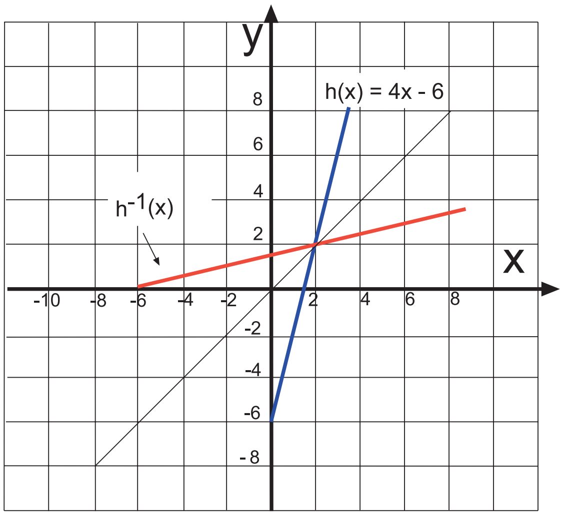
Q23: $g^{-1}(x) = 2 - \frac{1}{x}$ where $x \in \mathbb{R}$, $x \neq 0$. The diagram shows a sketch of this inverse function using a graphics calculator.



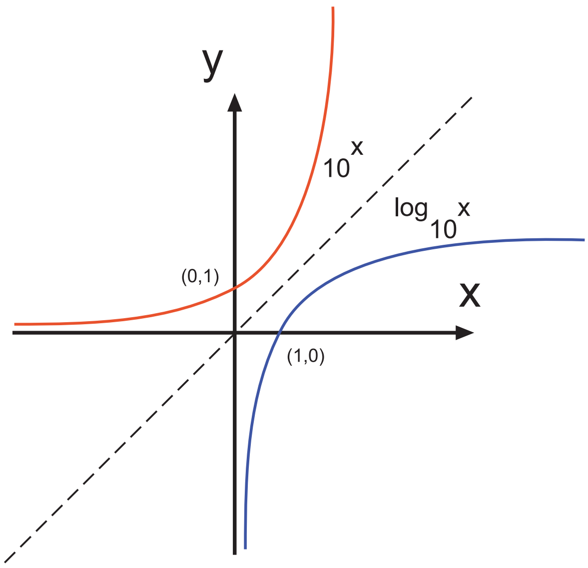


Q24: $h^{-1}(x) = \frac{x+6}{4}$ where $x \in \mathbb{R}$, $x > 6$. The diagram shows the inverse function sketch from a graphics calculator. The calculator image shows a dotted line of the original function and a solid line for its inverse.

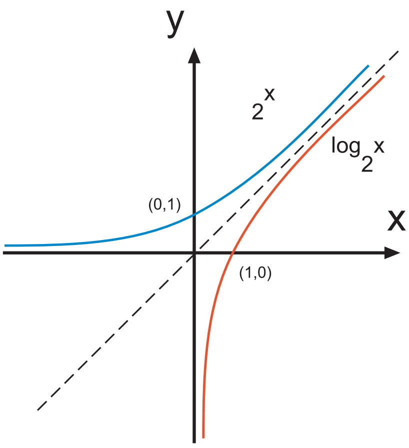




Q25:



Q26:



Note that $y = \log_2 x$ is the inverse of $y = 2^x$

Composite function exercise (page 47)

Q27:

a) f (x) = 1 - x and g (x) = x^2
b) f (x) = x - 2 and g (x) = 2^x
c) f (x) = x^2 and g (x) = cos (x)

Q28:

a) j (x) = x + 4 and k (x) = x^2
b) j (x) = x^3 and k (x) = x - 2
c) j (x) = 4x and k (x) = sin (x)

Q29:

a) $f(g(x)) = f(1 - 2x) = (1 - 2x)^2 - 4(1 - 2x) = 1 - 4x + 4x^2 - 4 + 8x = 4x^2 + 4x - 3$
$g(f(x)) = g(x^2 - 4x) = 1 - 2(x^2 - 4x) = 1 - 2x^2 + 8x$

b) $f(g(x)) = f(-5x) = -5x(-5x - 2) = 25x^2 + 10x$
$g(f(x)) = g((x(x - 2)) = -5(x(x - 2)) = -5x^2 + 10x$

c) $f(g(x)) = f(1 - x^2) = \frac{1}{2 + (1 - x^2)} = \frac{1}{3 - x^2}$
$g(f(x)) = g\left(\frac{1}{2 + x}\right) = 1 - \left(\frac{1}{2 + x}\right)^2 = 1 - \frac{1}{x^2 + 4x + 4}$

d) $f(g(x)) = f(x - 1) = \sin(x - 1)$
$g(f(x)) = g(\sin(x)) = \sin(x) - 1$

Multiple composite functions exercise (page 48)

Q30:

a) $\sin(2x^2 + 1)$
b) $-\sin^2 x + 2\sin^4 x$
c) $2\sin(-x + 2x^2) + 1$

Q31:

a) $r(x) = x^2$, $q(x) = 2x + 1$, $p(x) = -\cos x$
b) $r(x) = x + 1$, $q(x) = x^3$, $p(x) = 2x$
c) $r(x) = x - 1$, $q(x) = x^2$, $p(x) = 2x - 1$

Finding - f (x) exercise (page 49)

Q32:

a) $-2x + 1$
b) $-\sin(x)$
c) $\tan(x)$
d) $-3x^2$
e) $x + 2$
f) $x^2 - 1$

Identify - f (x) exercise (page 50)

Q33: The graph B represents $y = -f(x)$

Q34: The graph B represents $y = -f(x)$

Finding f (-x) exercise (page 52)

Q35:

a) $3(-x) - 7 = -3x - 7$
b) $\cos(-x) = \cos(x)$
c) $\tan(-x)$
d) $3(-x)^3 = -3x^3$
e) $-(-x) - 2 = x - 2$
f) $-(-x)^2 + 4 = -x^2 + 4$

Identify f (-x) exercise (page 53)

Q36: The graph A represents $y = f(-x)$

Q37: The graph D represents $y = f(-x)$

Vertical translation exercise (page 56)

Q38: The line $y = 6x$ is moved downwards by 3 units.

Q39: The line $y = 2x$ is moved upwards by 8 units.

Q40: The second graph moves down the y-axis by 3 units.

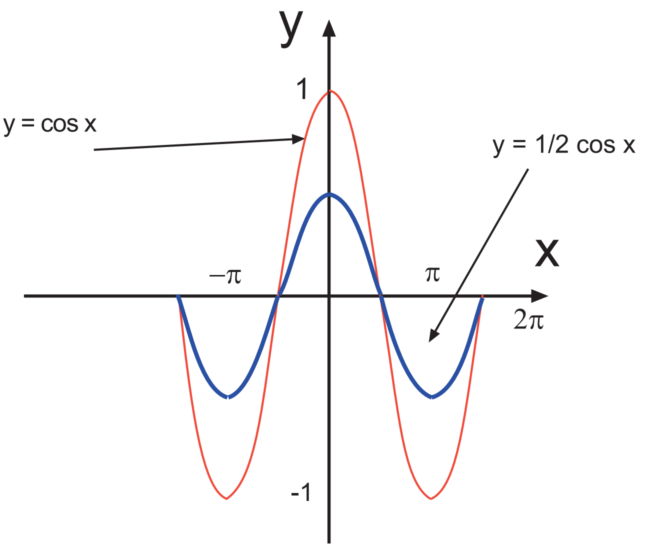
Q41: The second graph moves up the y-axis by 2 units.

Q42: The second graph moves up the y-axis by 4 units.

Q43: The second graph moves down the y-axis by 3 units (in this case by 3 radians).

Vertical scaling exercise (page 57)

Q44:



Q45: The first graph is scaled vertically by a factor of 3 to give the second graph.
(It stretches by a factor of 3)

Q46: The graph of sin x is scaled vertically by a factor of $^1/_4$ to give the second graph.
(It shrinks to one quarter of its size)

Q47: The graph of y = cos 2x is scaled vertically by a factor of $^1/_2$

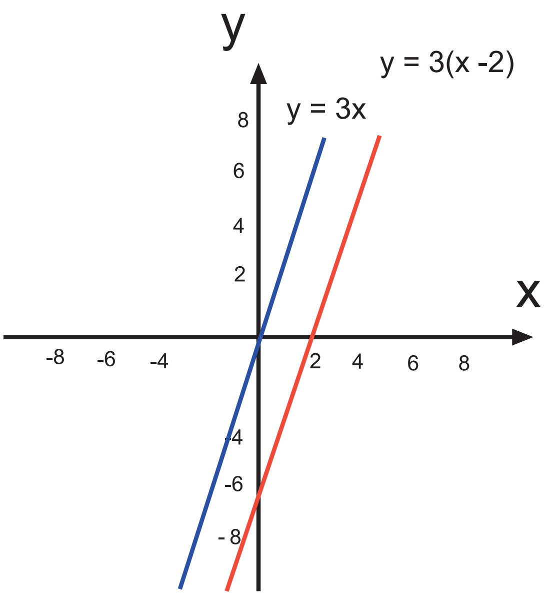
(It shrinks to half the size)

Q48: The graph of y = ln x is scaled vertically by a factor of 3

Q49: The graph of y = $^1/_2 e^x$ is scaled vertically by a factor of 4

Horizontal translation exercise (page 59)

Q50: The graph of y = 3 (x - 2) sits two units along the x-axis to the right of the graph of y = 3x



Q51: The graph of y = cos (x + 30)° sits 30° to the left of the graph of y = cos x°

Q52: The graph will lie $^1/_2$ a unit along the x-axis to the right of the graph of $y = (2x)^2$

$(y = (2x - 1)^2 = (2(x - {}^1/_2))^2$ so $k = -{}^1/_2)$

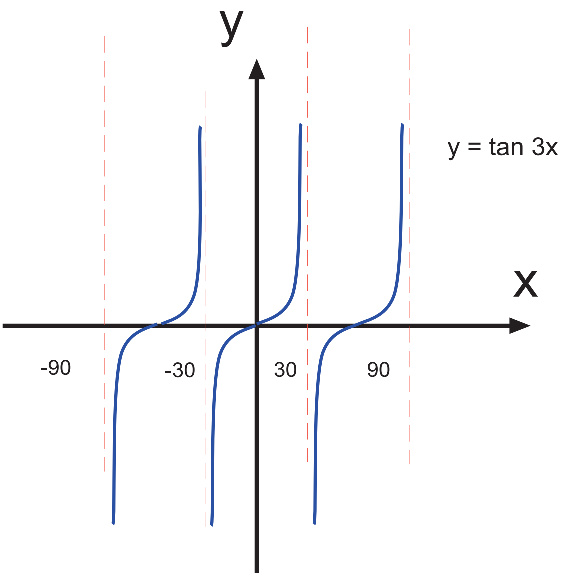
Q53: The graph of $y = \sqrt{x + 2}$ moves to the left of the graph of $\sqrt{x}$ by 2 units.

Horizontal scaling exercise (page 61)

Q54: The graph of y = sin x is scaled horizontally by a factor of 3 to give the graph of y = sin $^1/_3$ x

It stretches horizontally.

Q55: The graph of y = tan x is scaled horizontally by a factor of $^1/_3$

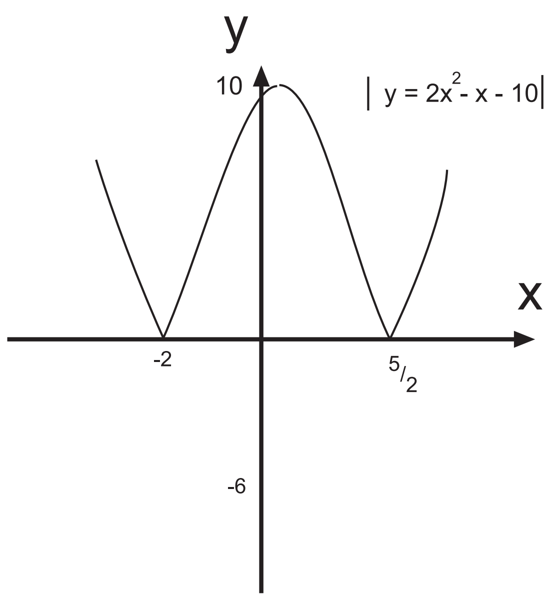


Q56: The graph of y = cos x° is scaled horizontally by a factor of $^4/_3$ to give the graph of y = cos $^3/_4$x°

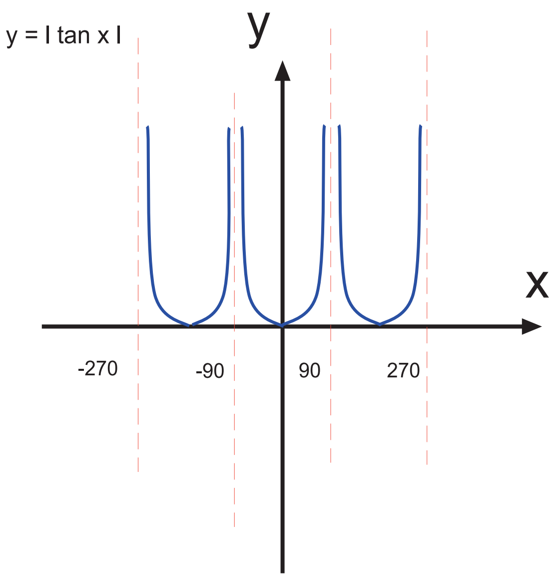
Q57: The graph of y = $\sqrt{x}$ has been scaled horizontally by a factor of $^1/_3$ to give the graph of y = $\sqrt{3x}$

Answers from page 62.

Q58:



Q59:



Multiple relationships exercise (page 63)

Q60:

a) The graph of f (x) is reflected in the x-axis, and stretched vertically by a factor of 4
b) The graph of f (x) is moved down the y-axis by 2 units and stretched vertically by a factor of 6
c) The graph of f (x) is moved 30° to the right and stretched vertically by a factor by 3
d) The graph of f (x) is moved to the right by 2 units and squeezed to half the size horizontally. This is because f (2x - 4) = f (2(x - 2))

Completing the square exercise (page 66)

Q61:

a) $(x - 2)^2 - 3$
b) $(x + 4)^2 - 18$
c) $(x - 3)^2 -15$
d) $(x + 1)^2 + 7$

Q62:

a) $(x - 2)^2 - 4$
b) $(x - 1)^2$
c) $(x - 2)^2 + 3$

Q63:

a) $2 (x - 1)^2 - 2$
b) $-2 (x + 2)^2 + 14$

c) $-3(x + 2)^2 - 3$

d) $-2(x - {}^3/_4)^2 + {}^{33}/_8$

e) $3(x + 1)^2 - 4$

Maximum/minimum values exercise (page 67)

Q64:

a) $x^2 - 4x + 9 = (x - 2)^2 + 5$
This has a turning point at (2, 5) and this is a minimum.

b) This has a minimum when $x = 60^\circ$
This gives $\cos x = {}^1/_2$ and $(2 \cos x - 1) = 0$
The function has a minimum value of 1
There is also a maximum value when $x = \pm 180^\circ$ which gives $(2 \cos x - 1) = -3$ and the function has a maximum value of 10

c) In completed square form this is $-2(x + 3)^2 - 3$ and the turning point is (-3, -3) which is a maximum.

d) $-4x^2 - 16x - 8 = -4(x + 2)^2 + 8$
This has a turning point at (-2, 8) which is a maximum.

Radian and degrees exercise (page 71)

Q65:

a) $\frac{\pi}{2}$

b) $\frac{3\pi}{4}$

c) $\frac{3\pi}{2}$

d) $\frac{7\pi}{12}$

e) $\frac{7\pi}{4}$

f) $\frac{\pi}{6}$

Q66:

a) 135°

b) 330°

c) 300°

d) 165°

e) 252°

f) 260°

Exact values exercise (page 77)

Q67:

a) -1
b) 0
c) $-^1/_2$
d) $-\frac{1}{\sqrt{3}}$
e) $-\sqrt{3}$
f) $-\frac{\sqrt{3}}{2}$
g) $-^1/_2$
h) 1
i) 0

Q68:

a) $\frac{1}{\sqrt{2}}$
b) $-\frac{1}{\sqrt{2}}$
c) $\frac{1}{\sqrt{2}}$
d) $\sqrt{3}$
e) $-\frac{1}{\sqrt{2}}$
f) $^1/_2$
g) 0
h) $-\frac{1}{\sqrt{2}}$
i) $-\frac{1}{\sqrt{3}}$

Amplitude and period exercise (page 80)

Q69:

a) The amplitude is 2 as the graph stretches vertically being of the form k f (x)
The period is the normal period for sin x which is 360°
b) The amplitude is $^1/_4$ and the period is 360°
c) The amplitude is 3 and the period 180° since the graph is squashed by a factor of 2
d) The amplitude is 1 and the period is 90°
e) The amplitude is 1 and the period is 720°

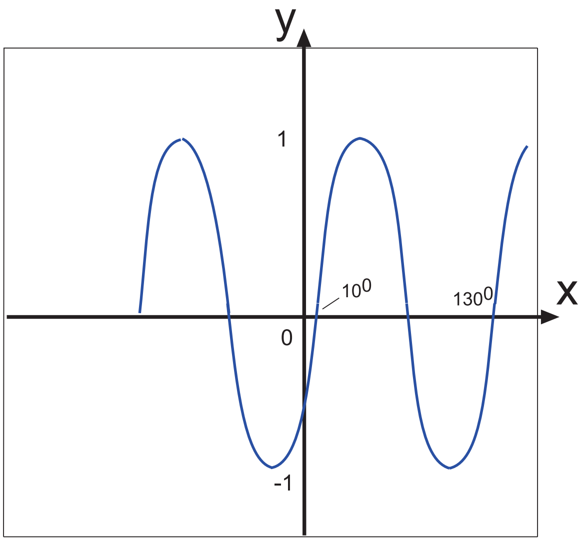
f) The amplitude is 3 and the period is 90°

g) The amplitude is 2 and the period is 120°

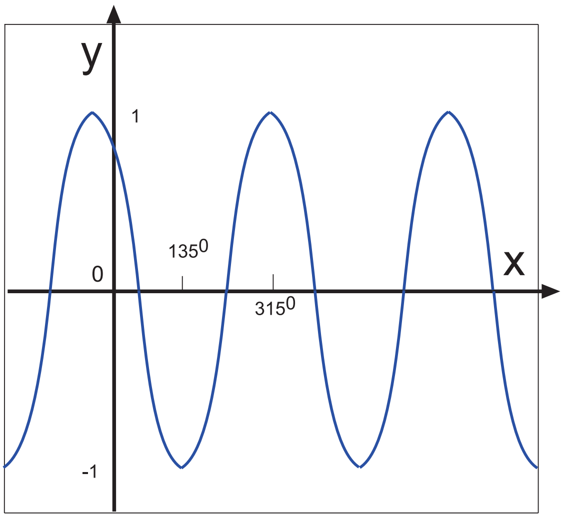
Trig. functions exercise (page 83)

Q70:

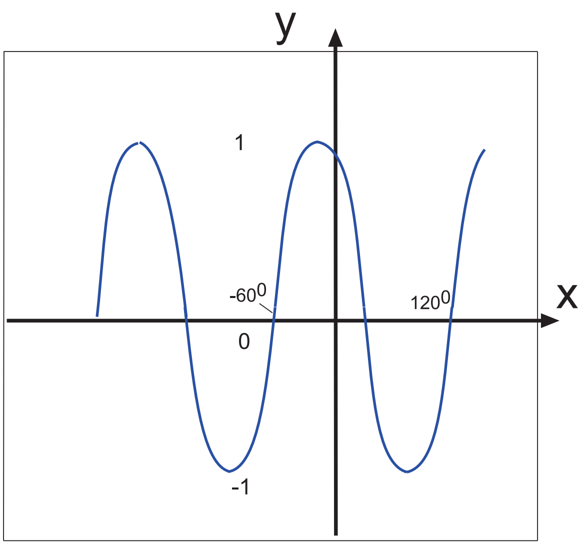
a)



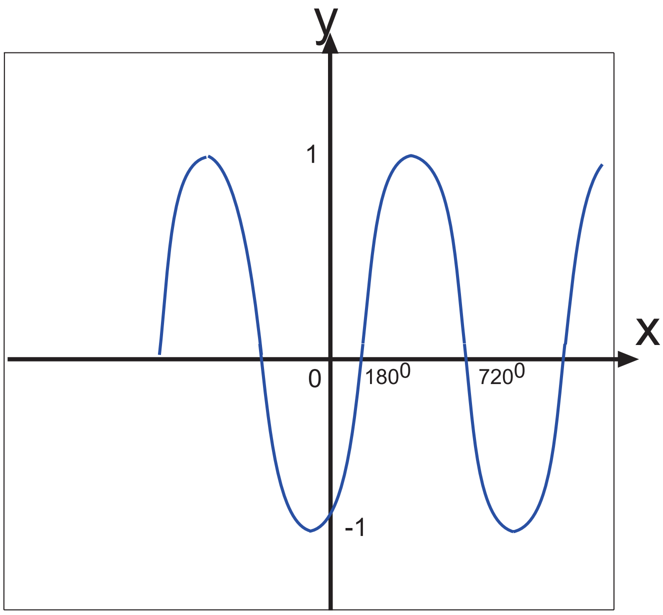
b)



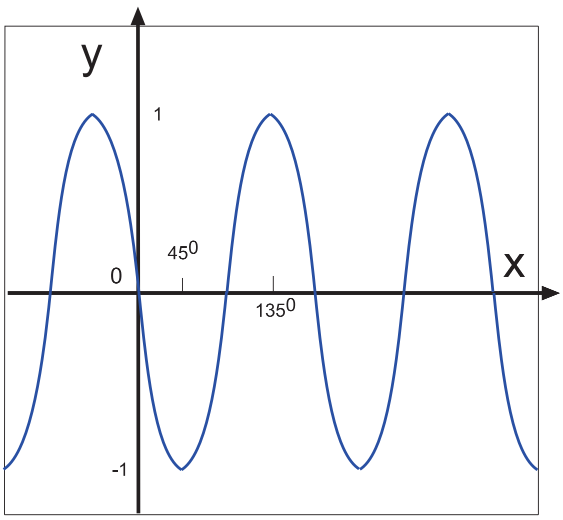
c)



d)



e)

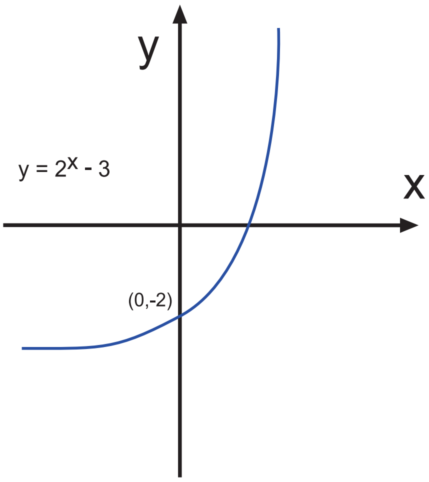


Exponential exercise (page 86)

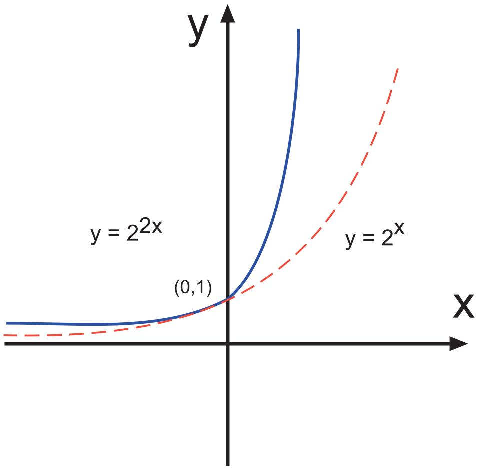
Q71: There are two ways to obtain the answer.

Either use coordinates and the shape of an exponential graph. When x = 0 y = -2 and plot.

Or use the rules for related graphs, then this graph is 3 units down the y-axis from the graph of $y = 2^x$

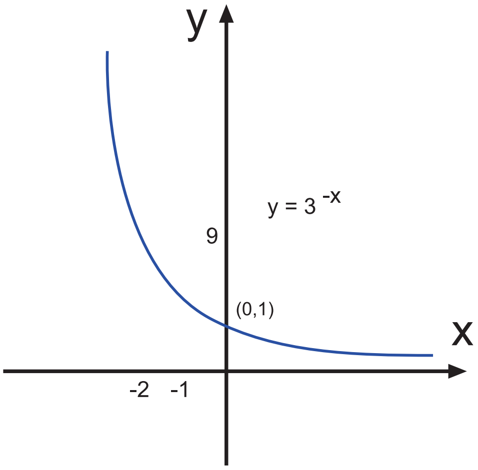


Q72: The graph squashes in along the x-axis to half the distance (or the scale is halved)



Q73: The graph is a reflection in the y-axis of the function f (x) = 3^x

(remember f (-x) being a reflection of f (x) in the y-axis)



Q74:

Graph A:

Using the coordinates of the point (2, 16) in the general equation gives 16 = a^2

So a = 4 and the graph has equation y = 4^x

Graph B: Again using substitution gives 27 = a^3

So a = 3 and the graph has equation y = 3^x

Graph C: Note the shape. This involves the identity $a^{-x} = (^1/_a)^x$

However using algebra and substituting gives

$9 = \frac{1}{a^2} \Rightarrow a^2 = \frac{1}{9} \Longrightarrow a = \frac{1}{3}$

Only the positive values of a are being considered by definition.

In the form y = a^x the graph has equation y = $(^1/_3)^x$

Graph D: Note the points.

Substituting the coordinates (0, 2) into the equation gives 2 = a^0 + b

$\Rightarrow$ b = 2 - 1 = 1 (This can be seen from the graph in this case, but it is safer to check.)

The equation has the form y = a^x + 1

Now substitute the coordinates of the second point and b = 1 to give

17 = $a^4 + 1 \Rightarrow a^4 = 16 \Rightarrow$ a = 2

The equation of the graph is y = 2^x + 1

Q75:

Graph A:

Using the coordinates of the point (0, 4) in the general equation gives

4 = $ka^0 \Rightarrow$ k = 4

Using the coordinates of the point (2, 16) and k = 4 in the general equation gives

16 = $4a^2$

So $a^2 = 4 \Rightarrow$ a = 2 and the graph has equation y = 4×2^x

Graph B: Using substitution with the point (0, 3) gives

3 = $a^0 + k \Rightarrow 3 = 1 + k \Rightarrow$ k = 2

Note that k could be found by visual methods and relating it to y = a^x

Substituting again with the point (3, 29) and k = 2 gives

29 = $a^3 + 2 \Rightarrow 27 = a^3 \Rightarrow$ a = 3

The graph has equation y = 3^x + 2

Graph C: Note the shape.

Substituting with the point (0, 3) gives 3 = $a^{-0} + k \Rightarrow$ k = 2

(Again k could be determined by relating it to y = a^x)

Now using the point (-2, 6) and k = 2 the equation becomes

6 = $a^{-(-2)} + k \Rightarrow a^2 = 4 \Rightarrow$ a = 2

The equation is y = 2^{-x} + 2

Graph D: Substituting the coordinates (0, -2) into the equation gives

$-2 = a^0 + k \Rightarrow k = -2 - 1 = -3$ (This can be seen from the graph in this case, but it is safer to check.)

Substituting the coordinates (3, 5) and k = -3 into the equation gives

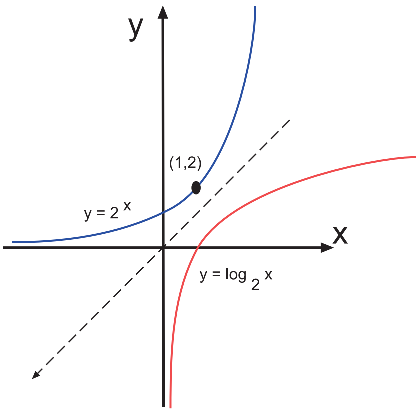
$5 = a^3 - 3 \Rightarrow a^3 = 8 \Rightarrow a = 2$

The equation has the form $y = 2^x - 3$

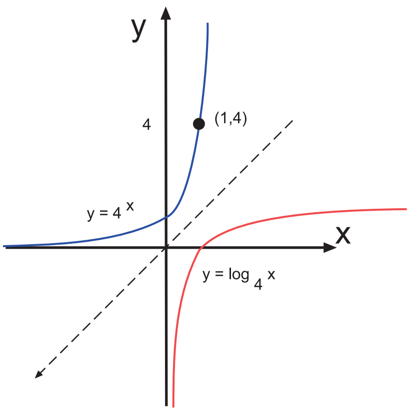
Logarithmic function exercise (page 89)

Q76: The graphs (shown with the exponential) are as follows:

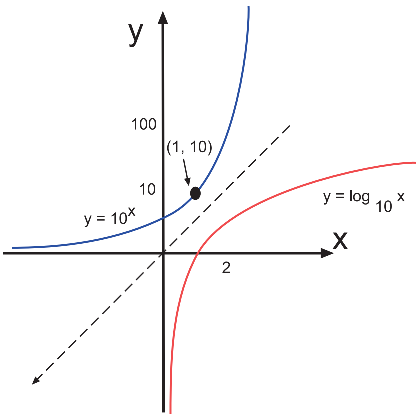
a)



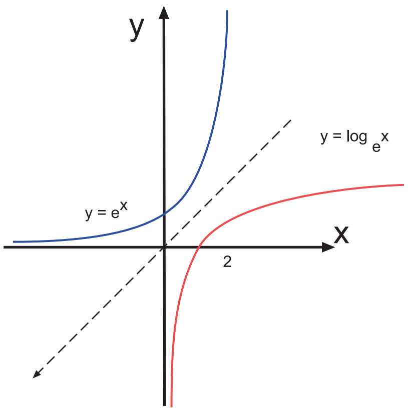
b)



c)



d)



Q77:

Graph A: For the related exponential the formula will take the form $y = 2^x + k$

Reflection in the line y = x shows that $y = 2^x + k$ will pass through the point (0, 3) and this gives $3 = 2^0 + k \Rightarrow k = 2$

The equation of the exponential graph is $y = 2^x + 2$ and the logarithmic function is $f(x) = \log_2 x + 2$

Graph B: In a similar manner to graph A, k can be found and is equal to -3

The equation is $f(x) = \log_3 x - 3$

Graph C: The related exponential passes through the points (0, -2) and (2, 1)

The point (0, -2) when substituted into $y = a^x + k$ gives $k = -3$

Then using (2, 1) gives $1 = a^2 - 3 \Rightarrow a = 2$

Thus the equation of the log graph is $f(x) = \log_2 x - 3$

Graph D: The related exponential graph, $y = a^x + k$ passes through the points (0, -1) and (3, 6)

Using (0, -1) gives $k = -2$ and subsequently using (3, 6) gives $6 = a^3 - 2 \Rightarrow a = 2$

The equation of the logarithmic graph is $f(x) = \log_2 x - 2$

Identify graphs exercise (page 91)

Q78:

a) $-x^2 + 2$

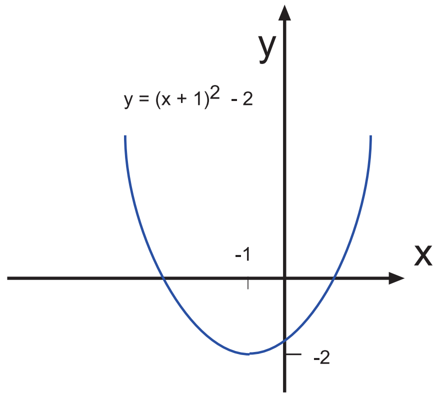
b) $(x + 1)^2$

c) $(x - 1)^2 - 4$

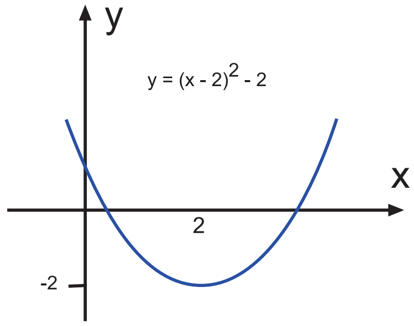
d) $-x^2 + 4$

Q79:

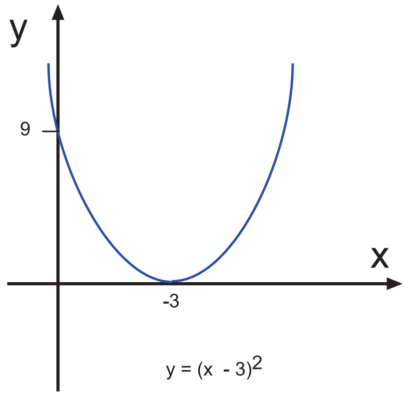
a)



b)



c)



Review exercise (page 93)

Q80:

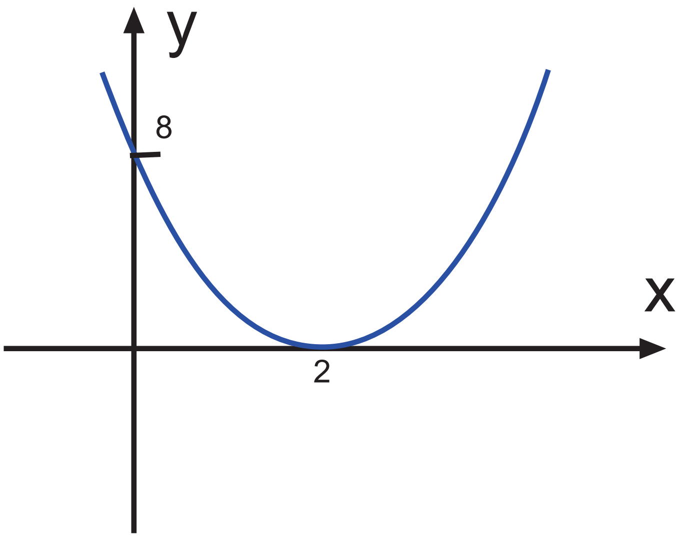
a) $f(x) = 2^x$
   The shape suggest the general formula of a^x
   Use the point (1, 2) to find a

b) f (x) = cos (x - 30) (or this is also sin (x + 60))
   Remember that the amplitude gives the value of a in the equation a = 1
   The period can be found by noticing that $^3{}_4$of one cycle takes place between 45° and 315° and so the period is 360° which indicates that b = 1
   The graph has a maximum at 30° to the right of the origin giving the formula shown.

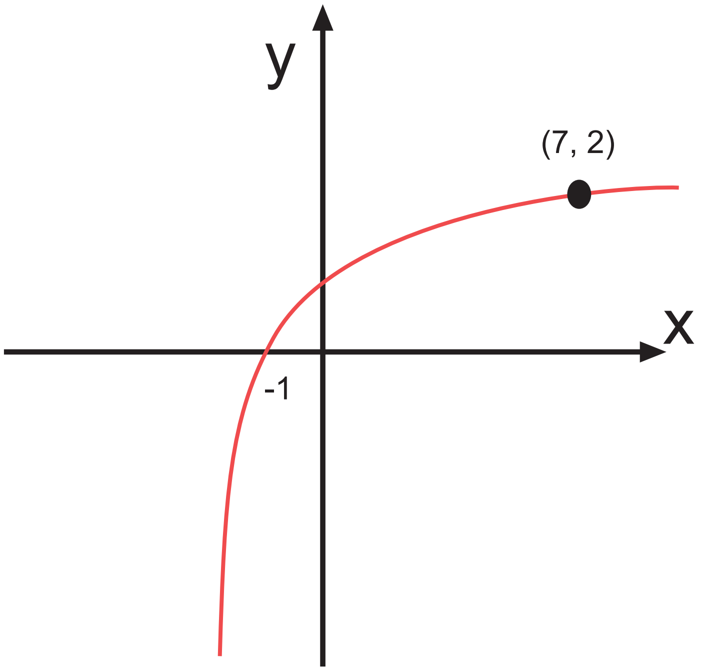
c) The shape indicates a quadratic of the form x^2
   The values shown that it has moved to the right by 5 units giving the form $(x - 5)^2$
   The value on the y-axis of 25 confirms that there has been no scaling.

Q81:

a)



b)



Remember to use the related graph of a^x - k

Q82:

a) $(3 \sin x - 1)^2$

b) $3 \times 3\sin x + 4$

c) $3 \sin (3x + 4)$

Advanced review exercise (page 95)

Q83:

a) When x = -1 then (x + 1) = 0 and there is a minimum value of 0
There is no maximum value.

b) When cos x = 1 then (cos x - 1) = 0 to give a minimum value of -2
But when cos x = -1 then (cos x - 1) = -2 and there is a maximum value at 10

c) If x = -3 then (x + 3) = 0 and the maximum value is -4. There is no minimum value since $(x + 3)^2$ increases as x increases making the value of the expression smaller and smaller.

Q84:

a) $3\left(x - \frac{1}{6}\right)^2 - \frac{59}{60}$

b) $2\left(x - \frac{3}{4}\right)^2 - \frac{1}{16}$

Q85:

a) $f(x) = 2^x - 3$
The shape suggest the general formula of $a^x + k$
Use the point (0, -2) to find k = -3 and then substitute the coordinates of the point (2, 1) into the equation $y = a^x - 3$ to give a = 2

b) f (x) = 2 cos (x - 45) (or this is also 2 sin (x + 45))
Remember that the amplitude gives the value of a in the equation a cos (bx + c) so a = 2
The period can be found by noticing that $^3{}_4$of one cycle takes place between 45° and 315° and so the period is 360° which indicates that b = 1
The graph moves by $^c/_b$ but in this case this is equal to the value of c. The graph shows that the movement is 45° to the right which indicates that c = -45

c) The shape indicates a quadratic of the form x^2
The values shown that it has moved to the right by 4 units giving the form $(x - 4)^2$ but it has also moved up by 3 units changing this to $(x - 4)^2 + 3$
Note that there is no indication of any scaling with the information given. In such cases, accept that there is none.

Set review exercise (page 96)

Q86: The answer is only available on the web.

Q87: The answer is only available on the web.

Q88: The answer is only available on the web.

Q89: The answers are only available on the web.

3 Basic Differentiation

Revision - Exercise 1 (page 101)

Q1:

a) $4x^7$
b) y^3
c) $\frac{1}{3}a^3$
d) b^6
e) $8y^3$
f) $2x^2$
g) m
h) $\frac{1}{x^6}$
i) 1
j) $3x^{1/2} = 3\sqrt{x}$

Q2:

a) x^{-2}
b) $5x^{-1}$
c) $\frac{1}{3}x^{-4}$
d) $\frac{1}{2}x^{-1}$
e) $5x^{1/2}$
f) $x^{1/3}$
g) $x^{4/3}$
h) $x^{-1/2}$
i) $\frac{1}{3}x^{-1/2}$
j) $2x^{-1/3}$

Q3:

a) $x + 2x^{3/2}$
b) $1 + x^{-1}$
c) $x^{2/3} - 1$
d) $\frac{1}{2}x - 3 + \frac{9}{2}x^{-1}$
e) $x^2 + 2 + x^{-2}$
f) $3x^2 - x^3$
g) $3x^2 + \frac{1}{2}$
h) $3x^{3/2} + 5x^{1/2}$
i) $4x^{-1} - x$
j) $x^{7/2} - 3x^{1/2}$

Answers from page 103.

Q4: The car is travelling faster at t = 4

Q5: At t = 4 the graph is steeper, therefore the car is travelling faster at this point.

Q6: Since the distance-time graph given is a curve then the speed is not constant. It is changing continuously and so we can only give an average for the first three seconds.

Exercise 2 (page 107)

Q7:

$$\begin{aligned} f(x) &= x^3 \\ f(x+h) &= (x+h)^3 \\ f'(x) &= \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} \\ &= \lim_{h\to 0} \frac{h(3x^2 + 3xh + h^2)}{h} \\ &= 3x^2 \end{aligned}$$

Q8:

$$\begin{aligned} f(x) &= 5x^2 \\ f(x+h) &= 5(x+h)^2 \\ f'(x) &= \lim_{h\to o} \frac{5x^2 + 10xh + 5h^2 - 5x^2}{h} \\ &= \lim_{h\to o} \frac{h(10x + 5h)}{h} \\ &= 10x \end{aligned}$$

Exercise 3 (page 109)

Q9:

a) $f'(x) = 8x^7$

b) $f'(x) = 3x^2$

c) $f'(t) = 7t^6$

d) $f'(a) = 20a^{19}$

Q10:

a) $f'(x) = -2x^{-3}$

b) $g'(x) = -x^{-2}$

c) $h'(t) = -19t^{-20}$

d) $f'(s) = -50s^{-51}$

Q11:

a) $f'(x) = \frac{5}{4}x^{1/4}$

b) $f'(t) = \frac{3}{4}t^{-1/4}$

c) $f'(x) = -\frac{2}{3}x^{-5/3}$

d) $g'(x) = -\frac{3}{7}x^{-10/7}$

e) $f'(x) = \frac{2}{7}x^{-5/7}$

f) $f'(t) = -\frac{8}{5}t^{-13/5}$

Q12:

a) $f'(x) = -\frac{2}{x^3}$

b) $f'(x) = -\frac{2}{3x^{5/3}}$

c) $f'(x) = \frac{1}{3x^{2/3}}$

d) $f'(x) = -\frac{7}{4x^{11/4}}$

e) $f'(x) = \frac{5}{2}x^{3/2}$

f) $f'(x) = -\frac{1}{4x^{5/4}}$

g) $f'(x) = \frac{2}{3x^{1/3}}$

h) $f'(x) = -\frac{2}{5x^{7/5}}$

Exercise 4 (page 112)

Q13: $\frac{1}{3}$

Q14: 500

Q15: $\frac{1}{6}$

Q16: $\frac{16}{3}$

Q17: -16

Q18: $-\frac{1}{16}$ cm/s

Q19: $-\frac{1}{16}$ m^3/s

Q20: $\frac{5}{2}$ bacteria per minute.

Exercise 5 (page 114)

Q21: $f'(x) = 14x^6$

Q22: $f'(x) = 4x^5$

Q23: $f'(y) = -\frac{10}{y^3}$

Q24: $f'(x) = -\frac{3}{4x^4}$

Q25: $g'(x) = -\frac{3}{5x^2}$

Q26: $f'(x) = \frac{5}{2\sqrt{x}}$

Q27: $f'(x) = -\frac{2}{x^{3/2}}$

Q28: $f'(x) = \frac{6}{\sqrt[3]{x}}$

Q29: $f'(t) = 10t - 2$

Q30: $f'(x) = 14x - \frac{1}{x^4}$

Q31: $f'(x) = \frac{1}{4\sqrt{x}} + \frac{1}{x^{3/2}}$

Q32: $g'(x) = \frac{4}{3x^{3/2}} + 7$

Q33: 11

Q34: $23\frac{3}{4}$

Q35: $-\frac{1}{9}$

Exercise 6 (page 116)

Q36: $f'(x) = 6x + 14$

Q37: $f'(x) = \frac{3}{2}\sqrt{x} + \frac{1}{\sqrt{x}} - 3$

Q38: $f'(x) = 2x - \frac{2}{x^3}$

Q39: $f'(x) = 9x^2 - 1 - \frac{7}{x^2}$

Q40: $f'(x) = 1 - \frac{1}{x^2} + \frac{2}{x^3}$

Q41: $f'(x) = 14x^{5/2} - \frac{3}{2x^{3/2}}$

Q42: $f'(x) = \frac{1}{2\sqrt{x}} + \frac{3}{x^{3/2}}$

Q43: $f'(x) = -\frac{2}{x^2} + \frac{10}{x^3} + \frac{9}{x^4}$

Q44: $f'(x) = 4 - \frac{1}{x^2}$

Q45: $f'(x) = \frac{5}{2}x^{3/2} + 9\sqrt{x}$

Exercise 7 (page 118)

Q46: 18x + 12

Q47: 4

Q48: 1

Q49:

a) $\frac{1}{x^2}$
b) $12t^{1/3} + 1$
c) $\frac{3}{2}\sqrt{s}$

Q50: (2, 3)

Q51: (-1, 3) and (1, 7)

Exercise 8 (page 120)

Q52:

a) y = 8x - 8
b) y = 6x - 1
c) x - 6y + 9 = 0
d) y = 6x + 3
e) y = 4x + 6
f) 5x - 4y - 4 = 0

Q53:

a) (1, 14)
b) x + y - 13 = 0

Q54:

a) At $x = 0$ the tangent has equation $y = 3x$
   At $x = 2$ the tangent has equation $x + y = 4$

b) (1,3)

Q55:

a) The equation of the tangent at A is $y = 4x - 4$
   The equation of the tangent at B is $y = 3x - 3$

b) (1,0)

Q56:

Q57: $y = \sqrt{3}x - \frac{1}{2}$

Exercise 9 (page 123)

Q58:

a) Increasing
b) Decreasing
c) Decreasing
d) Increasing

Q59:

a) Increasing for $x > 0$ and decreasing for $x < 0$
b) Increasing for $x > -\frac{3}{2}$ and decreasing for $x < -\frac{3}{2}$
c) Increasing for $x < -2$ and $x > 0$ and decreasing for $-2 < x < 0$
d) Increasing for $0 < x < 4$ and decreasing for $x < 0$ and $x > 4$
e) Increasing for $x < -1$ and $x > 3$ and decreasing for $-1 < x < 3$
f) Increasing for $x < -6$ and $x > -2$ and decreasing for $-6 < x < -2$

Q60:

a) $\frac{dy}{dx} = 6x^2$
   Since $x^2 \geqslant 0 \Rightarrow 6x^2 \geqslant 0$
   Hence the function is never decreasing.

b)
$$\begin{aligned}\frac{dy}{dx} &= 3x^2 - 18x + 27\\ &= 3\left(x^2 - 6x + 9\right)\\ &= 3(x-3)^2\end{aligned}$$
   $3(x-3)^2 \geqslant 0$ for all values of x.
   Hence the function is never decreasing.

Q61:

a)

$\frac{dy}{dx} = -\frac{1}{x^2}, \quad (x \neq 0)$

$-\frac{1}{x^2} < 0$ for all values of x, $(x \neq 0)$.

Hence the function is never increasing.

b)

$$\begin{aligned}\frac{dy}{dx} &= 12x - 6 - 6x^2\\ &= -6\left(x^2 - 2x + 1\right)\\ &= -6(x-1)^2\end{aligned}$$

$-6(x-1)^2 \leqslant 0$ for all values of x.

Hence the function is never increasing.

Exercise 10 (page 126)

Q62: (3,-5), minimum

Q63: (2,11), maximum

Q64: (0,2), rising point of inflection

Q65:

(-1,-4), minimum

(1,4), maximum

Q66:

(-1,7), maximum

(1,3), minimum

Q67:

(-3,27), maximum

(1,-5), minimum

Q68:

(1,0), maximum

(5,-32), minimum

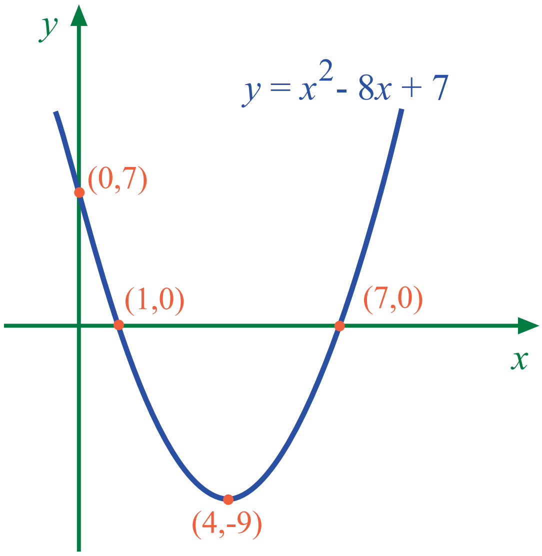
Q69:

(0,0), rising point of inflection

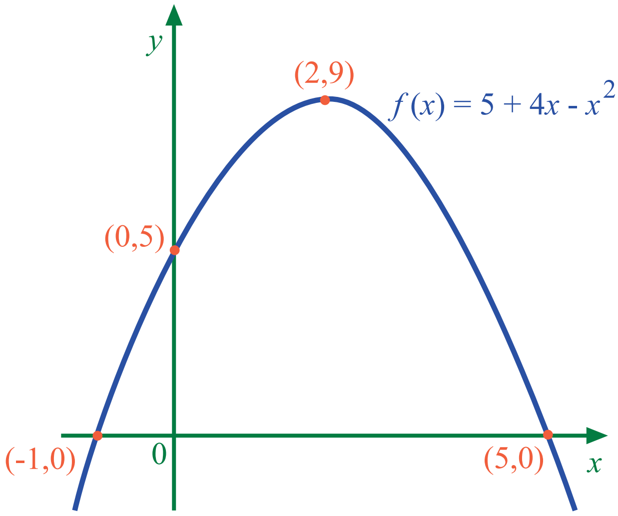
(3,27), maximum

Exercise 11 (page 128)

Q70:



Q71:



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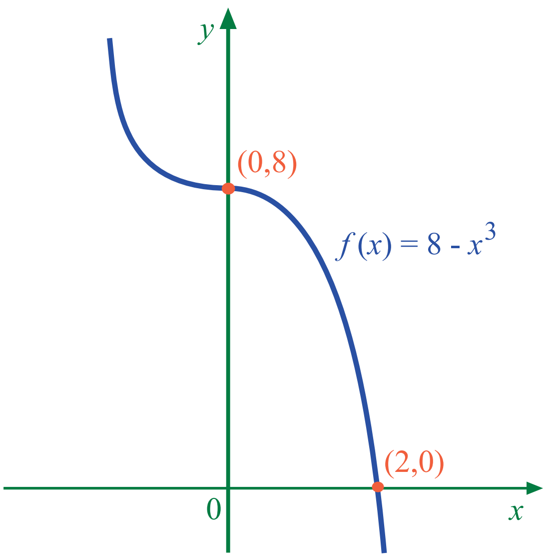
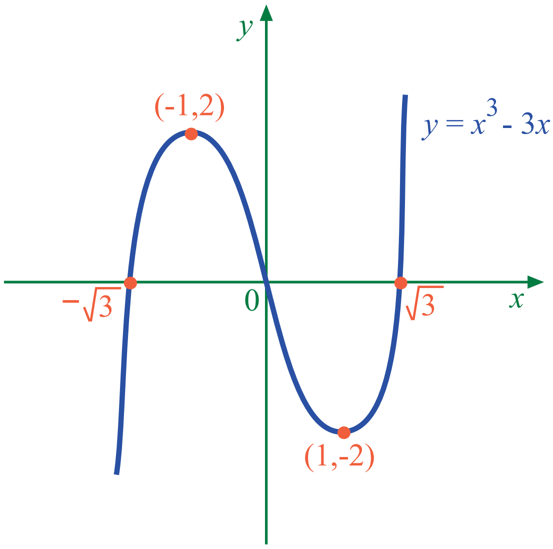
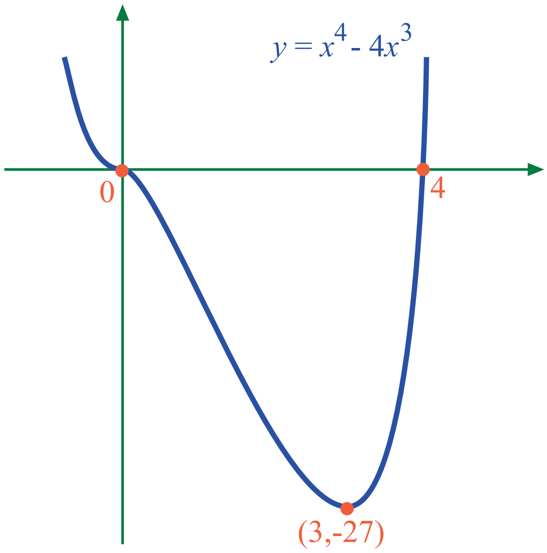


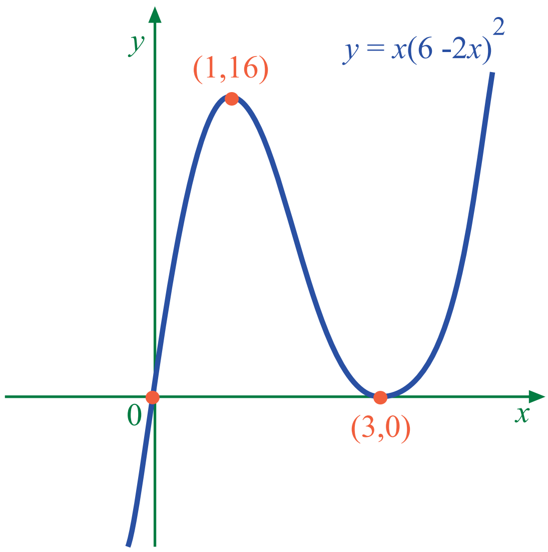
Q73:



Q74:



Q75:



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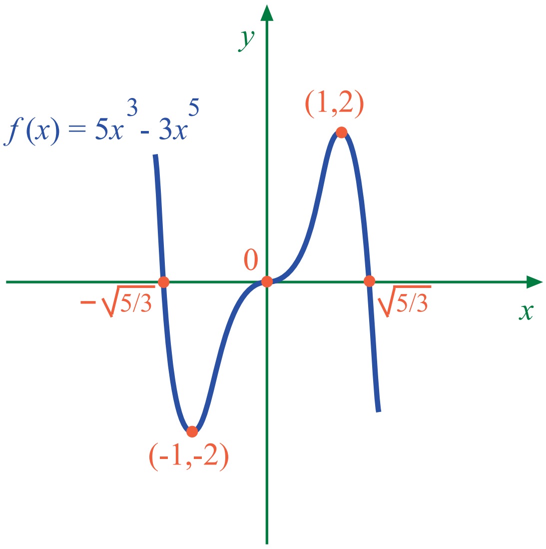
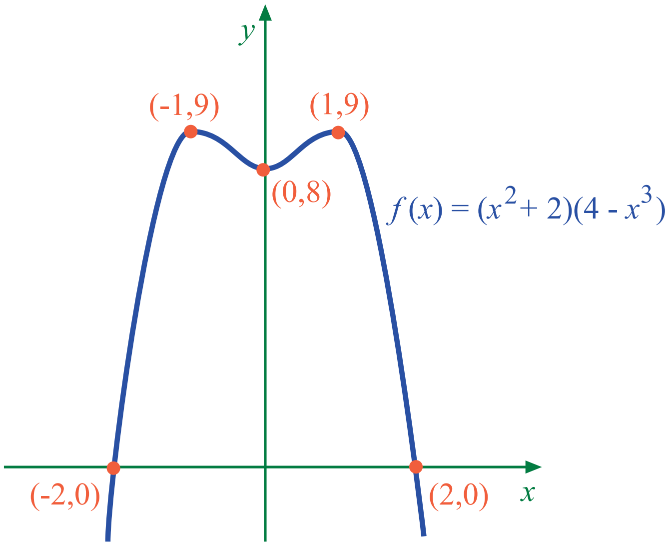


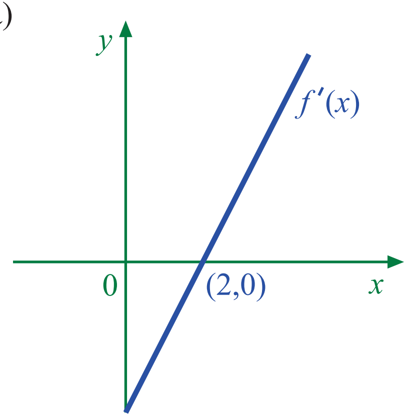
Q77:



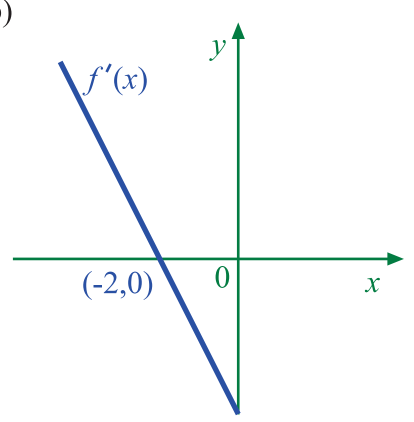
Exercise 12 (page 130)

Q78:

a)



b)



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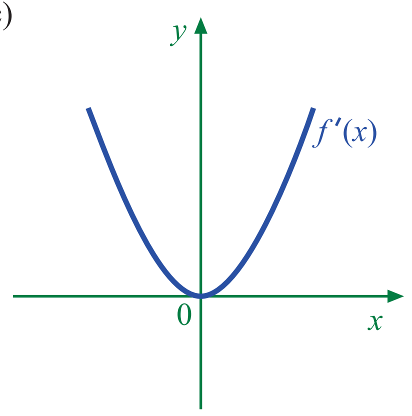

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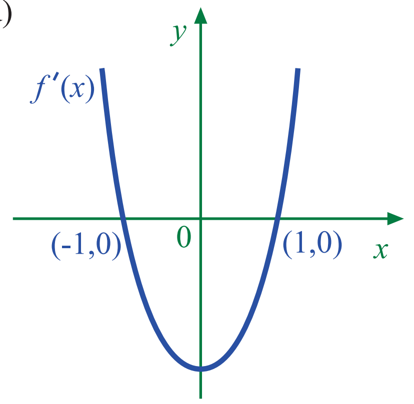

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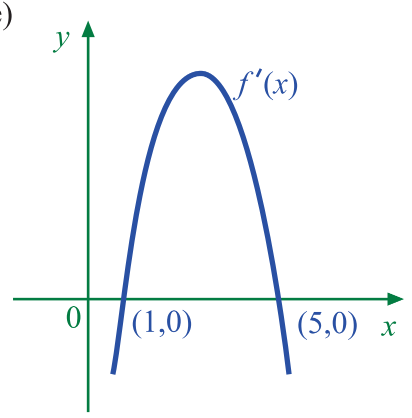

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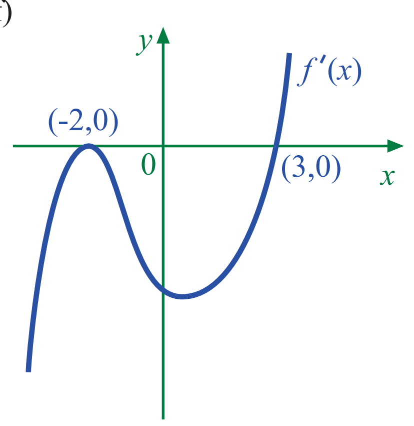
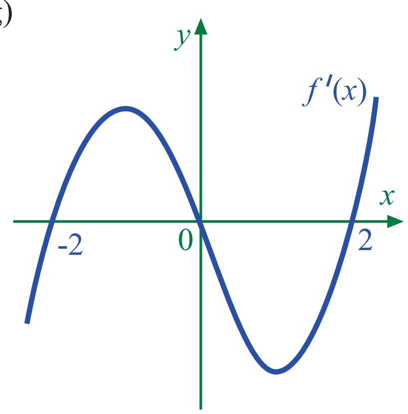
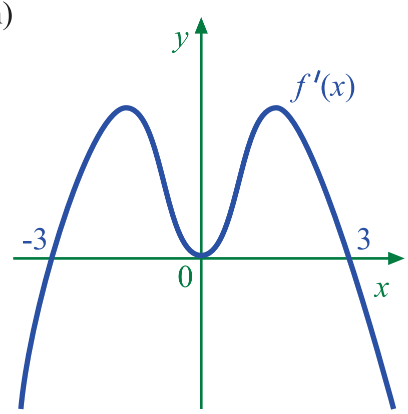


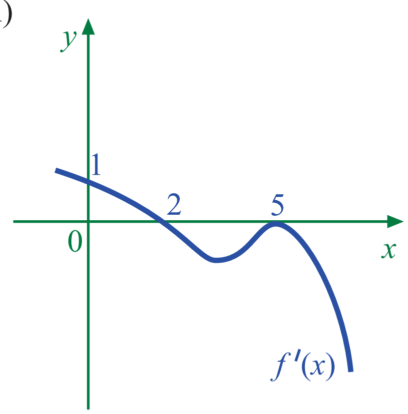
g)



h)



i)



Exercise 13 (page 134)

Q79:

a) Maximum of 6 at end point (7,6)
Minimum of -2 at stationary point (4,-2)

b) Maximum of 3 at stationary point (7,3)
Minimum of -4 at end point (0,-4)

Q80:

a) Maximum value of 4 at the end point (-1,4)
Minimum value of -5 at the stationary point (2,-5)

b) Maximum value of 20 at the end point (5,20)
Minimum value of 0 at the end point (0,0) and at the stationary point (3,0)

c) Maximum value of 20 at the end point (2,20)
Minimum value of -16 at the end point (-4,-16)

d) Maximum value of -1 at the stationary points (-1,-1) and (1,-1)
Minimum value of -10 at the end points (-2,-10) and (2,-10)

Exercise 14 (page 137)

Q81: 30 m/s

Q82: 440 bacteria/second

Q83: -3

Q84: $\frac{3}{4}$ cm/second

Q85:

a) $v = 120 - 3t^2$ and $a = -6t$
b) $v = 72$ cm/s and $a = -24$ cm/s^2

Q86: 2000 cm^3/s

Exercise 15 (page 140)

Q87:

a) $P = 14x - x^2$
b) $x = y = 7$
c) $P = 49$

Q88:

a) At t =25
b) 72 km/h

Q89:

a) $A = 24x - x^2$
b) Breadth = Length = 12 cm
c) $A = 144$ cm^2

Q90:

a) Let y represent the length of the rose bed then
$xy = 36 \Rightarrow y = \frac{36}{x}$
Hence the perimeter is given by
$P = 2x + 2y$
$= 2x + \frac{72}{x}$
b) Breadth = length = 6 metres.
c) Minimum perimeter = 24 metres.

Q91:

a) y = 24 -2x

b) The volume of the box is

$$\begin{aligned} V &= xy^2 \\ &= x\,(24 - 2x)^2 \\ &= x\left(576 - 96x + 4x^2\right) \\ &= 576x - 96x^2 + 4x^3 \end{aligned}$$

c) x = 4 cm

d) Maximum volume = 1024 cm^3

Q92:

a) area = $24x - \frac{1}{2}x^3$

b) x = 4

c) 64 m^2

Q93:

a) $h = \dfrac{4000}{x^2}$

b) The surface area is given by

$$\begin{aligned} A &= x^2 + 4xh \\ &= x^2 + 4x \times \frac{4000}{x^2} \\ &= x^2 + \frac{16000}{x} \end{aligned}$$

c) x = 20 cm and h = 10 cm

Q94:

a) $h = \dfrac{400}{\pi x^2}$

b) The surface area of the can is given by

$$\begin{aligned} S &= 2\pi rh + 2\pi r^2 \\ &= 2\pi x \times \frac{400}{\pi x^2} + 2\pi x^2 \\ &= \frac{800}{x} + 2\pi x^2 \end{aligned}$$

c) $x = \sqrt[3]{\dfrac{200}{\pi}} \approx 4$ cm and $h \approx 8$ cm

Review exercise on basic differentiation (page 148)

Q95:

a) f'(x) = 4x + 5

b) $f'(x) = \dfrac{3}{2\sqrt{x}} + \dfrac{5}{2x^{3/2}}$

Q96: f'(2) = -2

Q97:

a) (1,0) and (3,-4)

b)

There is a maximum turning point at (1,0)

x	1^-	1	1^+
$f'(x)$	+	0	−
slope	↗	→	↘

There is a minimum turning point at (3,-4)

x	3^-	3	3^+
$f'(x)$	−	0	+
slope	↘	→	↗

Advanced review exercise in basic differentiation (page 149)

Q98: $f'(x) = \frac{7x^{5/2}}{4} - \frac{3}{4\sqrt{x}} - \frac{1}{4x^{3/2}}$

Q99: f'(2) = 13

Q100: y = -5x + 1

Q101:

a) (-3,0),(0,-27),(3,0)

b)

There is a maximum turning point at (-3,0)

x	-3^-	-3	-3^+
$f'(x)$	+	0	−
slope	↗	→	↘

There is a minimum turning point at (1,-32)

x	1^-	1	1^+
$f'(x)$	−	0	+
slope	↘	→	↗

Q102:

Maximum value is 5 at the end point (0,5)

Minimum value is -22 at the minimum turning point (3,-22)

Q103:

a) 3x + 2y = 60

b) Since 3x + 2y = 60 then $y = 30 - \frac{3}{2}x$

$$\begin{aligned} A &= xy \\ &= x\left(30 - \frac{3}{2}x\right) \\ &= 30x - \frac{3}{2}x^2 \end{aligned}$$

c) Length = 15 metres and breadth = 10 metres.

Set review exercise in basic differentiation (page 150)

Q104: This answer is only available on the course web site.

Q105: This answer is only available on the course web site.

Q106: This answer is only available on the course web site.

4 Recurrence relations

Revision exercise (page 154)

Q1: u $= 10 \times (-5) + 30 = -50 + 30 = -20$

Q2: L $= \frac{20}{1-0.8} = \frac{20}{0.2} = \frac{200}{2} = 100$

Q3: $u_{n=1} = 2 \times 6 + 100 = 112$

Q4: x = 3, y = -4

Q5: x = -0.5, y = 7

Exercise 1 (page 156)

Q6:

a) $u_1 = 3$, $u_2 = 6$, $u_3 = 9$, $u_4 = 12$, $u_5 = 15$, ..., $u_{10} = 30$
b) $u_1 = 3$, $u_2 = 7$, $u_3 = 11$, $u_4 = 15$, $u_5 = 19$, ..., $u_{10} = 39$
c) $u_1 = -1$, $u_2 = 5$, $u_3 = 15$, $u_4 = 29$, $u_5 = 47$, ..., $u_{10} = 197$
d) $u_1 = 2$, $u_2 = 4$, $u_3 = 8$, $u_4 = 16$, $u_5 = 32$, ..., $u_{10} = 1024$
e) $u_1 = {}^1/_2$, $u_2 = {}^1/_4$, $u_3 = {}^1/_6$, $u_4 = {}^1/_8$, $u_5 = {}^1/_{10}$, ..., $u_{10} = {}^1/_{20}$
f) $u_1 = 1$, $u_2 = 3$, $u_3 = 6$, $u_4 = 10$, $u_5 = 15$, ..., $u_{10} = 55$
g) $u_1 = 0$, $u_2 = {}^1/_2$, $u_3 = {}^2/_3$, $u_4 = {}^3/_4$, $u_5 = {}^4/_5$, ..., $u_{10} = {}^9/_{10}$
h) $u_1 = 1\frac{1}{2}$, $u_2 = 1\frac{1}{4}$, $u_3 = 1\frac{1}{8}$, $u_4 = 1\frac{1}{16}$, $u_5 = 1\frac{1}{32}, \ldots u_{10} = 1\frac{1}{1024}$,
i) e), g) and h) are convergent.

Q7:

a) $u_1 = {}^1/_2$, $u_2 = {}^2/_3$, $u_3 = {}^3/_4$, $u_4 = {}^4/_5$, $u_5 = {}^5/_6$
b) $u_{10} = 0.9$, $u_{100} = 0.99$, $u_{1000} = 0.999$
c) $u_n \Rightarrow 1$ as $n \Rightarrow \infty$

Q8:

1. 0
2. 0
3. 3
4. 5

Exercise 2 (page 158)

Q9:

a) 2, 7, 12, 17, 22, $u_{10} = 47$
b) 1, 3, 9, 27, 81, $u_{10} = 19683$

c) 5, -7, 17, -31, 65, u_{10} = -2047

d) 200, 164, 135.2, 112.16, 93.728, u_{10} = 44.2 (to 1 decimal place)

e) 5, -5, 5, -5, 5, u_{10} = -5

f) 10, 7, 6.4, 6.28, 6.256, u_{10} = 6.25 (to 2 decimal places)

Q10:

a) $u_{n+1} = u_n + 5$, $u_1 = 2$

b) $u_{n+1} = 3u_n$, $u_1 = 4$

c) $u_{n+1} = 2u_n + 1$, $u_1 = 3$

d) $u_{n+1} = -2u_n + 2$, $u_1 = 8$

Q11:

a) 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ...

b) 1, 2, 1.5, 1.667, 1.6, 1.625, 1.615, 1.619, 1.617, ... This sequence seems to tend to a limit. The limit has a value of approximately 1.61804 and is called the golden ratio. It has special significance in many areas including nature and architecture.

Exercise 3 (page 161)

Q12:

a) $C_{n+1} = 0.8C_n$, C_0 = £18000

b) £7372 (to the nearest £1)

c) $C_n = (0.8)^n C_0$

Q13:

a) $T_{n+1} = T_n + 12$, $T_0 = 15$

b) 75 trees.

Q14:

a) $P_{n+1} = P_n + 19$, $P_0 = 20$

b) £248

c) $P_n = P_0 + 19n$

Q15:

a) $M_{n+1} = 1.06M_n$, $M_0 = 1000$

b) Â£1338.23 (to the nearest penny)

c) $M_n = (1.06)^n M_0$

Q16:

a) $V_{n+1} = 0.94V_n$, $V_0 = 2100$ km^3

b) 1449 km^3 (to nearest whole number)

c) $V_n = (0.94)^n V_0$

d) After 12 years the iceberg is less than 1000 km^3

Q17:

a) $B_{n+1} = 1.13B_n$, $B_0 = 120$

b) 196 bacteria (to the nearest whole number)

c) 6 minutes.

Exercise 4 (page 164)

Q18:

a) Let M_0 represent the amount of money he has in his account on his 12th birthday. $M_{n+1} = 1.06M_n + 50$

b) £419.69

Q19:

a) Let F_0 represent the size of the shoal at the start. $F_{n+1} = 0.7F_n + 25$

b)

$$F_5 = 120$$
$$F_{10} = 89$$

Q20:

a) Let C_0 be the initial amount of chemical discharged into the river then $C_0 = 60$ and $C_{n+1} = 0.2C_n + 60$

b) After 7 days there is approximately 75 kg of chemical waste in the river.
After 15 days there is also approximately 75 kg of chemical waste in the river.

c) Yes it is safe because the level of waste chemical in the river never exceeds 75 kg

Q21:

a) Let R_0 represent the inital rabbit population then $R_0 = 120$ and $R_{n+1} = 1.15R_n - 30$

b) $R_6 = 15$

c) The rabbit popultion cannot keep up with sales. Eventually there will be no more rabbits left to sell.

Q22:

a) Let D_0 be the initial amount of the drug in the patient's bloodstream and let D_n represent the amount of the drug in the patient's blood stream after 8n hours then $D_0 = 90$ and $D_{n+1} = 0.7D_n + 20$

b) 69.4 units (1d.p.)

Q23:

a) Let M_0 represent the initial mortgage. $M_{n+1} = 1.06M_n - 6000$

b) $M_5 = £73235$
$M_{10} = £64183$

c) During the 28th year.

Exercise 5 (page 166)

Q24:

a) $u_2 = 20.8$

b) n = 4

Q25:

a) $u_2 = 298, \quad u_4 = 583.12$

b) 7 terms, u_0, u_1, u_2, u_3, u_4, u_5, and u_6

Q26:

a) $u_2 = 200, \quad u_5 = 862.025$

b) 4 terms.

Q27:

a) $u_2 = 20.02, \quad u_3 = 20.002, \quad u_4 = 20.0002$

b) As $n \rightarrow \infty$ $u_n \rightarrow 20$

Exercise 6 (page 168)

Q28:

a)

- i Diverges
- ii Diverges
- iii Converges to 30
- iv Converges to 15
- v Converges to 10
- vi Converges to 8
- vii Diverges
- viii Diverges

b) a = 0.6,0.2,-0.2 and -0.5 all give convergent sequences.

c) In general the sequence will be convergent for $-1 < a < 1$

Q29:

a)

- i Converges to 16
- ii Converges to 8
- iii Converges to 4
- iv Converges to 0.4
- v Converges to -1.2
- vi Converges to -8

b) All the recurrence relations converge to a limit.

c) Changing b does not effect whether the sequence will converge or not but it does change the value of the limit and the rate of convergence.

Q30:

a) The sequences all converge to 20

b) The value of u_0 does not effect the value of the limit.

Exercise 7 (page 172)

Q31:

i

a) $u_0 = 3$
$u_1 = 11.5$
$u_2 = 15.75$
$u_3 = 17.875$
$u_4 = 18.9375$
$u_5 = 19.46875$

b) The sequence tends to a limit because -1 $<$ 0.5 $<$ 1

c) L = 20

ii

a) $u_0 = 100$
$u_1 = 24$
$u_2 = 8.8$
$u_3 = 5.76$
$u_4 = 5.152$
$u_5 = 5.0304$

b) The sequence tends to a limit because -1 $<$ 0.2 $<$ 1

c) L = 5

iii

a) $u_0 = 200$
$u_1 = 180$
$u_2 = 160$
$u_3 = 140$
$u_4 = 120$
$u_5 = 100$

b) This sequence does not have a limit because a = 1 and $1 \notin -1 < x < 1$

iv

a) $u_1 = -4$

$u_2 = 6.8$
$u_3 = 4.64$
$u_4 = 5.072$
$u_5 = 4.9856$
$u_6 = 5.00288$

b) The sequence tends to a limit because -1 $<$ 0.2 $<$ 1

c) L = 5

v

a) $u_0 = 30$
$u_1 = 2.5$
$u_2 = 9.375$
$u_3 = 7.65625$
$u_4 = 8.0859375$
$u_5 = 7.978515625$

b) The sequence tends to a limit because -1 $<$ -0.25 $<$ 1

c) L = 8

vi

a) $u_0 = 0$
$u_1 = 25$
$u_2 = -12.5$
$u_3 = 43.75$
$u_4 = -40.625$
$u_5 = 85.9375$

b) This sequence does not have a limit because a = -1.5 and $-1.5 \notin -1 < x < 1$

Q32:

a) Since -1 $<$ 0.3 $<$ 1 then the sequence will converge.

b) $L = \frac{5}{0.7} = \frac{50}{7} = 7\frac{1}{7}$

Q33:

a) $W_{n+1} = 0.25W_n + 15, \quad W_0 = 80$

b) 20.2 kg.

c) Since -1 $<$ 0.25 $<$ 1

d) $L = \frac{15}{0.75} = 20$ kg.

Q34:

a) $F_{n+1} = 0.38F_n + 20$ with $F_0 = 70$

b) 33 files (to nearest whole number)

c) 32 files (to nearest whole number)

Q35: Pestkill is more effective in the long term. Using Pestkill there are 765 pests on the trees and using Killpest there are 769 pests on the trees in the long term.

Q36:

a) 23.2 units
b) Yes the doctor can prescribe this course of treatment. The level of antibiotic in the patient's body will always be less than 93.4 units.

Exercise 8 (page 175)

Q37:

a) a = 3, b = -2
b) a = 2.5, b = -10
c) a = 0.5, b = -5
d) a = -0.8, b = 20

Q38:

a) a = 0.75, b = 500
b) $u_0 = 10000$
c) L = 2000

Q39: b = 12

Q40: £500 is invested each year at an interest rate of 10%

Q41: x = 80% and y = 500

Q42: x = 6y

Review exercise in recurrence relations (page 177)

Q43:

a) $G_{n+1} = 0.6G_n + 2$
b) L = 5, which implies that over a long period of time the amount of "Growmore" plant food in the geraniums will stabilise at 5g

Q44:

a) $S_{n+1} = 0.25S_n + 15$
b) L = 20, which implies that over a long period of time the number of students in the queue will stabilise at 20

Advanced review exercise in recurrence relations (page 178)

Q45:

a) The recurrence relation $K_{n+1} = 0.1K_n + 2$ with $K_0 = 1$ approaches a limit as $n \to \infty$ because $-1 < 0.1 < 1$.

However the recurrence relation $G_{n+1} = 2G_n + 0.5$ with $G_0 = 0$ does not approach a limit because $2 > 1$

b) $L = \frac{2}{0.9} = \frac{20}{9} = 2\frac{2}{9}$

Q46:

a) $D_{n+1} = 0.15D_n + 1.2$ with $D_0 = 0.5$

b) $D_4 = 1.411$ metres (3 d.p.)

c) Yes the harbour can remain open because as $n \to \infty$, $D_n \to L = 1.412$ metres (3 d.p.) and $1.412 < 1.5$

Q47:

a) At this level of discharge the recurrence relation is $u_{n+1} = 0.6u_n + 2.5$ and as $n \to \infty$, $u_n \to 6.25$
Since $6.25 > 5$ the fish would be endangered at this level of discharge.

b) When the chemical released is reduced by 30% to 1.75 mg/l then the recurrence relation becomes $u_{n+1} = 0.6u_n + 1.75$ and as $n \to \infty$, $u_n \to 4.375$
Since $4.375 < 5$ the fish would not be endangered and so the Local Authority can grant permission at this level of discharge.

Q48:

a) 13.05 mg

b) 3 doses.

c) 4 hours after the n^{th} dose the level of serum in his body will have dropped to $(0.85)^4 = 0.522$ (to 3 d.p.) of what it was previously. He is then given another 25 mg. of serum. Thus the recurrence relation is $u_{n+1} = 0.522u_n + 25$

d) They can continue giving the serum for as long as is necessary because as $n \to \infty$, $u_n \to 52.3$ and $52.3 < 55$

Set review exercise in recurrence relations (page 180)

Q49: This answer is only available on the course web site.

Q50: This answer is only available on the course web site.