SCHOLAR Study Guide

SQA Higher Mathematics Unit 2

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SCHOLAR Study Guide Unit 2: Mathematics

1. Mathematics

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Contents

Topic 1

Factor/Remainder theorem and quadratic theory

Contents

Learning Objectives

- *Use the Factor/Remainder theorem and apply quadratic theory*

Minimum performance criteria:

- *Apply the Factor/Remainder theorem to a polynomial function*
- *Determine the nature of the roots of a quadratic using the discriminant*

Prerequisites

A sound knowledge of the following subjects is required for this unit:

- Algebraic manipulation
- Fraction manipulation
- Surd arithmetic
- Straight line graphs

1.1 Revision exercise

Revision exercise

15 min

Q1: Factorise $3x^2 - 4x + 1$

Q2: Simplify $\frac{2ab - 3cd}{6ad}$ and express as separate fractions.

Q3: Evaluate $\frac{-a - 2c}{4b^2 - 3}$ when a = -2, b = -1 and c = -3

Q4: Where does the graph of the equation y = -2x + 6 cross the axes?

Q5: Simplify fully the following:

a) $\frac{4\sqrt{12}}{7} \times \frac{\sqrt{147}}{\sqrt{8}}$

b) $\frac{\sqrt{18}}{6} + \frac{5}{\sqrt{50}}$

1.2 Polynomials

Learning Objective

Determine the roots of a polynomial

There are a few terms which will help to define the expressions and equations in this topic.

Polynomial of degree n.

An expression of the form $a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + ... + a_2 x^2 + a_1 x^1 + a_0$
where $a_0, ..., a_n$ are constants with $a_n \neq 0$
is called a polynomial of degree n.

Degree of a polynomial

The degree of a polynomial is the value of the highest power of x in the expression.

Here are some examples:

$x + 3$ has degree 1
$x^2 - 2x + 3$ has degree 2
$4x^3 + 2x^2 - 5$ has degree 3
$x^4 - x$ has degree 4
and of course a constant such as 7 has degree 0

A polynomial of degree 0 is called a *constant*.
Examples are 3 or -4

A polynomial of degree1 is called a *linear* expression.
Examples are $3x$ or $2x + 1$

A polynomial of degree 2 is a *quadratic* expression.
Examples are $3x^2$, $2x^2 - 1$, $x^2 - 3x$, $4x^2 + 2x + 1$ and $(x - 1)^2$

A polynomial of degree 3 is a *cubic* expression.
Examples are $3x^3$, $3x^3 - 1$, $3x^2 + 2x$, $4x^3 + 2x^2$, $x^3 + 2x^2 - 3x - 1$ and $(x + 2)^3$

A polynomial of degree 4 is a *quartic* expression.
Examples are $3x^4$, $-x^4 + 1$, $2x^4 + 3x^2$, $4x^4 - x^3$, $x^4 + x^2 - x - 1$ and $(x - 1)^4$

Other polynomials of higher degree have specific names but those shown here will be used frequently in this course.

It is also worth noting the difference in the meaning of the following terms.

Equations may have *solutions*: $2x - 5 = 3$ has a solution of $x = 4$

Polynomials may have *roots*: $2x - 4$ has a root $x = 2$

Root of a polynomial

A root of a polynomial p (x) is a solution to the equation p (x) = 0

Example Find p (1), p (2), p (3) and p (4) for the polynomial $p(x) = 2x^2 - 3x + 1$ and state which value of x, if any, is a root of p (x)

Substitute the values shown for x in the polynomial and evaluate.

p (1) = 0

p (2) = 3

p (3) = 10

p (4) = 21

Since p (1) = 0, by definition, x = 1 is a root of the polynomial p (x)

Terminology exercise

5 min

Q6: For each of the following:

1. Identify whether it has roots or solutions
2. State the degree of the expression in x
3. Give the roots or solutions as appropriate

a) $2x^2 = 8$

b) $3x - 3$

c) $-2x^2 - 3x + 9$

Q7: Find p (1), p (2), p (3) and p (-1) for the polynomial $p(x) = x^3 + 4x^2 - 3x + 1$

Factorisation of a polynomial is one method of finding a solution. Here is an example.

Example : Solution of an equation

Find the solutions of the equation $x^2 + 4x - 5 = 0$

Factorise the left hand side to give:

$(x + 5)(x - 1) = 0 \Rightarrow$

$(x + 5) = 0$ OR $(x - 1) = 0 \Rightarrow$

$x = -5$ OR $x = 1$

The example shows that the equation $x^2 + 4x - 5 = 0$ has solutions $x = -5$ or $x = 1$

In other words p (-5) = 0 or p (1) = 0

The *factors* of the polynomial are (x + 5) and (x - 1)

Factor Theorem

The **Factor theorem** states that if p (h) = 0 then (x - h) is a factor of p (x)

and if (x - h) is a factor of p (x) then p (h) = 0

The factor theorem can be used to find unknown coefficients in a polynomial.

Examples

1. Finding one unknown coefficient

State the value of b in the polynomial $x^3 - 2x^2 + bx + 6$ given that x - 1 is a factor.

If x - 1 is a factor then p (1) = 0 $\Rightarrow$

$1 - 2 + b + 6 = 0 \Rightarrow$

$b = -5$

2. Finding two unknown coefficients

Find the coefficients a and b in the polynomial $2x^3 + ax^2 + bx - 6$ given that x + 3 and x + 2 are factors.

By the factor theorem p (-3) = 0 and p (-2) = 0

Substitution of these values into the polynomial gives

$-54 + 9a - 3b - 6 = 0 \Rightarrow 9a - 3b = 60 \Rightarrow 3a - b = 20$ for $x = -3$

and $-16 + 4a - 2b - 6 = 0 \Rightarrow 4a - 2b = 22 \Rightarrow 2a - b = 11$ for $x = -2$

These two equations 3a - b = 20 and 2a - b = 11 can be solved simultaneously to give a = 9 and b = 7

Finding coefficients exercise

10 min

Q8: Find the coefficient k for the following polynomials:

a) $x^3 - x^2 + kx + 1$ where (x + 1) is a factor.
b) $2x^3 + kx^2 + 3x - 10$ where (x - 2) is a factor.
c) $3x^3 + bx - 8$ where (x + 2) is a factor.
d) $kx^2 - 8x + 16$ where (x + 4) is a factor.

Q9: Find the coefficients c and d for the following polynomials:

a) $x^3 + cx^2 + dx + 2$ given that (x - 1) and (x + 1) are factors.
b) $cx^3 + x^2 + dx - 4$ given that (x + 1) and (x - 2) are factors.
c) $x^3 + cx^2 + dx + 6$ given that (x + 3) and (x - 1) are factors.
d) $cx^3 - 13x + d$ given that (x - 3) and (x + 4) are factors.

Returning to the previous example of finding the solutions of the equation $x^2 + 4x - 5 = 0$, note that the polynomial $x^2 + 4x - 5$ factorises to give (x + 5)(x - 1)

So $x^2 + 4x - 5 = (x + 5)(x - 1)$

This in turn can be rearranged to give $\frac{x^2 + 4x - 5}{x + 5} = x - 1$

In words this means that dividing $x^2 + 4x - 5$ by x + 5 gives another polynomial (x - 1) with no remainder.

In fact any polynomial p (x) can be divided by another polynomial q (x) of lesser or equal degree to produce a third polynomial r (x) and a remainder (= s (x)/(q (x)).

If q (x) is a factor of p (x) then there will be no remainder.

Remainder theorem

If a polynomial p (x) is divided by (x - h) the remainder is p (h)

The proof of this formula is given in the section headed Proofs near the end of this topic.

Suppose that the polynomial $3x^3 + 6x^2 + 11x + 6$ has a root of x = -1

Then $3x^3 + 6x^2 + 11x + 6 = 0$ when x = -1

Since x = -1 is a root, (x + 1) is a factor of the polynomial.

By the remainder theorem,
$3x^3 + 6x^2 + 11x + 6$ divides by (x + 1) and leaves a remainder p (-1).

The task now is to find the remainder p (-1).

For a moment think about division of numbers.

There are certain terms associated with this operation.

Dividend

The dividend in a long division calculation is the expression which is being divided. As a fraction it is the numerator.

Divisor

The divisor is the expression which is doing the dividing.
It is the expression outside the division sign. As a fraction it is the denominator.

Quotient

The quotient is the answer to the division **but not including** the remainder.

Example $35 \div 8 = 4$ r 3

The dividend is 35

The divisor is 8

The quotient is 4 and the remainder is 3

In long division style this is written as

$$\begin{array}{r} 4\,3 \\ 8\overline{)\,3\,5\,1} \\ \underline{3\,2}\downarrow \\ 3\;\;1 \\ \underline{2\;\;4} \\ 7 \end{array}$$

This would be written in fraction terms as $43\ ^7/_8$

Returning to the problem of finding p (h) the same technique can be used.

Examples

1. Divide $3x^3 + 6x^2 + 11x + 6$ by $x + 1$

$$x + 1\overline{)\,3x^3 + 6x^2 + 11x + 6}$$

STEP 1: Lay out the division leaving gaps for 'missing terms' if any. Start with the highest power of x.

$$\begin{array}{r} 3x^2\qquad\qquad\qquad \\ x + 1\overline{)\,3x^3 + 6x^2 + 11x + 6} \end{array}$$

STEP 2: Divide the first term of the divisor (x) into the first of the dividend ($3x^3$) and write the answer at the top ($3x^2$).

$$\begin{array}{r} 3x^2\qquad\qquad\qquad \\ x + 1\overline{)\,3x^3 + 6x^2 + 11x + 6} \\ \underline{3x^3 + 3x^2}\qquad\qquad \end{array}$$

STEP 3: Multiply each of the terms in the divisor by the first term of the quotient and write underneath the dividend.

$$\begin{array}{r} 3x^2\qquad\qquad\qquad \\ x + 1\overline{)\,3x^3 + 6x^2 + 11x + 6} \\ \underline{3x^3 + 3x^2}\qquad\qquad \\ 3x^2 + 11x + 6 \end{array}$$

STEP 4: Subtract to give a new last line in the dividend.

$$\begin{array}{rrrrr}
 & & 3x^2 & +\,3x & \\ \cline{2-5}
x+1\,\big) & 3x^3 & +\,6x^2 & +\,11x & +\,6 \\ \cline{2-3}
 & 3x^3 & +\,3x^2 & & \\
 & & 3x^2 & +\,11x & +\,6
\end{array}$$

STEP 5: Divide the first term of the divisor (x) into the first term of the last line ($3x^2$) and write the answer at the top.

$$\begin{array}{rrrrr}
 & & 3x^2 & +\,3x & \\ \cline{2-5}
x+1\,\big) & 3x^3 & +\,6x^2 & +\,11x & +\,6 \\ \cline{2-3}
 & 3x^3 & +\,3x^2 & & \\
 & & 3x^2 & +\,11x & +\,6 \\ \cline{3-4}
 & & 3x^2 & +\,3x &
\end{array}$$

STEP 6: Multiply each of the terms in the divisor by the 2nd term of the quotient (3x) and write underneath the divisor.

$$\begin{array}{rrrrr}
 & & 3x^2 & +\,3x & \\ \cline{2-5}
x+1\,\big) & 3x^3 & +\,6x^2 & +\,11x & +\,6 \\ \cline{2-3}
 & 3x^3 & +\,3x^2 & & \\
 & & 3x^2 & +\,11x & +\,6 \\ \cline{3-4}
 & & 3x^2 & +\,3x & \\
 & & & 8x & +\,6
\end{array}$$

STEP 7: Subtract to give a new last line in the dividend.

$$\begin{array}{rrrrr}
 & & 3x^2 & +\,3x & +\,8 \\ \cline{2-5}
x+1\,\big) & 3x^3 & +\,6x^2 & +\,11x & +\,6 \\ \cline{2-3}
 & 3x^3 & +\,3x^2 & & \\
 & & 3x^2 & +\,11x & +\,6 \\ \cline{3-4}
 & & 3x^2 & +\,3x & \\
 & & & 8x & +\,6
\end{array}$$

STEP 8: Divide the first term of the divisor (x) into the first term of the last line (8x) and write the answer at the top.

$$\begin{array}{rrrrr}
 & & 3x^2 & +\,3x & +\,8 \\ \cline{2-5}
x+1\,\big) & 3x^3 & +\,6x^2 & +\,11x & +\,6 \\ \cline{2-3}
 & 3x^3 & +\,3x^2 & & \\
 & & 3x^2 & +\,11x & +\,6 \\ \cline{3-4}
 & & 3x^2 & +\,3x & \\
 & & & 8x & +\,6 \\ \cline{4-5}
 & & & 8x & +\,8
\end{array}$$

STEP 9: Multiply each of the terms in the divisor by the 3rd term of the quotient (8) and write underneath the divisor.

$$
\begin{array}{rl}
 & 3x^2 + 3x \;\; + \;\; 8 \\ \cline{2-2}
x + 1 \big) & 3x^3 + 6x^2 + 11x + 6 \\
 & \underline{3x^3 + 3x^2} \\
 & \qquad 3x^2 + 11x + 6 \\
 & \qquad \underline{3x^2 + \;\; 3x} \\
 & \qquad\qquad\quad 8x + 6 \\
 & \qquad\qquad\quad \underline{8x + 8} \\
 & \qquad\qquad\qquad\; - 2
\end{array}
$$

STEP 7: Subtract to give a new last line in the dividend (-2). The division stops here in this case as the degree of the divisor (1) is greater than the degree of the last line (0).

Therefore $\dfrac{3x^3 + 6x^2 + 11x + 6}{x + 1} = 3x^2 + 3x + 8$ remainder $- 2$

2. Divide $3x^3 - 2x^2 + 6$ by $x + 4$.

$$
\begin{array}{rl}
 & \\ \cline{2-2}
x + 4 \big) & 3x^3 - 2x^2 \qquad\quad + 6
\end{array}
$$

STEP 1: Lay out the division leaving gaps for 'missing terms'. Start with the highest power of x.

$$
\begin{array}{rl}
 & 3x^2 \\ \cline{2-2}
x + 4 \big) & 3x^3 - 2x^2 \qquad\quad + 6
\end{array}
$$

STEP 2: Divide the first term of the divisor into the first of the dividend and write the answer at the top.

$$
\begin{array}{rl}
 & 3x^2 \\ \cline{2-2}
x + 4 \big) & 3x^3 \; - \; 2x^2 \qquad\quad + 6 \\
 & \underline{3x^3 + 12x^2}
\end{array}
$$

STEP 3: Multiply each of the terms in the divisor by the first term of the quotient and write underneath the dividend.

$$
\begin{array}{rl}
 & 3x^2 \\ \cline{2-2}
x + 4 \big) & 3x^3 \; - \; 2x^2 \qquad\quad + 6 \\
 & \underline{3x^3 + 12x^2} \\
 & \qquad - 14x^2 \qquad\quad + 6
\end{array}
$$

STEP 4: Subtract to give a new last line in the dividend.

$$
\begin{array}{rl}
 & 3x^2 \; - 14x \\ \cline{2-2}
x + 4 \big) & 3x^3 \; - \; 2x^2 \qquad\quad + 6 \\
 & \underline{3x^3 + 12x^2} \\
 & \qquad - 14x^2 \qquad\quad + 6
\end{array}
$$

STEP 5: Divide the first term of the divisor into the first term of the last line and write the answer at the top.

$$
\begin{array}{rl}
 & 3x^2 \; - 14x \\ \cline{2-2}
x + 4 \big) & 3x^3 \; - \; 2x^2 \qquad\quad + 6 \\
 & \underline{3x^3 + 12x^2} \\
 & \qquad - 14x^2 \qquad\quad + 6 \\
 & \qquad \underline{- 14x^2 - 56x}
\end{array}
$$

STEP 6: Multiply each of the terms in the divisor by the 2nd term of the quotient and write underneath the divisor.

$$
\begin{array}{r|llll}
 & 3x^2 & -14x & & \\
\hline
x+4 & 3x^3 & -2x^2 & & +6 \\
 & \underline{3x^3} & \underline{+12x^2} & & \\
 & & -14x^2 & & +6 \\
 & & \underline{-14x^2} & \underline{-56x} & \\
 & & & 56x & +6
\end{array}
$$

STEP 7: Subtract to give a new last line in the dividend.

$$
\begin{array}{r|llll}
 & 3x^2 & -14x & +56 & \\
\hline
x+4 & 3x^3 & -2x^2 & & +6 \\
 & \underline{3x^3} & \underline{+12x^2} & & \\
 & & -14x^2 & & +6 \\
 & & \underline{-14x^2} & \underline{-56x} & \\
 & & & 56x & +6
\end{array}
$$

STEP 8: Divide the first term of the divisor into the first term of the last line and write the answer at the top.

$$
\begin{array}{r|llll}
 & 3x^2 & -14x & +56 & \\
\hline
x+4 & 3x^3 & -2x^2 & & +6 \\
 & \underline{3x^3} & \underline{+12x^2} & & \\
 & & -14x^2 & & +6 \\
 & & \underline{-14x^2} & \underline{-56x} & \\
 & & & 56x & +6 \\
 & & & \underline{56x} & \underline{+224}
\end{array}
$$

STEP 9: Multiply each of the terms in the divisor by the 3rd term of the quotient and write underneath the divisor.

$$
\begin{array}{r|llll}
 & 3x^2 & -14x & +56 & \\
\hline
x+4 & 3x^3 & -2x^2 & & +6 \\
 & \underline{3x^3} & \underline{+12x^2} & & \\
 & & -14x^2 & & +6 \\
 & & \underline{-14x^2} & \underline{-56x} & \\
 & & & 56x & +6 \\
 & & & \underline{56x} & \underline{+224} \\
 & & & & -218
\end{array}
$$

STEP 10: Subtract to give a new last line in the dividend. The division stops here since the degree of the last line (0) is less than the degree of the divisor (1)

Long division and the factor theorem can be useful in determining the factors of a polynomial.

Example : Factorise a polynomial

Fully factorise the polynomial $3x^3 + x^2 - 3x - 1$

The trick is to search for a value for x which gives p (x) = 0
Try, for example, 0, 1, -1, 2, -2 and so on until one is found.

Here x = 1 gives p (1) = 0 and so x - 1 is one factor.

Dividing $3x^3 + x^2 - 3x - 1$ by this factor $x - 1$ gives $3x^2 + 4x + 1$

This in turn factorises further to give $3x^2 + 4x + 1 = (3x + 1)(x + 1)$

The factors are $x - 1$, $x + 1$ and $3x + 1$

Note that not all polynomials have only linear factors.

Example Factorise $x^3 + 3x^2 + 6x + 4$

By trial $x = -1$ gives $p(-1) = 0$ and so $x + 1$ is a factor.

Dividing $x^3 + 3x^2 + 6x + 4$ by $(x + 1)$ leaves $x^2 + 2x + 4$ which has no further factors. This polynomial has one linear and one quadratic factor.

It can also happen that once a quotient is obtained from using the first factor, a further search and another division is needed to find other factors.

Example Factorise $x^4 + 2x^3 - 7x^2 - 8x + 12$

By trial one factor is $x - 1$. On division this gives:

$x^4 + 2x^3 - 7x^2 - 8x + 12 = (x - 1)(x^3 + 3x^2 - 4x - 12)$

At this stage since $x^3 + 3x^2 - 4x - 12$ is not easy to factorise, look for another factor by trial and find, say $x - 2$

(Note that a factor of the quotient will be a factor of the original polynomial which makes this second factor easier to find.)

Divide the quotient $x^3 + 3x^2 - 4x - 12$ by $x - 2$ to give $x^2 + 5x + 6$ which is then easily factorised to $(x + 2)(x + 3)$

The four factors are

$(x - 2)$, $(x - 1)$, $(x + 2)$ and $(x + 3)$

Division of a polynomial exercise

15 min

Q10: Fully factorise the following:

a) $x^4 + x^3 - 7x^2 - x + 6$
b) $x^3 + 5x^2 + 11x + 10$
c) $x^4 - 5x^2 + 4$

Q11: (Harder) Fully factorise the following:

a) $8x^3 + 32x^2 + 26x + 6$
b) $30x^3 + 29x^2 - 22x + 3$ given that $3x - 1$ is a factor.
c) $12x^3 + 8x^2 - 13x + 3$ given that $2x + 3$ is a factor.

1.3 The quadratic formula

Learning Objective

Use the quadratic formula to find the roots of a quadratic

Consider the equation $2x^2 + 4x + 1$ and try to factorise it. It is certainly not obvious what the factors might be.

There is however, a formula that can be used when factorisation of a quadratic is not easy to find.

Quadratic formula

The roots of the quadratic $ax^2 + bx + c$ are given by the formula

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The proof of this formula is given in the section headed Proofs near the end of the topic.

Here is an example using this formula.

Example Find the roots of the quadratic $2x^2 + 5x + 1$ using the quadratic formula.

The quadratic is in the form $ax^2 + bx + c$ where a = 2, b = 5 and c = 1

Substituting these values into the formula gives the roots as x =

$$\frac{-2 \pm \sqrt{5^2 - 4 \times 2 \times 1}}{2 \times 2} =$$

$$\frac{-2 \pm \sqrt{17}}{4} =$$

$$-\frac{1}{2} + \frac{\sqrt{17}}{4} \text{ or } -\frac{1}{2} - \frac{\sqrt{17}}{4}$$

so x = $-\frac{1}{2} + \frac{\sqrt{17}}{4}$ or $-\frac{1}{2} - \frac{\sqrt{17}}{4}$

The answer can be left like this or it can be evaluated when necessary to give 0.53 and -1.53 to 2 d.p.

Quadratic formula exercise

10 min

Q12: Find the roots of the following quadratics leaving your answer in surd form:

a) $3x^2 + 5x - 1$

b) $4x^2 + 8x - 3$

c) $x^2 - 4x + 1$

d) $x^2 + 3x - 1$

There are now two methods of finding the roots of a quadratic expression:

1. Factorisation.
2. Using the quadratic formula.

The method to use depends on the quadratic expression but the quadratic formula will work in all cases where there are roots.

Recall that the roots of a quadratic expression p (x) are the solutions to the quadratic equation p (x) = 0. These can also be found from the graph of the quadratic by finding the value at which the graph crosses the x-axis.

10 min

Calculator exercise

With a graphics calculator find the roots of the quadratic $3x^2 - 5x - 1$ to 2 d.p. by plotting the graph of this function.

In the TI 83 use the 'zero' facility under the 'calc' menu.

Other calculators will have a similar menu that allows the cursor to trace the graph and input two values, one either side of the point at which the graph crosses the x-axis.

10 min

Calculator program

Program your graphics calculator to prompt for the coefficients a, b and c and calculate the roots using the quadratic formula. There is a version for a TI 83 given in the answers if you find this difficult.

1.4 The discriminant and its properties

Learning Objective

Use the discriminant and its properties to establish results about quadratics

When the quadratic formula is introduced the first examples and questions using it give 2 distinct answers

However in some cases there will be only one solution and in others, there will be no solution at all.

Examples

1. Two distinct real roots

Find the roots of the quadratic $x^2 - 2x - 1$

Answer:

Using the formula this gives x =

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{+2 \pm \sqrt{4 + 4}}{2} = 1 + \sqrt{2} \text{ or } 1 - \sqrt{2}$$

2. Equal real roots

Find the roots of the quadratic $x^2 + 6x + 9$

Using the formula gives x =

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-6 \pm \sqrt{36 - 36}}{2} = \frac{-6 \pm \sqrt{0}}{2} = -3 + 0 \text{ or } -3 - 0$$

That is, x = -3 twice.

3. No real roots

Find the roots of the quadratic $x^2 - x + 4$

In this case, the formula gives x =

$$\frac{+1 \pm \sqrt{1 - 4}}{2} = \frac{+1 \pm \sqrt{-3}}{2}$$

There is a problem here as the formula has given the square root of a negative number. In such cases the answer should state that the quadratic does not have any real roots. Note that there may be complex roots - a topic for study at higher levels.

A close examination of these three examples shows that it is one particular part of the formula which determines (or discriminates between) the different types of answer.

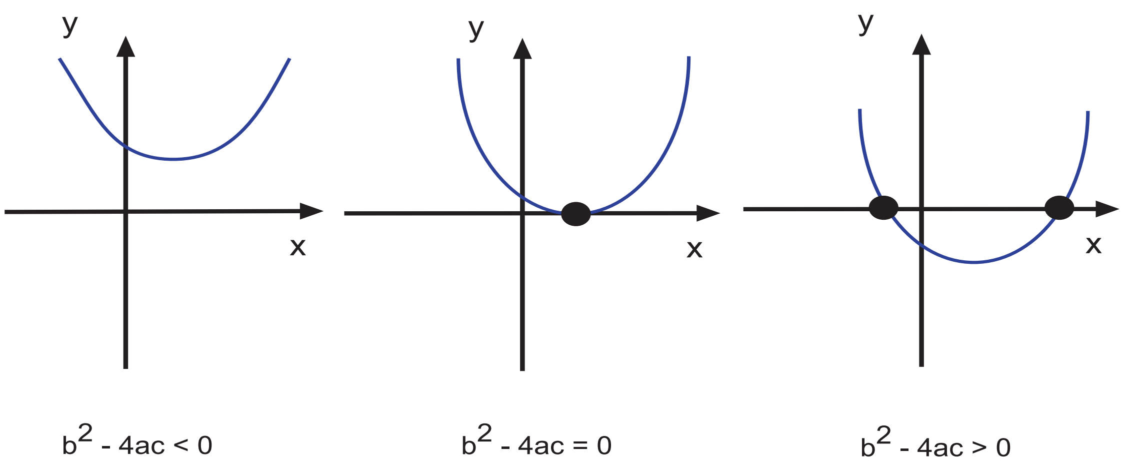
Discriminant

The discriminant of the quadratic equation $ax^2 + bx + c = 0$ is $b^2 - 4ac$

The following conditions on the discriminant hold:

- If $b^2 - 4ac < 0$, there are no real roots.
- If $b^2 - 4ac = 0$, the roots are real and equal.
- If $b^2 - 4ac > 0$, the roots are real and distinct.

These conditions can be related to the position of the graph of a quadratic.



Examples

1. Find the value of q such that the quadratic $4x^2 - 12x + q$ has equal roots.

When this quadratic has real and equal roots, the discriminant is equal to zero.

a = 4, b = -12 and c = q

therefore $b^2 - 4ac = 144 - 16q = 0$

$\Rightarrow 16q = 144 \Rightarrow q = 9$

2. Find the range of values for s such that $x^2 + sx + 9$ has real distinct roots.

When the quadratic has real and distinct roots, $b^2 - 4ac > 0$

So with a = 1, b = s and c = 9 this gives

$s^2 - 36 > 0 \Rightarrow s^2 > 36 \Rightarrow s > 6$ or $s < -6$

Discriminant exercise

10 min

Q13: Find the discriminant for the following quadratics and determine the nature of their roots. (distinct real, equal real, no real)

a) $2x^2 - 4x + 5$

b) $3x^2 + 4x + 1$

c) $4x^2 - 4x + 1$

d) $2x^2 + x - 1$

e) $3x^2 - 4x + 5$

f) $2x^2 - 2$

Q14: For the following quadratics find the value or range of values for q which satisfy the condition given:

a) $x^2 + qx + 4$ such that there are no real roots.

b) $2x^2 - 5x + q$ such that the roots are distinct real.

c) $4x^2 + qx + 9$ such that there are equal roots.

Q15: Do the graphs of the following quadratics touch, cross or avoid contact with the x-axis?

a) $2x^2 - 4x + 1$

b) $-x^2 - 3x + 2$

c) $3x^2 - 4x + 5$

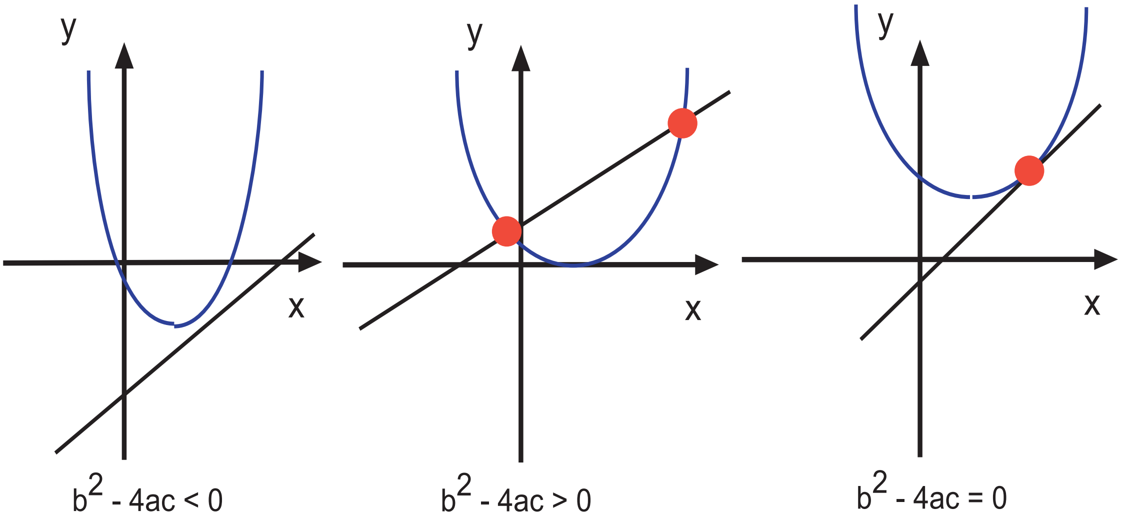
Extra Help: The discriminant

An online exercise is provided to help you if you require additional assistance with this material, or would like to revise this subject.

The discriminant conditions followed by application of the quadratic formula can also be used to determine the intersection points, if any, of a straight line and a parabola.

The following conditions apply to the quadratic formed from the equation of a straight line and the equation of a parabola at intersection:

- If $b^2 - 4ac < 0$, the line does not come in contact with the parabola.
- If $b^2 - 4ac > 0$, the line cuts the parabola at two distinct points.
- If $b^2 - 4ac = 0$, the line touches the parabola at one (common) point and the straight line is a tangent to the parabola.



Examples

1. Does the line $y = 2x - 3$ intersect with the graph of the quadratic $-2x^2 - x + 1$? If it does find the points of intersection.

If the line and parabola intersect then $2x - 3 = -2x^2 - x + 1$

This can be rearranged into a quadratic equation to give

$2x^2 + 3x - 4 = 0$

Use the discriminant conditions for roots.

$b^2 - 4ac = 41$

So $b^2 - 4ac > 0$ and there are two intersection points.

Using the quadratic formula to find solutions of $2x^2 + 3x - 4 = 0$ gives

$$x = \frac{-3 \pm \sqrt{9 + 32}}{4} =$$

$$\frac{-3 + \sqrt{41}}{4} \text{ or } \frac{-3 - \sqrt{41}}{4}$$

In this case evaluate these to give x = 0.85 and -2.35 to two decimal places.

Substitution of these values for x in the equation of the line y = 2x - 3 gives
y = -1.3 and y = -7.7

The two points of intersection are (0.85, -1.3) and (-2.35, -7.7)

2. Find the equation of the tangent to the curve $x^2 - 3x + 1$ that has gradient 3

A tangent is a straight line and with a gradient of 3 will have an equation of the form
y = 3x + c

At intersection $3x + c = x^2 - 3x + 1 \Rightarrow x^2 - 6x + 1 - c = 0$

The discriminant of $x^2 - 6x + 1 + c$ is 36 - 4(1 - c) = 32 + 4c

If the line is a tangent, then it only touches the curve and will have only one point of intersection.

The discriminant condition for this to happen is $b^2 - 4ac = 0$

So $32 + 4c = 0 \Rightarrow c = -8$

The equation of the tangent is y = 3x - 8

10 min

Intersection exercise

Q16: Find the equation of the tangent to the curve $2x^2 - 2x - 3$ that has a gradient of 2

Q17: Find the equation of the tangent to the curve $-3x^2 - 5x + 5$ with a gradient of 1

Q18: Do the following lines and quadratics intersect. If they do, find the intersection point(s).

a) The quadratic $x^2 - 4x + 2$ and the line y = 2x - 3
b) The quadratic $2x^2 - 5x - 1$ and the line y = -2x + 1
c) The quadratic $-x^2 - 3x + 2$ and the line y = -4x + 5
d) The quadratic $x^2 + 3x - 2$ and the line y = 5x - 3

1.5 Quadratic inequalities

Learning Objective

Solve quadratic inequalities

The general shape of the graph of a quadratic is a parabola:

If the coefficient of the x^2 term is positive then the parabola is U-shaped and the graph has a minimum turning point.

If the coefficient of the x^2 term is negative then the parabola is $\cap$-shaped and there is a maximum turning point.

The position within the four quadrants depends on the full equation but from the graph sketching techniques of previous topics, the points at which the graph crosses both axes can be found. If the quadratic is converted into completed square form, the turning point is also straightforward to find.

The x-coordinate of the turning point, however, is easy to obtain from a standard quadratic formula.

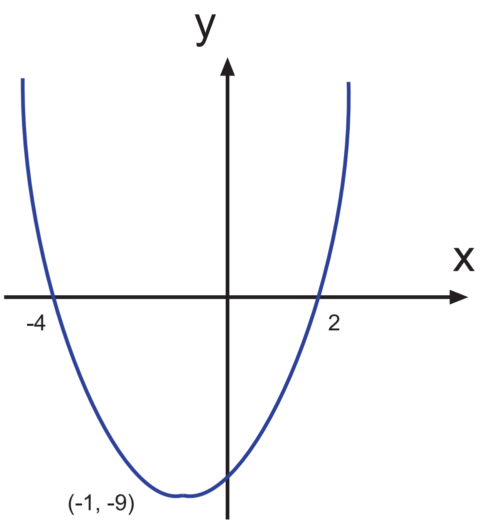
x-coordinate of the turning point of a quadratic graph
The x-coordinate of the turning point of the graph of the quadratic $ax^2 + bx + c$ is given by $^{-b}/_{2a}$

Being able to sketch the graph of a quadratic can help considerably with the solutions to inequalities.

Example Solve $x^2 + 2x - 8 < 0$

Sketch the curve using the techniques already given. The quadratic factorises to give (x + 4)(x - 2) and this means that the graph crosses the x-axis at x = -4 and x = 2

The formula for the x-coordinate of the turning point gives $^{-b}/_{2a} = -1$. The y-coordinate is found by substituting this value in the equation of the quadratic $x^2 + 2x - 8$ to give -9



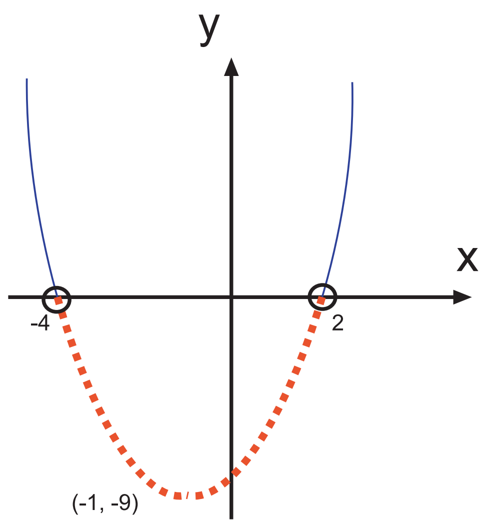
The graph clearly shows those values where the quadratic has a positive value, a negative value or is equal to zero.

Above the x-axis: $x^2 + 2x - 8 > 0$

Below the x-axis: $x^2 + 2x - 8 < 0$

At the x-axis: $x^2 + 2x - 8 = 0$

The solution in this case is that part of the graph which lies under the x-axis.



Then in terms of the values of x, the graph is below the x-axis when $-4 < x < 2$

Remember that a hollow circle indicates that the point is not included in the solution to the inequality whereas a filled -in circle indicates that the point is part of the solution.

10 min

Inequality demonstration

There is a web demonstration of inequalities.

10 min

Inequalities exercise

Q19: Find the values of x for which the following hold:

a) $2x^2 - 2x - 12 > 0$

b) $x^2 - x - 2 < 0$

c) $-x^2 + 3x - 2 > 0$

d) $x^2 + 3x - 10 = 0$

e) $x^2 - 2x - 24 \geq 0$

Q20: (harder) Solve the following to two decimal places:

a) $x^2 - 2x - 12 > 0$

b) $-x^2 - x + 1 \leq 0$

c) $2x^2 + 3x - 7 \geq 0$

d) $x^2 + 3x - 6 = 0$

e) $-x^2 - 2x + 12 \geq 0$

1.6 Approximate roots in a given interval

Learning Objective

Find an approximate root of a polynomial

The quadratic formula is extremely useful in finding the roots of a quadratic. For higher order polynomials (for example, cubics), it is not so easy to find a root unless it is an integer value (found by trial factors etc).

In these cases, an approximate value can be found by using iteration. One strategy is:

- Find by trial, two integers x_1 and x_2 between which the root lies (a sign change indicates this).
- Take the midpoint value x_3 between them and evaluate the polynomial for this value.
- Replace the former value (either x_1 or x_2) which has the same sign as this midpoint value (x_3) by it.
- Rename and recalculate as in the last two steps.

The iteration may take a long time and there are shortcuts but this will always converge to the root if there is one. The point at which the iteration can stop depends on the accuracy of the answer required. However, be sure to check far enough to allow the correct number of decimal places.

The following example will help to make this method clearer.

Example Show that the polynomial p (x) = x^3 - $2x^2$ + 3x - 1 has a root between 0 and 1. Find an approximate value for this root to 1 decimal place.

p (0) = -1 and p (1) = 1

There is a sign change from negative to positive which indicates that the graph crosses the x-axis.

That is, there is a root between 0 and 1

To find an approximation to the root, construct a table as shown:

The initial values are x_1 = 0, x_2 = 1 and x_3 is the midpoint value between them = 0.5

x_1	x_2	x_3	f (x_1)	f (x_2)	f (x_3)	Replace with x_3
0	1	0.5	-1	1	0.125	x_2
0	0.5	0.25	-1	0.125	-0.359	x_1
0.25	0.5	0.375	-0.359	0.125	-0.104	x_1
0.375	0.5	0.4375	-0.104	0.125	0.0134	x_2
0.375	0.4375	0.406	-0.104	0.0134	-0.045	x_1
0.406	0.4375	0.422	-0.045	0.0134	-0.015	x_1

Note that on the sixth row of calculations the values of x_1 and x_2 to one decimal place are the same at 0.4 and the iteration can stop.

So the root is x = 0.4 approximated to 1 d.p.

The values in the columns f (x_1) and f (x_2) indicate further accuracy.
From the first two columns it can be seen that the value of the root lies between 0.406 and 0.4375. However, notice that column 5 has a value much closer to zero than column 4. This indicates that the root actually lies closer to 0.4375

In the table construction it is best not to round the values of x too far as it can compromise the accuracy.

These calculations can be cumbersome and using a programmable calculator can help.

10 min

Approximate roots demonstration

There is a demonstration of the process on the web.

10 min

Calculator program

This program is designed for a TI83 graphics calculator and will help in calculating the values for iterations. If you do not have this calculator but do have a programmable one, look at this one and try to amend it to suit your own.

The example expression used in this program is $x^3 - 2x^2 + 3x - 1$

Simply substitute the expression in the line on which this appears for other calculations.

: Prompt X

: X^3 - 2*X^2 + 3*X - 1 $\rightarrow$ R

: Disp R

When this program runs it will prompt for the value of X and return the value of the expression.

CHALLENGE: Try to program the calculator to carry out the full iteration if there is time.

20 min

Approximate roots exercise

Q21: Find the approximate root of the polynomial $2x^2 - 5x - 2$ which lies between -1 and 0 to 1 d.p.

Q22: Find the approximate root of the polynomial $3x^2 + 6x - 1$ which lies between 0 and 1 to 1 d.p.

Q23: Show that a root of the polynomial $x^2 - 7x - 2$ lies between 7 and 8 and find an approximate value for it to 1 d.p.

Q24: Show that a root of the polynomial $4x^3 - 2x^2 + x - 2$ lies between 0 and 1 and find an approximate value for it to 2 d.p.

1.7 Summary

The following points and techniques should be familiar after studying this topic:

- Using the correct terminology for expressions and equations.
- Using the factor/remainder theorems to factorise and find roots or coefficients of polynomials.
- Finding the roots of a quadratic using the quadratic formula.
- Determining if a quadratic has roots by using discriminant properties.
- Solving quadratic inequalities.
- Finding approximate roots using iteration.

1.8 Proofs

Proof 1: The quadratic formula

Let $ax^2 + bx + c = 0$ then

$$a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) = 0$$

$$\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) = 0$$

$$\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a} = 0$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Proof 2: The remainder theorem

Let p (x) be a polynomial which on division by (x - h) gives a quotient of q (x) and a remainder R

Then $p(x) = (x - h)q(x) + R$

But if $x = h$

then $p(h) = (h - h)q(h) + R \Rightarrow$

$p(h) = 0 + R \Rightarrow$

$p(h) = R$

1.9 Extended information

Learning Objective

Display a knowledge of the additional information available on this subject

There are links on the web which give a selection of interesting sites to visit. Browsing the web under 'algebra' will lead to many other good sites which cover this topic.

There is a long history attached to equations and polynomials.
The St. Andrews web site at
http://www-groups.dcs.st-and.ac.uk/~history/HistTopics/Quadratic_etc_equations.html
gives one of the best accounts and consequently only a few comments are given here on Mathematicians, mainly of the modern era.

Euler

The results of previous work on quartics (degree 4) and cubics inspired Euler to examine quintics (of degree 5) to see if he could relate them in a similar way to quartics. He failed.

Henrik

In 1824 after attempts by many mathematicians at finding solutions for quintics, Henrik published a proof that this was not possible.

Galois

He continued to work on the solutions for high degree equations and group theory came to prominence. His work on radicals also goes by the name of Galois theory.

From this time onwards, many new and exciting areas of mathematics were discovered and developed.

Finding solutions to equations or roots of polynomials was greatly enhanced with the introduction of computers.

Modern day mathematicians worth tracing include: Alan Turing, Johann Von Neumann and Stephen Wolfram.

1.10 Review exercise

Review exercise

15 min

Q25: Show that (x - 2) is a factor of the polynomial p (x) = $x^3 + x^2 - 10x + 8$ and express p (x) in fully factorised form.

Q26: Determine the nature of the roots of the polynomial $2x^2 - 3x + 5$ by examining the discriminant.

Q27: Find the roots of the polynomial p (x) which is given as $2x^3 - 3x^2 + 1$

Q28: Determine the nature of the roots of the polynomial $x^3 - 9x^2 + 27x - 27$

1.11 Advanced review exercise

Advanced review exercise

15 min

Q29: Find the values of k for $k \in \mathbb{R}$ such that $\frac{(x - 5)^2}{x^2 + 5} = k$ has real equal roots..

Q30: Find p and q in the polynomial $f(x) = x^3 + px^2 - 4x + q$ given that $x - 3$ is a factor and $q = -4p$.

Q31: Use the discriminant to determine the nature of the roots of the expression $3x^2 - 2x + 1$

Q32: Determine if $y = x^2 - 4x + 1$ and $y = -x^2 - 5x - 6$ intersect by using the determinant properties. For each quadratic determine the nature of the roots.

1.12 Set review exercise

Set review exercise

15 min

Work out your answers and then access the web to input your answers in the online test called set review exercise.

Q33: Show that $(x + 3)$ is a factor of the polynomial $p(x) = x^3 + 3x^2 - x - 3$ and express $p(x)$ in fully factorised form

Q34: Determine the nature of the roots of the polynomial $3x^2 - 9x + 5$ by examining the discriminant.

Q35: Find the roots of the polynomial $p(x)$ which is given as $x^3 + 6x^2 + 11x + 6$

Q36: With the aid of long division and factorising, determine the nature of the roots of the polynomial $x^3 - 18x^2 + 108x - 216$

Topic 2

Basic Integration

Contents

Learning Objectives

Use basic integration

Minimum performance criteria:

- *Integrate functions reducible to the sums of powers of x (definite and indefinite).*
- *Find the area between a curve and the x-axis using integration.*
- *Find the area between two curves using integration.*

Prerequisites

You should be familiar with algebraic operations on indices. If necessary you should revise this work before commencing this topic.

2.1 Introduction

Learning Objective

Obtain a formula for the area under a curve

Many practical problems require the evaluation of an area.

Example A cyclist is travelling along a path at a steady speed of 8 metres/second. How far has she travelled after 5 seconds?

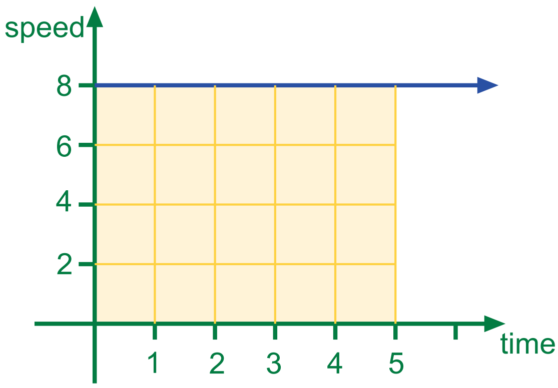
Distance = Speed $\times$ Time

8×5

= 40 metres

Notice this is the same as the answer that you obtain by calculating the area under the speed-time graph.

The cyclist will travel 40 metres in 5 seconds.

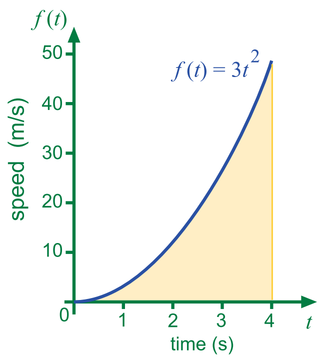


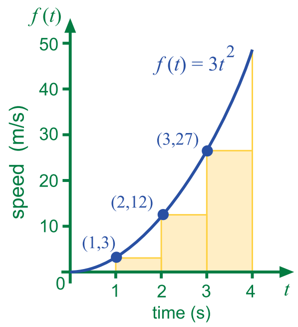
In general, it is true that the area under a speed-time graph represents distance travelled. This is easy to calculate when speed remains constant but is more complicated when speed varies.

Consider the following example.

Example The speed of an aircraft accelerating along a straight stretch of runway is given by the formula f (t) = $3t^2$.
How far has the aircraft travelled during the first four seconds?

The shaded area in the graph shown here represents the distance travelled by the aircraft in the first 4 seconds. We can obtain an estimate for this area by dividing the shaded area into rectangles as follows.





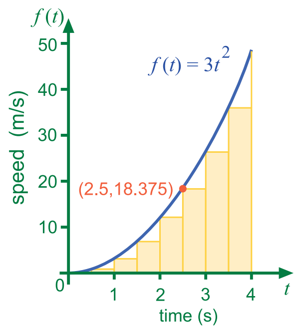
This first estimate for the area under the curve gives

1 × 3 = 3

1 × 12 = 12×1 × 27 = 27

Total area = 42

Clearly this answer is too small as there are large gaps under the curve that have not been included.



We can obtain a better estimate if we make the rectangles half as wide and so fill up more of the space below the curve.

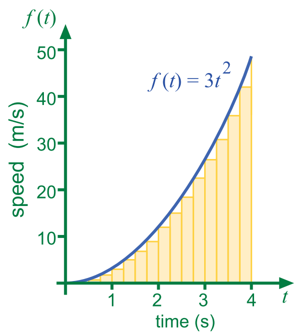
Note that the height of each rectangle is equal to the y-coordinate of a point on the curve.

Thus when x = 2.5 the height of the corresponding rectangle is y = 3 x $(2.5)^2$ = 18.375

Adding up areas of rectangles of width 0.5 x height gives

Total area = 52.5

Check this answer for yourself. It is more accurate but still too small.



An even better estimate is obtained by making the rectangles even narrower. This time the base of the rectangle is 0.25 units wide.

Total area = 58.125

Again you should check this for yourself. This answer is the most accurate estimate we have obtained so far. However, it will still be too small as there are uncounted gaps beneath the curve.

By increasing the number of rectangles the area can be found more accurately.
The total area of the rectangles when there are 100 rectangles is 63.0432 square units. In fact the more rectangles we use the nearer the total area approaches a limit of 64 square units. Thus we can conclude that the aircraft travels a distance of 64 metres in the first 4 seconds of taking off.

Obviously it would be better if we had a quicker method for determining the area under the curve and thus the distance travelled. The questions in Exercise 1 should lead you to some conclusions.

---

20 min

An area formula

This activity is available on-line.

30 min

Exercise 1

There is an on-line exercise at this point which you might find helpful.

The following table lists the results for the area under the curve f (t) = $3t^2$ for various values of t.

t	0	1	2	3	4	5
Area, A(t)	0	1	8	27	64	125

Perhaps you have already spotted that when the curve has equation f (t) = $3t^2$ then the area under the curve is given by A(t) = t^3 (Starting from t = 0)

Q1: How many metres has the aircraft travelled after 3 seconds?

Q2: How many metres has the aircraft travelled after 5 seconds?

Q3: How many metres has the aircraft travelled after 6 seconds?

Q4:

The following tables list the results for the area under a variety of curves for different values of t. Use these tables to find a formula for A (t) in each case.

f (t) = 2t

t	0	1	2	3	4	5
Area, A(t)	0	1	4	9	16	25

Q5:

f(t) = $4t^3$

t	0	1	2	3	4	5
Area, A(t)	0	1	16	81	256	625

Q6:

$f(t) = 5t^4$

t	0	1	2	3	4	5
Area, A(t)	0	1	32	243	1024	3125

Q7:

Copy and complete the following table for A(t)

f(t)	$2t$	$3t^2$	$4t^3$	$5t^4$	$6t^5$	...	$100t^{99}$
Area, A(t)	t^2	t^3				...	

Q8: In general when the curve has equation $f(t) = (n + 1)\ t^n$, where $n \neq -1$ then the area under the curve can be calculated from the formula A(t) = ?

2.2 Anti-derivatives

Learning Objective

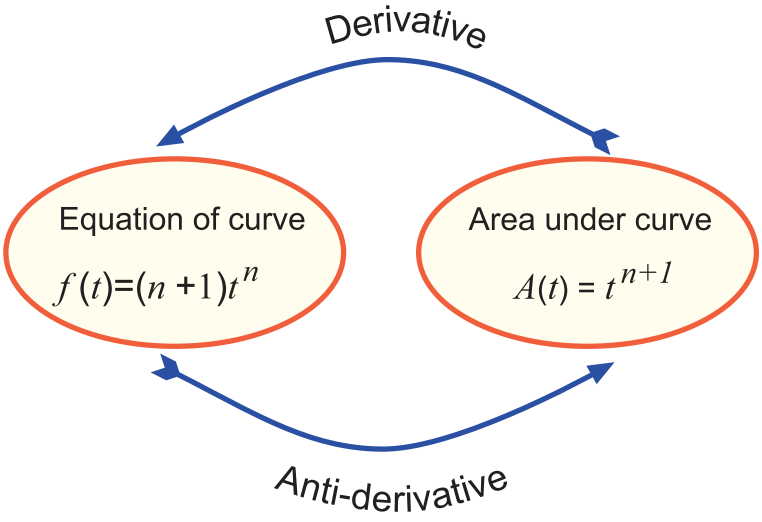
Find anti-derivatives of simple functions

In the previous exercise you should have made the conclusion that when the equation of a curve is $f(t) = (n + 1)t^n$ then the area under the curve is $A(t) = t^{n+1}$

Perhaps you have already spotted the connection between f (t) and A(t). In fact f (t) is the derivative of A(t) and we can write $f(t) = A'(t)$.

However, given f (t) we want to find A(t) which is the reverse to differentiating. We say that A(t) is an anti-derivative of f (t).

This information is summarised in the following diagram.



Examples

1.

The derivative of f (x) = x^4 is f ' (x) = $4x^3$
$\Leftrightarrow$ An anti-derivative of f '(x) = $4x^3$ is f (x) = x^4

2.

The derivative of g (x) = x^{-5} is g' (x) = - $5x^{-6}$
$\Leftrightarrow$ An anti-derivative of g' (x) = - $5x^{-6}$ is g (x) = - $5x^{-5}$

3. The derivative of f (x) = $x^{1/3}$ is f' (x) = $\frac{1}{3}x^{-2/3}$
$\Leftrightarrow$ An anti-derivative of f' (x) = $\frac{1}{3}x^{-2/3}$ is f (x) = $x^{1/3}$

20 min

Exercise 2

There is an on-line exercise at this point which you might find helpful.

Given the following derivatives find an anti-derivative.

Q9:

a) f '(x) = $9x^8$
b) f '(x) = $15x^{14}$
c) f '(x) = $20x^{19}$
d) f '(x) = $2x$

Q10:

a) f '(x) = - $7x^{-8}$
b) f '(x) = $-21x^{-22}$
c) f '(x) = - $9x^{-10}$

d) $f'(x) = -x^{-2}$

Q11:

a) $f'(x) = \frac{3}{2}x^{1/2}$
b) $f'(x) = \frac{5}{4}x^{1/4}$
c) $f'(x) = \frac{3}{4}x^{-1/4}$
d) $f'(x) = \frac{2}{5}x^{-3/5}$
e) $f'(x) = -\frac{1}{4}x^{-5/4}$
f) $f'(x) = -\frac{5}{6}x^{-11/6}$

2.3 Integration

Learning Objective

Know the notation and vocabulary associated with process of integration

Examine the derivatives in the following table.

f (x)	f ' (x)
x^6	$6x^5$
$x^6 + 3$	$6x^5$
$x^6 - 11$	$6x^5$
$x^6 + C$	$6x^5$

(C is a constant)

We can see that $f(x) = x^6$, $f(x) = x^6 + 3$ and $f(x) = x^6 - 11$ are all anti-derivatives of $f'(x) = x^6$ and in general, the anti-derivative of $f'(x) = 6x^5$ is $f'(x) = x^6 + C$, where C is a constant.

integration

Integration is the process of finding anti-derivatives.

The German mathematician Gottfried Leibniz devised the notation that we use for integration. Thus we write $\int 6x^5\,dx = x^6 + C$

$\int 6x^5\,dx$ is read as "the indefinite integral of $6x^5$ with respect to x"

Similarly $\int (3t - 2)^5\,dt$ is read as "the indefinite integral of $(3t - 2)^5$ with respect to t"

In general, $\int f(x)\,dx = F(x) + C$ means $F'(x) = f(x)$

f (x) is called the **integrand**, F (x) is called the **integral** and C is the **constant of integration**.

integralWhen $\int f(x)\,dx = F(x) + C$ then F (x) is the **integral**.

constant of integration When $\int f(x)\,dx = F(x) + C$ then C is the

constant of integration.

integrand When $\int f(x)\,dx = F(x) + C$ then f (x),which is the function to be integrated, is called the **integrand**.

2.4 A rule for integration

Learning Objective

Integrate simple functions

Since

$$\frac{d}{dx}\left(x^n + C\right) = nx^{n-1}$$

then it follows that

$$\int nx^{n-1}\,dx = x^n + C$$

However, it is more useful to have a formula for integrating x^n

Notice that

$$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1} + C\right) = x^n$$

Hence by the reverse process

$$\boxed{\int x^n dx = \frac{x^{n+1}}{n+1} + C, \quad (n \neq -1)}$$

Examples

1. Find $\int x^7\,dx$

$$\int x^7\,dx = \frac{x^{7+1}}{7+1} + C$$
$$= \frac{1}{8}x^8 + C$$

2. Integrate x^{-3}

$$\int x^{-3}\,dx = \frac{x^{-3+1}}{-3+1} + C$$
$$= -\frac{1}{2}x^{-2} + C$$
$$= -\frac{1}{2x^2} + C$$

Notice that this answer has been expressed with a positive power f

Example Integrate $x^{-\frac{1}{3}}$

$$\begin{aligned}\int x^{-\frac{1}{3}}dx &= \frac{x^{-\frac{1}{3}+1}}{-\frac{1}{3}+1}+C\\ &= \frac{x^{\frac{2}{3}}}{\frac{2}{3}}+C\\ &= \frac{3}{2}x^{\frac{2}{3}}+C\end{aligned}$$

or x

Exercise 3

30 min

There is an on-line exercise at this point which you might find helpful.

Find the following integrals, expressing your answers with positive powers. Remember to include, C, the constant of integration.

Q12:

a) $\int x^4\ dx$
b) $\int x^9\ dx$
c) $\int t^{11}\ dt$
d) $\int x dx$
e) $\int y^{14}\ dy$
f) $\int s^{21}\ ds$

Q13:

a) $\int x^{-3}\ dx$
b) $\int p^{-5}\ dp$
c) $\int x^{-18}\ dx$
d) $\int u^{-6}\ du$

Q14:

a) $\int x^{2/3}\ dx$
b) $\int v^{5/4}\ dv$
c) $\int x^{-2/3}\ dx$
d) $\int x^{-7/5}\ dx$
e) $\int x^{-1/6}\ dx$
f) $\int r^{-7/6}\ dr$

2.5 Further rules for integration

Learning Objective

Integrate more complex functions

The following rules are useful when integrating more complex functions.

$$\int (f(x) \pm g(x))\, dx = \int f(x)\, dx \pm \int g(x)\, dx$$

$\int af(x)\, dx = a \int f(x)dx$ where a is a constant.

Examples

1. Find $\int 6x^2\, dx$

$$\int 6x^2\, dx = 6 \int x^2 dx$$
$$= 6\frac{x^3}{3} + C$$
$$= 2x^3 + C$$

2. Find $\int 4dx$

$$\int 4dx = \int 4x^0 dx \quad \text{(Remember that } x^0 = 1\text{)}$$
$$= 4 \int x^0 dx$$
$$= 4x + C$$

3. Integrate $7x^2 + 5x^{1/4} - 1$

$$\int \left(7x^2 + 5x^{1/4} - 1\right) dx = 7 \int x^2 dx + 5 \int x^{1/4} dx - 1 \int x^0 dx$$
$$= 7\frac{x^3}{3} + 5\,\frac{x^{5/4}}{\frac{5}{4}} - x + C$$
$$= \frac{7}{3}x^3 + 4x^{5/4} - x + C$$

Now try the questions in the following exercise

Exercise 4

30 min

There is an on-line exercise at this point which you might find helpful.

Find the following integrals, expressing your answers with positive powers. Remember to include, C, the constant of integration.

Q15:

a) $\int 5x^2\ dx$

b) $\int 8x^3\ dx$

c) $\int 9t^{-7}\ dt$

d) $\int 8dx$

e) $\int 12x^{1/2}\ dx$

f) $\int -7v^{-3}\ dv$

g) $\int -3dy$

h) $\int 4a^{-2/3}\ da$

i) $\int 8x^{-7/5}\ dx$

j) $\int -6x^{-7/4}\ dx$

Q16:

a) $\int \left(6x^2 + 5x\right) dx$

b) $\int \left(3x^5 - 6x^2 + 7\right) dx$

c) $\int \left(5t^{3/2} - 1\right) dt$

d) $\int \left(7y^{-4} + 2y + 8\right) dy$

e) $\int \left(4x^{-2/3} + x^{2/3}\right) dx$

f) $\int \left(3 - 10x^{-11/6} + 7x^{-3}\right) dx$

2.6 Integrating products and quotients

Learning Objective

Integrate functions that are expressed as products and quotients.

Before integrating more complex functions we must express the **integrand** as the sum of individual terms in the form ax^n

Examples

1. Find $\int (2p - 3)^2\, dp$

$$\int (2p - 3)^2 dp = \int \left(4p^2 - 12p + 9\right) dp$$
$$= \tfrac{4}{3}p^3 - 6p^2 + 9p + C$$

2. Find $\int \left(\dfrac{u^2 + 2}{\sqrt{u}}\right) du$

$$\int \left(\frac{u^2 + 2}{\sqrt{u}}\right) du = \int \left(\frac{u^2}{u^{1/2}} + \frac{2}{u^{1/2}}\right) du$$
$$= \int \left(u^{3/2} + 2u^{-1/2}\right) du$$
$$= \tfrac{2}{5}u^{5/2} + 4u^{1/2} + C$$
$$= \tfrac{2}{5}\sqrt{u}^5 + 4\sqrt{u} + C$$

Exercise 5

40 min

There is an on-line exercise at this point which you might find helpful.

Q17: Find the following integrals, giving your answer with positive indices.

a) $\int \sqrt[4]{x}dx$

b) $\int \sqrt[3]{x^2}\, dx$

c) $\int 7\sqrt{x^5}\, dx$

d) $\int \dfrac{dx}{x^3}$

e) $\int \dfrac{dx}{3x^2}$

f) $\int \dfrac{-\, dx}{2x^5}$

g) $\int \dfrac{5dx}{\sqrt[3]{x^2}}$

h) $\int \dfrac{dx}{5\sqrt{x^3}}$

Q18:

a) $\int (x - 5)^2 \, dx$

b) $\int (2t + 1)^2 \, dt$

c) $\int x \left(x^2 + \frac{2}{x}\right) \, dx$

d) $\int \sqrt{u} \left(u^2 - 4\right) \, du$

e) $\int \left(v + \frac{1}{v}\right)^2 \, dv$

f) $\int p \, (3p - 1)) \, (p + 2) \, dp$

Q19:

a) $\int \frac{3x^5 + x^4}{x^2} \, dx$

b) $\int \frac{5 - t^6}{t^3} \, dt$

c) $\int \frac{x + 1}{\sqrt{x}} \, dx$

d) $\int \frac{v^2 + v - 4}{\sqrt{v}} \, dv$

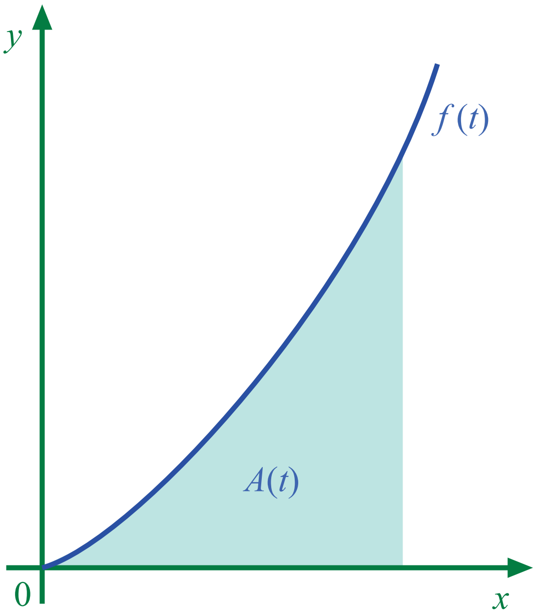
e) $\int \frac{\left(a^2 - 3\right)^2}{a^2} \, da$

f) $\int \left(t + \frac{5}{t}\right)^2 \, dt$

2.7 The Area under a Curve

Learning Objective

Identify the area under a curve as a definite integral and vice-versa

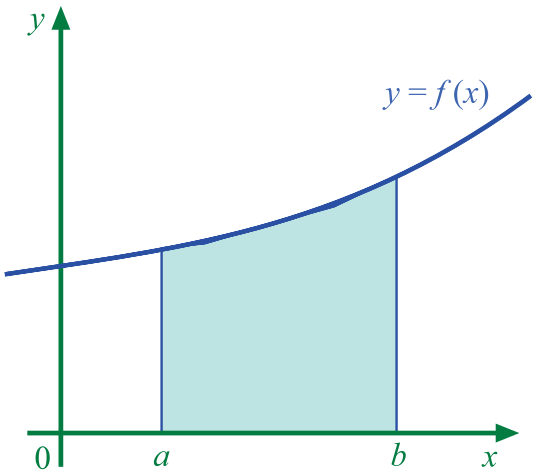


At the start of this topic we saw that when the equation of a curve is f (t) = (n + 1)t^{n} then the area under the curve can be calculated from the formula A (t) = t^{n+1}

We also noticed that A (t) is the anti-derivative of f (t) (apart from a constant) thus

$\int f(t)\, dt = A(t) + C$

This idea can be extended as follows.



The area between the graph y + f(x) and the x-axis from x + a to x = b is

$$\int_a^b f(x)\, dx$$

This is called a **definite integral**, a is the **lower limit of integration** and b is the **upper limit of integration**.

lower limit of integration a is the **lower limit of integration** in the definite integral

$$\int_a^b f(x)\, dx$$

upper limit of integrationb is the **upper limit of integration** in the definite integral

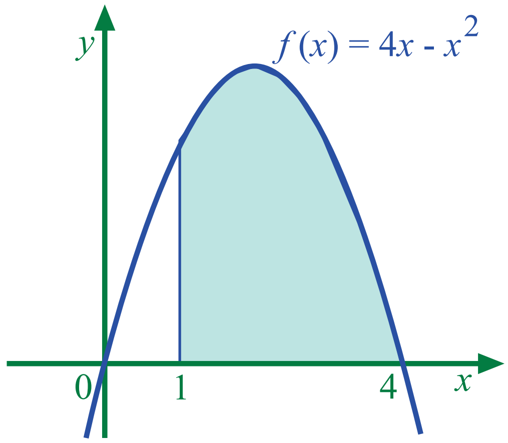
$$\int_a^b f(x)\, dx$$

definite integralA **definite integral** is one which has a numerical value thus when F (x) is the anti-derivative of f (x) then

$$\int_a^b f(x)\, dx = F(b) - F(a) \quad (a \leqslant x \leqslant b)$$

Example

For the diagram shown here write the shaded area as a definite integral.



The limits of integration are from x = 1 to x = 4 and f (x) = 4 x - x^2 thus

Area = $\int_1^4 \left(4x - x^2\right)\, dx$

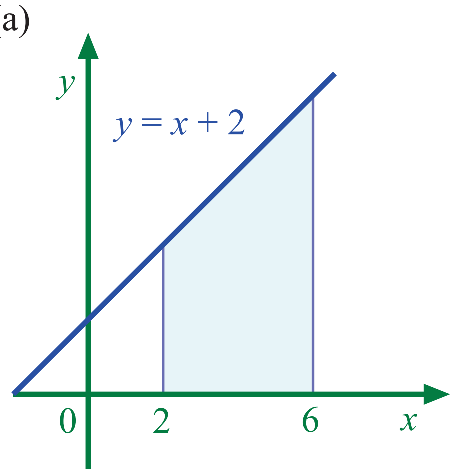
Exercise 6

There is an on-line exercise at this point which you might find helpful.

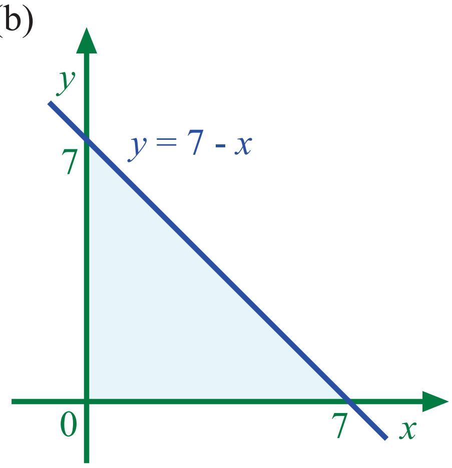
30 min

Q20: Write down the definite integrals associated with the shaded areas in the following diagrams.

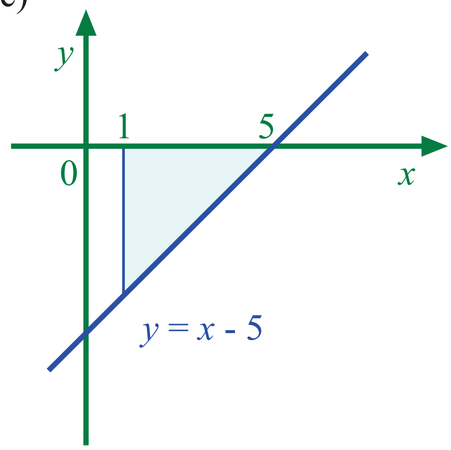
(a)



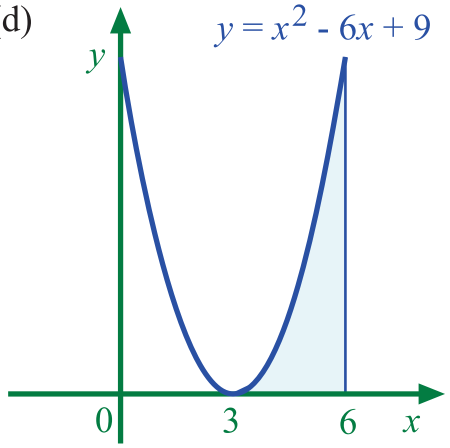
(b)



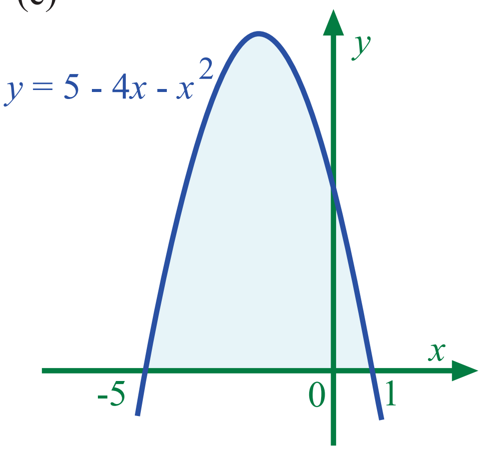
(c)



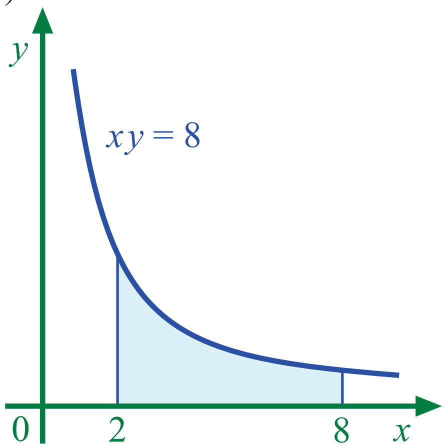
(d)



(e)



(f)



Q21: Sketch the areas associated with the following definite integrals.

a) $\int_{1}^{5} (x + 2)\, dx$

b) $\int_{0}^{8} \frac{1}{2} x dx$

c) $\int_{-2}^{2} (3 - x)\, dx$

d) $\int_{1}^{3} x^2\, dx$

e) $\int_{-1}^{2} \left(4 - x^2\right) dx$

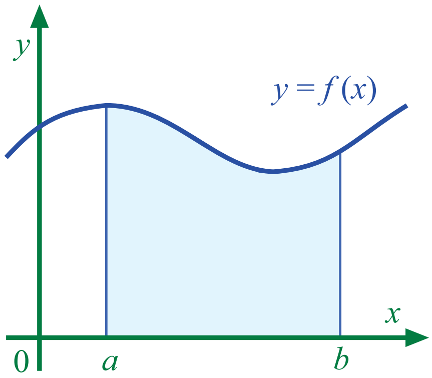
f) $\int_{0}^{5} \left(x^2 - 6x + 9\right)\, dx$

2.8 The fundamental theorem of calculus

Learning Objective

Evaluate definite integrals

The area between the graph of y = f (x) and the x-axis from x = a to x = b can be calculated as the area from x = 0 to x = b minus the area from x = 0 to x = a



Thus if F (x) is an anti-derivative of f (x) then we can write

$$\int_a^b f(x)\,dx = (F(b) + C) - (F(a) + C)$$
$$= F(b) - F(a) \qquad (a \leqslant x \leqslant b)$$

This result is known as the **fundamental theorem of calculus**.

Note that C is cancelled out when calculating a definite integral.

fundamental theorem of calculus The **fundamental theorem of calculus** states that if F (x) is an anti-derivative of f (x) then

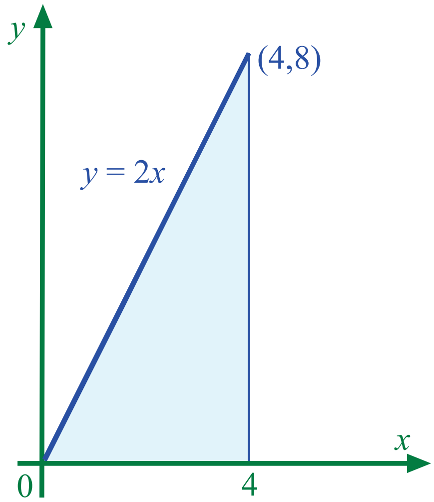
$$\int_a^b f(x)\,dx = (F(b) + C) - (F(a) + C)$$
$$= F(b) - F(a) \qquad (a \leqslant x \leqslant b)$$

The following examples may make this clearer.

Examples

1.

Find the area under the straight line $y = 2x$ between $x = 0$ and $x = 4$



Since this area is a triangle it can be calculated as

Area $= \frac{1}{2} \times$ base $\times$ height

$= \frac{1}{2} \times 4 \times 8$

$= 16$ units^2

Compare this with the integral

$$\int_0^4 2x\,dx = \left[x^2\right]_0^4$$

$$= \left(4^2\right) - \left(0^2\right)$$

$$= 16$$

Thus the definite integral gives the area under the line which is consistent with our formula for the area of a triangle.

2. Evaluate:

$\int_1^3 \left(x^2 + 2x + 1\right) dx$

$$\begin{aligned}\int_1^3 \left(x^2 + 2x + 1\right) dx &= \left[\frac{x^3}{3} + x^2 + x\right]_1^3 \\ &= \left(\frac{3^3}{3} + 3^2 + 3\right) - \left(\frac{1^3}{3} + 1^2 + 1\right) \\ &= (21) - \left(2\tfrac{1}{3}\right) \\ &= 18\tfrac{2}{3} \text{ units}^2\end{aligned}$$

Exercise 7

50 min

There is an on-line exercise at this point which you might find helpful.

Evaluate the following definite integrals.

Q22:

a) $\int_0^3 x^2 dx$

b) $\int_{-5}^2 3dx$

c) $\int_1^3 4t^3 dt$

d) $\int_1^4 6x dx$

e) $\int_{-2}^4 (2v + 5)\, dv$

f) $\int_0^3 \left(x^3 - 2\right) dx$

Q23: Evaluate

a) $\int_1^{16} x^{-1/2} dx$

b) $\int_1^9 3x^{1/2} dx$

c) $\int_2^5 u^{-2} du$

d) $\int_1^2 \frac{12}{x^3} dx$

e) $\int_4^{25} \sqrt{s} ds$

f) $\int_{1}^{125} 8\sqrt[3]{x}dx$

Q24:

a) $\int_{0}^{4} t\left(t^2 - 3\right) dt$

b) $\int_{2}^{6} (2x + 1)^2 dx$

c) $\int_{1}^{3} v^2 \left(2 - \frac{3}{v}\right) dv$

d) $\int_{2}^{4} \frac{3x^3 - 2}{x^2}dx$

e) $\int_{1}^{4} \left(x - \frac{1}{x}\right)^2 dx$

f) $\int_{0}^{9} \frac{5x^2 - 4x - 1}{\sqrt{x}}dx$

Q25: Find the value of a, (a $>$ 0) for which

a) $\int_{0}^{a} (3x - 5)\ dx = 4$

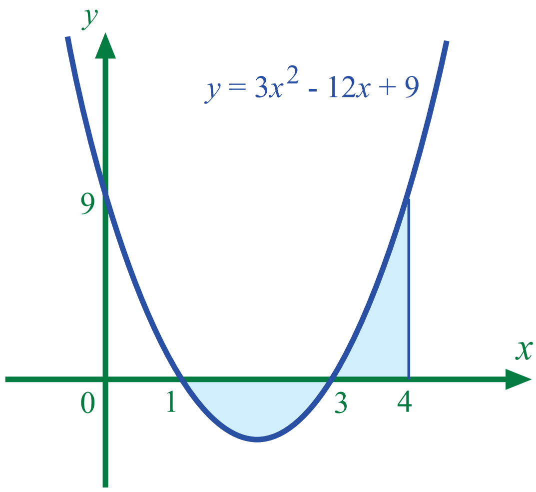
b) $\int_{0}^{a} \sqrt{x}dx\ = 18$

2.9 Calculating areas

Learning Objective

Calculate areas above and below the x-axis

The graph for $y = 3x^2 - 12x + 9$ is shown here.



Evaluate the following definite integrals for the above graph.

Q26:

$$\int_{1}^{3} \left(3x^2 - 12x + 9\right)\, dx$$

Q27:

$$\int_{3}^{4} \left(3x^2 - 12x + 9\right) dx$$

You should notice that the value of the integral is

- ***positive*** when the area is ***above*** the x- axis
- ***negative*** when the area is ***below*** the x- axis

Q28:

Evaluate the following integral.

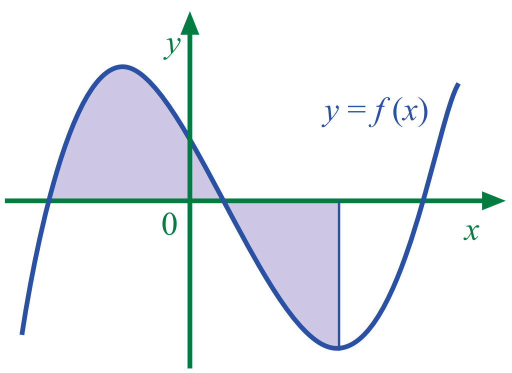
$$\int_{1}^{4} \left(3x^2 - 12x + 9\right)\, dx$$

Q29: Explain the answer to this last integration.

Q30: What is the total shaded area in the previous diagram?

To calculate an area between a curve and the x -axis

1. make a sketch, noting where the curve crosses the x -axis
2. calculate the areas above and below the axis seperately
3. ignore the negative sign for areas below the axis
4. add areas together.



Now try the questions in the following exercise.

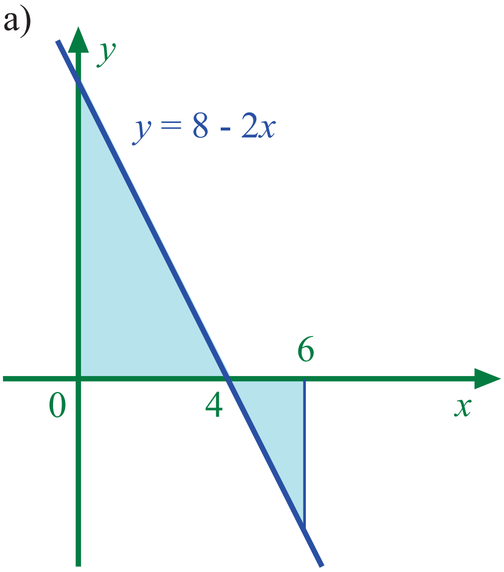
60 min

Exercise 8

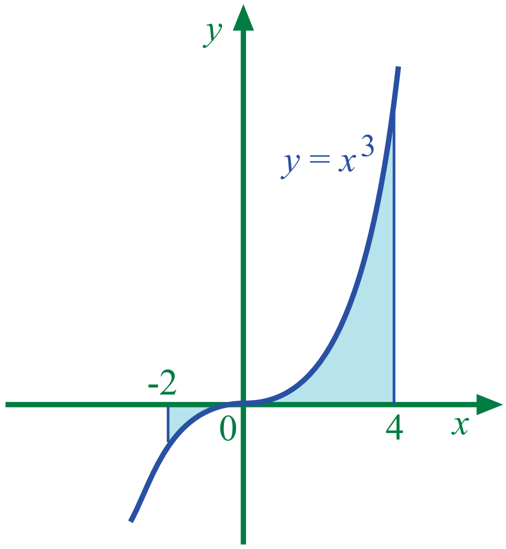
There is an on-line exercise at this point which you might find helpful.

Q31: Calculate the total shaded area for each of the following graphs.

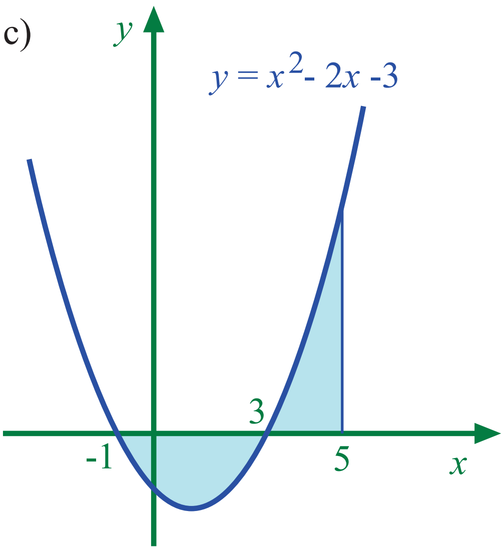
a)



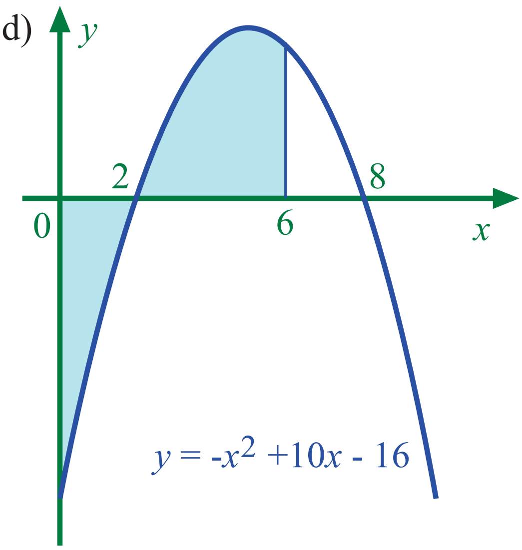
b)



c)

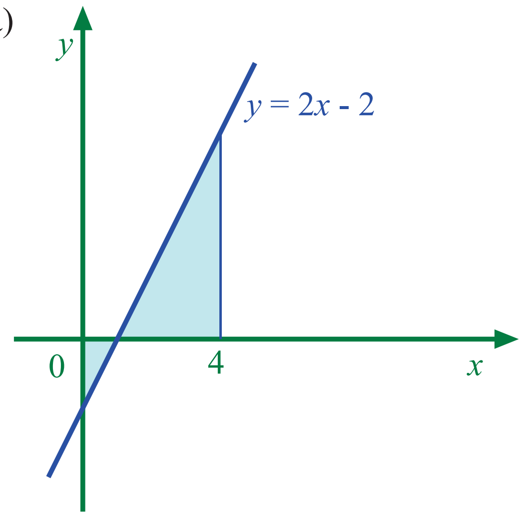


d)

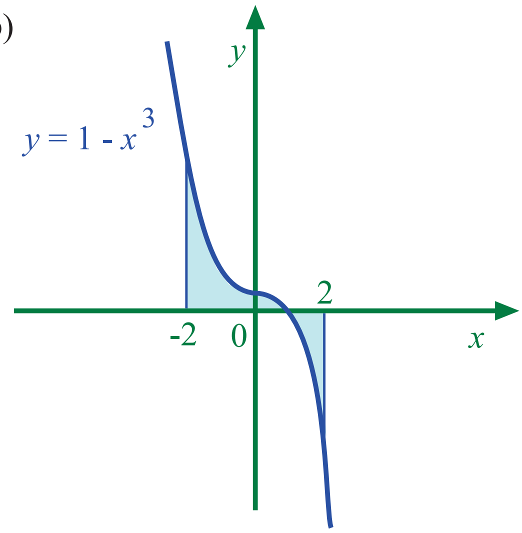


Q32: For each of the following graphs identify where the curve cuts the x-axis and hence calculate the total shaded area.

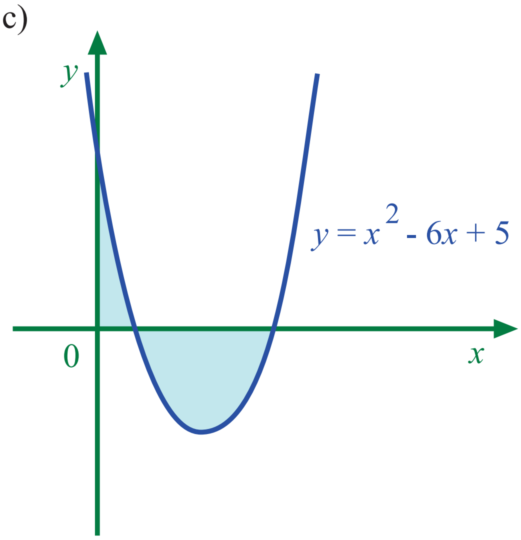
a)



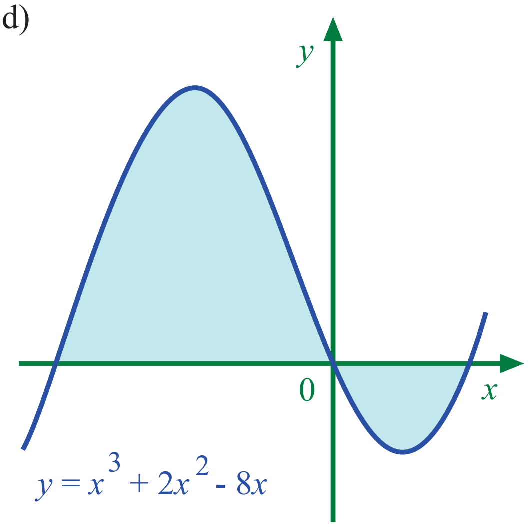
b)



c)



d)



Q33: Sketch the following curves and hence find the total area between the curve, the x-axis and the given lines.

a) $y = \frac{1}{2}x - 1, \quad x = 0, \quad x = 6$

b) $y = 12 - 4x - x^2, \quad x = -3, \quad x = 3$

c) $y = x^3 + 3x^2, \quad x = -4, \quad x = 2$

Q34: Calculate the area ***enclosed*** by the graph of the following functions and the x-axis

a) $y = x^2 - 9$

b) $y = 2x - x^2$

Q35:

a) Factorise the polynomial $x^3 - 3x - 2$ and hence find where the curve $y = x^3 - 3x - 2$ cuts the x-axis

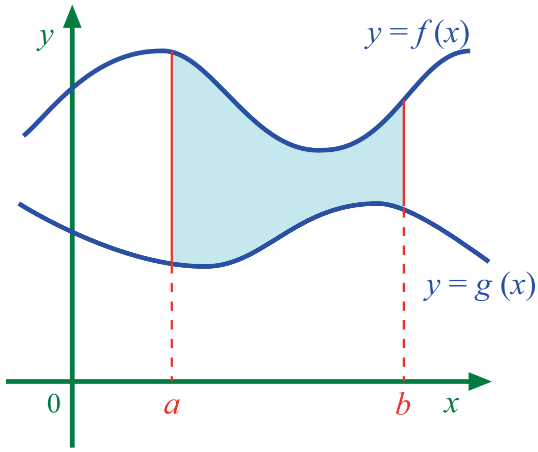
b) Find the stationary points and their nature and hence make a good sketch of the curve.

c) Find the total area between the curve , the x-axis and the lines x = -1 and x = 4

2.10 The area between two graphs

Learning Objective

Calculate the area between two graphs



The shaded area in the diagram opposite is between the curves y = f (x) and y = g (x) from x = a to x = b

The shaded area

= the area from the x-axis to y = f (x)

minus the area from the x-axis to y = g (x)

$$= \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

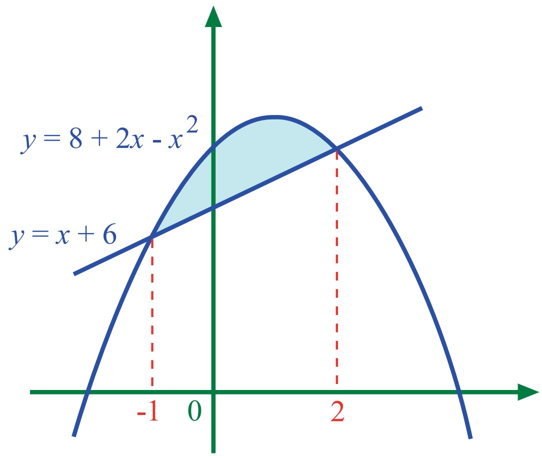
$$= \int_a^b \left(f(x) - g(x)\right)\,dx$$

Example Find the area enclosed between the line y = x + 6 and the curve y = 8 + 2x - x^2

First we should find the points of intersection between the line and the curve.

The graphs intersect where

$x + 6 = 8 + 2x - x^2$
$\Rightarrow x^2 - x - 2 = 0$
$\Rightarrow (x - 2)(x + 1) = 0$
$\Rightarrow x = 2$ or $x = -1$



Notice that between x = -1 and x = 2 the graph for y = 8 + 2x - x^2 is above that for y = x + 6
Thus the enclosed area is given by

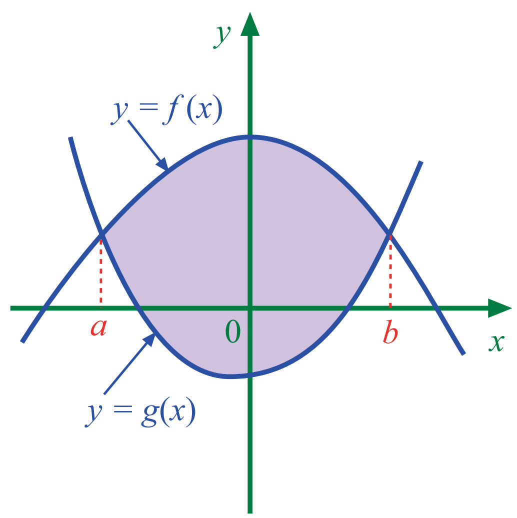
$$\begin{aligned}\text{Area} &= \int_{-1}^{2} \left(\left(8 + 2x - x^2\right) - (x + 6)\right) dx \\ &= \int_{-1}^{2} \left(2 + x - x^2\right) dx \\ &= \left[2x + \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3\right]_{-1}^{2} \\ &= 3\tfrac{1}{3} - \left(-1\tfrac{1}{6}\right) \\ &= 4\tfrac{1}{2} \text{ square units}\end{aligned}$$

To calculate an area between two graphs

1. make a sketch
2. calculate the points of intersection where f (x) = g (x)
3. note which graph is above the other between the points of intersection.

For the diagram shown here f (x) $\geq$ g (x) for a $\leq$ x $\leq$ b and thus the area enclosed by the curves is given by

$$\int_a^b (f(x) - g(x))\,dx$$



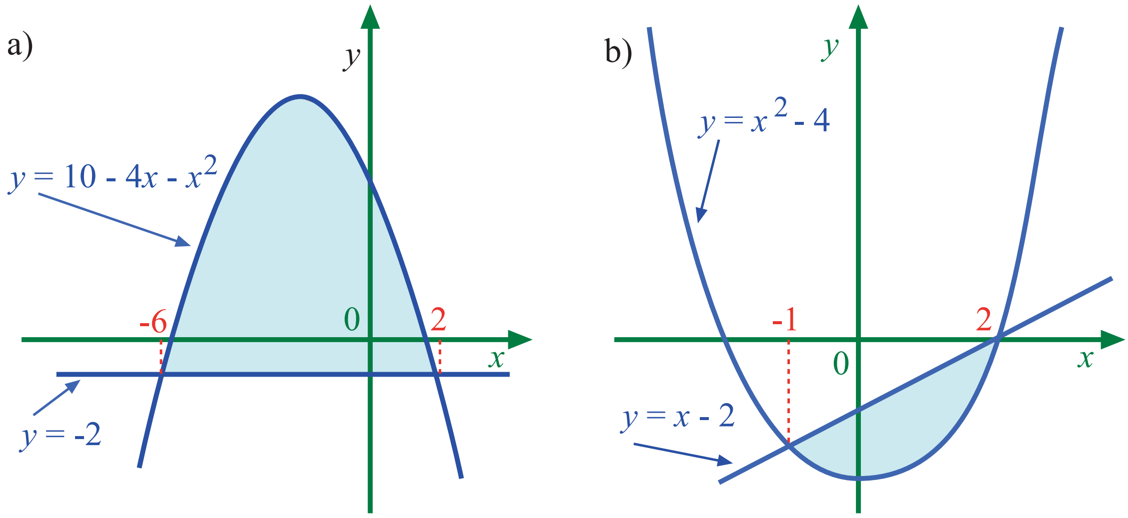
Now try the questions in the following exercise.

60 min

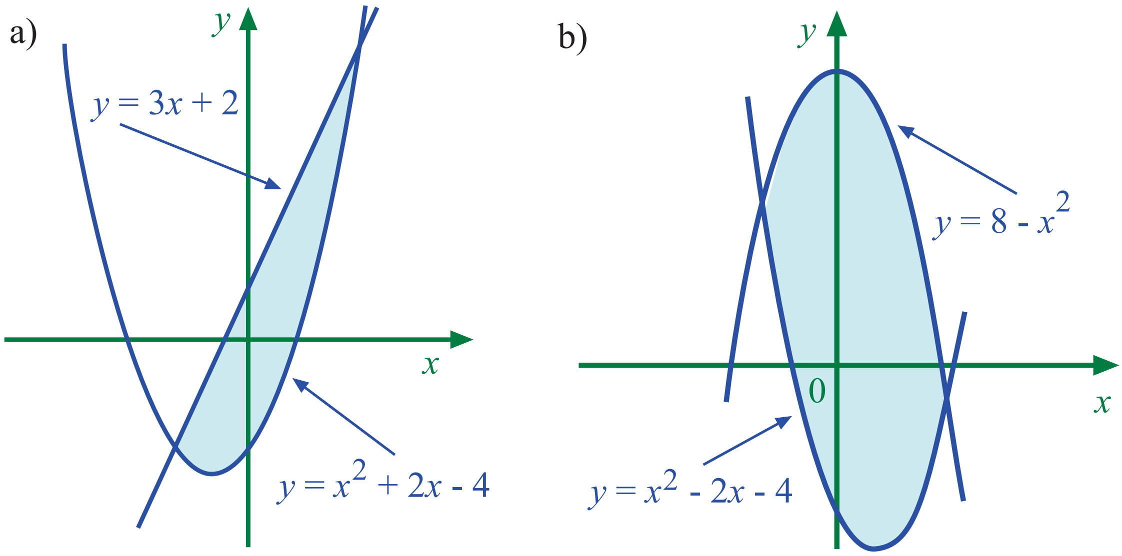
Exercise 9

There is an on-line exercise at this point which you might find helpful.

Q36: Calculate the shaded area in the following diagrams



Q37: Find the points of intersection in the following diagrams and hence calculate the shaded areas.



Q38: Calculate the area enclosed by each of the following pairs of graphs. Remember to make a sketch.

a) $y = 3$ and $y = x^2 - 2x$

b) $y = x - 1$ and $y = x^2 - 4x + 3$

c) $y = x^2 - 2x - 2$ and $y = 4 + 2x - x^2$

d) $y = 10 - 2x - x^2$ and $y = 2x^2 + x - 8$

Q39:

The curve $y = x^3 - 4x$ and the line

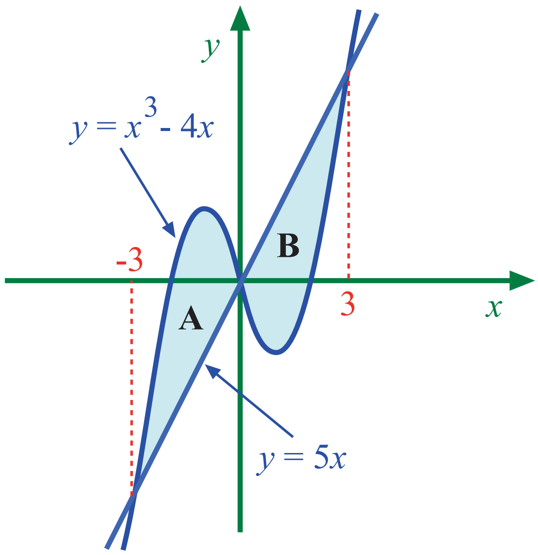
$y = 5x$ are shown in the diagram opposite. Notice that the enclosed area is in two parts.

a) Calculate area A

b) Calculate area B

c) Write down the total shaded area.

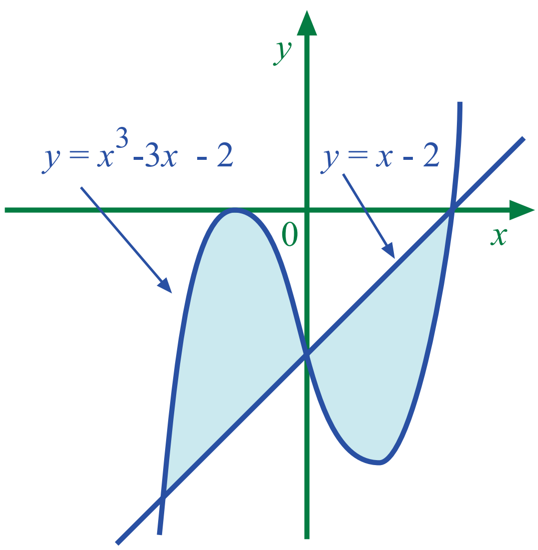
d) Why must these areas be calculated seperately?



Q40:

a) Calculate the coordinates of the points of intersection between the curve $y = x^3 - 3x - 2$ and the line $y = x - 2$

b) Calculate the total area enclosed between these two graphs.



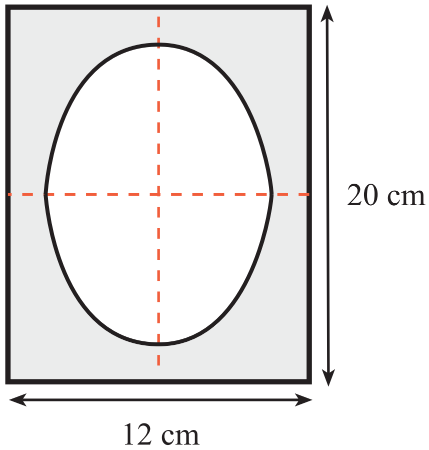
Q41:

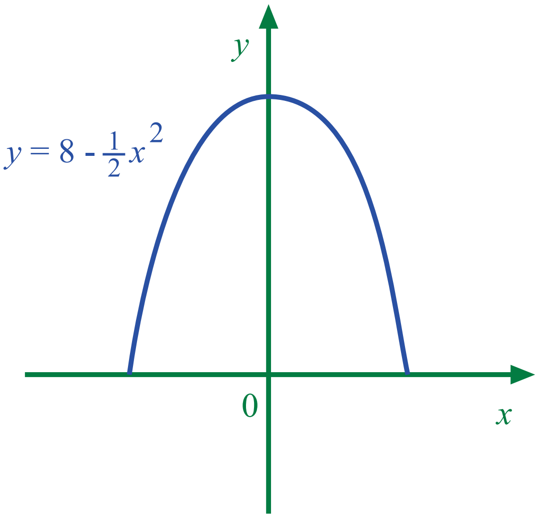
A silver photograph frame is rectangular in shape with a symmetric parabolic shaped space in the centre.

The coordinate diagram shows the outline of the top half of the parabola.

Calculate the surface area of silver in the photograph frame

(1 unit = 1 cm)





Q42: Two particles are released from the origin at t= 0(seconds). The two particles travel in a straight line with different speeds (v m/s) given by $v = 12t - t^2$ and $v = 0.5t^2$

a) How far has each particle travelled from the origin after 10 seconds.

b) After how many seconds will the particles have travelled the same distance from the origin?

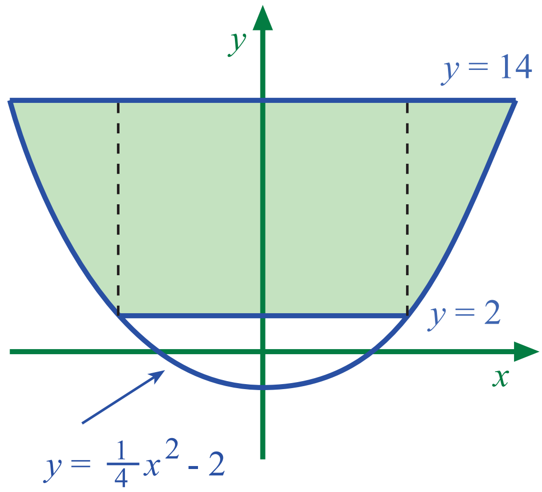
Q43:

The shaded part of the diagram shown opposite represents an area of ground that is to be planted with grass seed.

It is shaped like a parabola with equation $y = \frac{1}{4}x^2 - 2$ between the lines y = 2 and y = 14

Calculate the area to be planted.

(1 unit = 1 metre)



Float the Boat

10 min

2.11 Differential equations

Learning Objective

Solve differential equations of the form $\frac{dy}{dx} = f(x)$

$\frac{dy}{dx} = 2x^2 + 5$ and $\frac{ds}{dt} = 3t - 1$ are examples of differential equations. This type of equation can usually be solved by integration.

Differential equation A **differential equation** is an equation involving an unknown function and its derivatives.

Example The gradient of the tangent to a curve is given by the **differential equation** $\frac{dy}{dx} = 2x$

Find the equation of the curve.

We can solve this equation by integrating

$$y = \int \frac{dy}{dx} dx = \int 2x \, dx$$

$$y = x^2 + C$$

$y = x^2 + C$ gives a family of curves each of which satisfies the differential equation.

Thus, for example, the curves

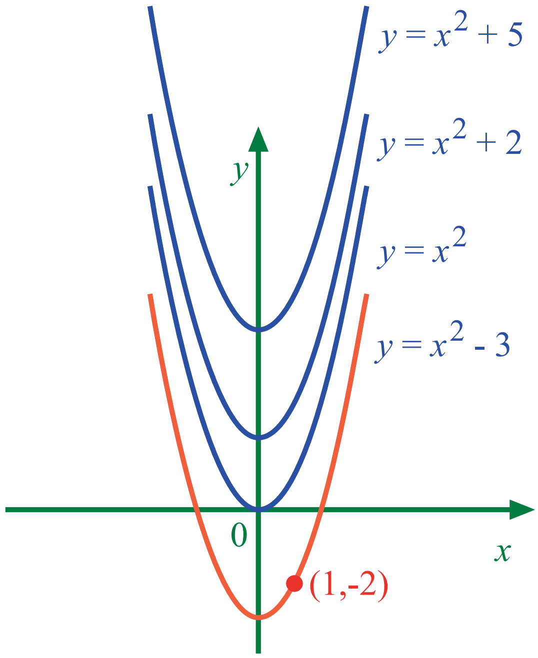
$y = x^2 + 5$

$y = x^2 + 2$

$y = x^2$

$y = x^2 - 3$

all satisfy $\frac{dy}{dx} = 2x$



If we are given more information then we can identify one curve in particular. For example if we are given the initial condition that the curve passes through the point (1,-2) then we can identify C and thus a particular curve.

Since the curve passes through the point (1,-2) then we can write

$$y = x^2 + C$$
$$-2 = 1^2 + C$$
$$\Rightarrow C = -3$$

Thus the curve that passes through the point (1,-2) has equation $y = x^2 - 3$

Note that:

- $y = x^2 + C$ is a general solution of the differential equation
- $y = x^2 - 3$ is a particular solution of the differential equation

general solution The **general solution** of a differential equation contains an arbitrary constant and gives infinitely many solutions that all satisy the differential equation.

particular solution The **particular solution** of a differential equation is a solution which is often obtained from the general solution when an initial condition is known.

initial condition For a differential equation an **initial condition** is additional information required to determine a particular solution. This could be a coordinate on a curve, a velocity at t = 0, the amount of money in a bank account on 1st January 2000, etc.

Example At the start of a race a cyclist accelerates so that his speed after t seconds is given by $v = t^2$
How far will he have travelled in the first 6 seconds?

Let s represent the distance that the cylist travels after t seconds then we can write

$v = \frac{ds}{dt} = t^2$

We can find the formula for the distance (s) by integrating

$$s = \int \frac{ds}{dt} dt = \int t^2 dt$$
$$s = \tfrac{1}{3}t^3 + C$$

Since the cyclist has not moved at the start of the race then we have the initial condition s = 0 when t = 0. Hence C = 0 and the **particular solution** is $s = \frac{1}{3}t^3$

Also when $t = 6,\quad s = \frac{1}{3}6^3 = 72$ metres

The cyclist travels 72 metres in the first 6 seconds

Exercise 10

30 min

There is an on-line exercise at this point which you might find helpful.

Q44: Find the general solutions for each of these differential equations.

a) $\frac{dy}{dx} = 4x$

b) $\frac{ds}{dt} = 6t^2 - 5$

c) $\frac{dy}{dx} = 3 - x$

d) $\frac{ds}{dt} = \frac{1}{2}t + 3t^2$

Q45: Find the particular solutions for each of these differential equations.

a) $\frac{ds}{dt} = 5t$ and s = 0 when t = 0

b) $\frac{dy}{dx} = 3x^2 + x - 5$ and y = 4 when x = 0

c) $\frac{ds}{dt} = -3t^2 + t$ and s = 0 when t = 2

d) $\frac{dy}{dx} = 9\sqrt{x}$ and y = 50 when x = 4

Q46: The gradient of the tangent to a curve is given by the differential equation $\frac{dy}{dx} = 2x + 4$. The curve passes through the point (2,9). Find the equation of the curve.

Q47: Find the equation of the function y = f(x) for which $\frac{dy}{dx} = \frac{1}{2}x - 2$ and f(2) = 0

Q48: A particle is moving in a straight line so that its acceleration (in m/s^2) is described by the formula, $a = 8 + 3t$
Find the velocity of the particle at t = 2 given that v = 0 when t = 0
(Acceleration = rate of change of velocity i.e. $a = \frac{dv}{dt}$)

Q49: A water tank is being filled so that the rate of change of volume (in m^3/s) is given by $\frac{dV}{dt} = 3 + \frac{1}{5}t$

Given that the water tank was initially empty calculate the volume of water in the tank after 5 seconds.

2.12 Summary

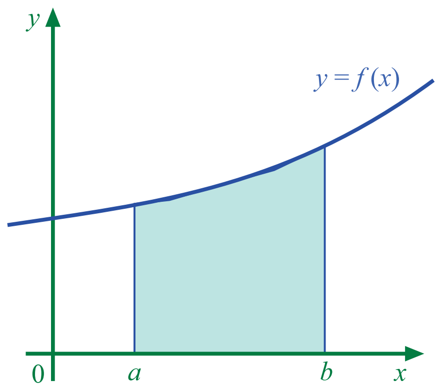
Learning Objective

Recall the main learning points from this topic

1. Anti-differentiation is the reverse process to differentiation.
2. f(x)dx is an indefinite integral.
   If F(x) is an anti-derivative of f(x) then $\int f(x)dx = F(x) + C$
   C is the constant of integration
3. ***Some rules for integration.***
   - $\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$
   - $\int (f(x) + g(x))\, dx = \int f(x)dx + \int g(x)dx$
   - $\int af(x)dx = a \int f(x)dx \quad$ (a is a constant)
4. ***The area under a curve***

   The area between the graph y = f(x) and the x-axis from x = a to x = b is given by the definite integral

   $$\int_a^b f(x)dx$$

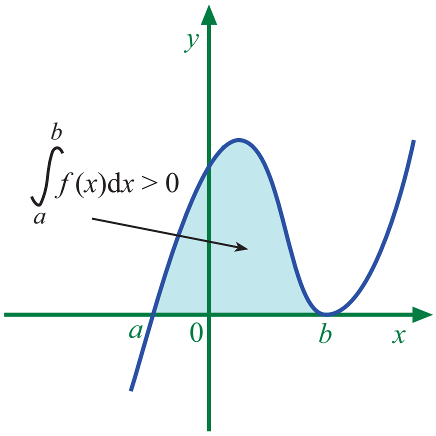


5. ***The Fundamental Theorem of Calculus.***

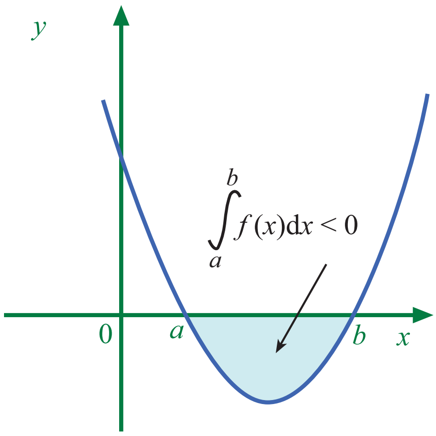
   If F(x) is an anti-derivative of f(x) then

   $$\int_a^b f(x)dx = F(b) - F(a) \quad (a \leqslant x \leqslant b)$$

6. An area ***above*** the x-axis will give a positive value for the integral.

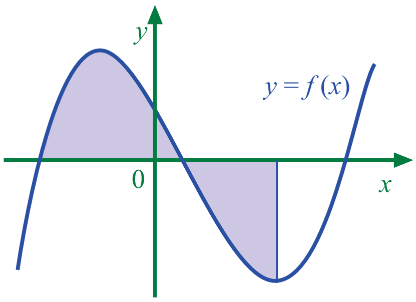


An area ***below*** the x-axis will give a negative value for the integral.



7. To calculate an area between a curve and the x-axis

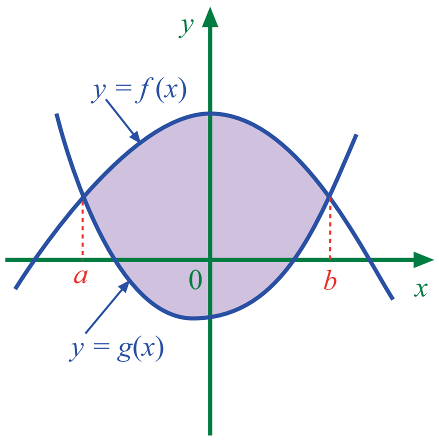
    1. make a sketch
    2. calculate the areas above and below the axis seperately
    3. ignore the negative sign for areas below the axis
    4. add areas together.



8. ***The area between two curves.***

To calculate an area between two graphs

1. make a sketch
2. calculate points of intersection
3. note which graph is above the other between the points of intersection.



For the diagram shown here $f(x) \geqslant g(x)$ for $a \leqslant x \leqslant b$ and thus the area enclosed by the curves is given by

$$\int_a^b (f(x) - g(x))\, dx$$

9. $\frac{dy}{dx} = f(x)$ is a differential equation.

The general solution of this differentiatal equation is

$y = \int f(x)dx$

2.13 Extended Information

There are links on the web which give a variety of web sites related to this topic.

2.14 Review exercise

Review exercise in basic integration

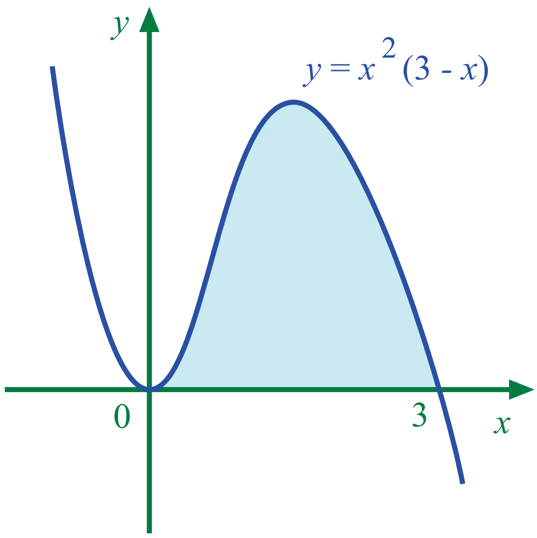
15 min

There is an on-line exercise at this point which you might find helpful.

Q50: Find $\int \frac{6}{x^4}dx$

Q51:

Calculate the shaded area shown in the diagram.

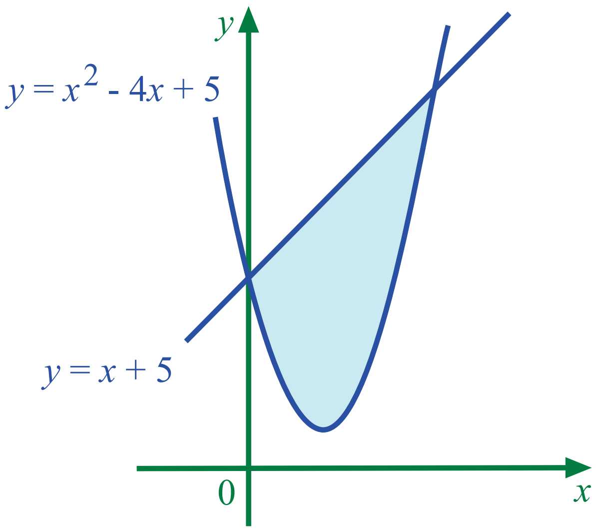


Q52:

The diagram shows the line with equation y = x + 5 and the curve with equation $y = x^2 - 4x + 5$

Write down the integral which represents the shaded area.

Do **not** carry out the integration.



2.15 Advanced review exercise

Advanced review exercise in basic integration

There is an on-line exercise at this point which you might find helpful.

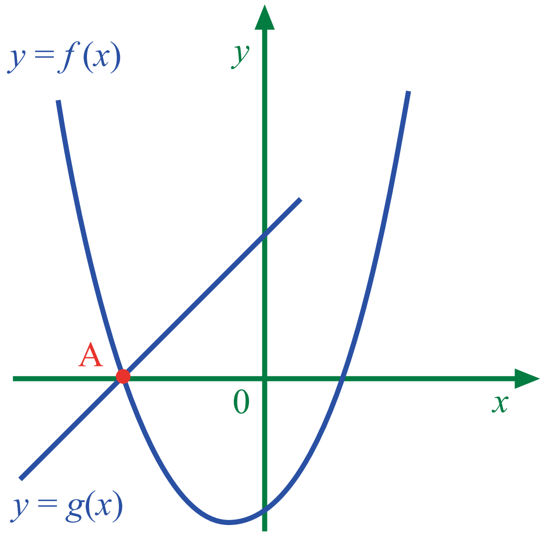
30 min

Q53: Evaluate $\int_{-1}^{1} \left(10 - 3x^2\right)$ dx and draw a sketch to illustrate the area represented by this integral.

Q54:

The graphs of y = f(x) and y = g(x) intersect at the point A on the x-axis, as shown in the diagram.

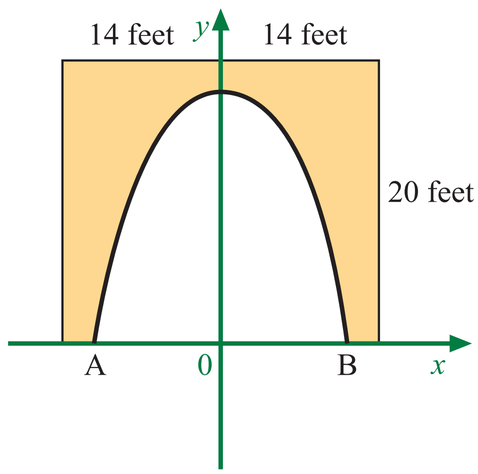
If g(x) = x + 2 and f'(x) = 4x + 1, find f(x)



Q55:

The concrete on the 20 feet by 28 feet rectangular facing of the entrance to an underground cavern is to be repainted.

Coordinate axes are chosen as shown in the diagram with a scale of 1 unit equal to 1 foot. The roof is in the form of a parabola with equation $y = 18 - \frac{1}{8}x^2$



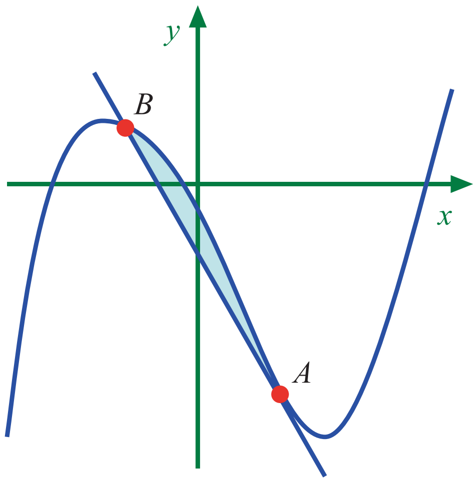
a) Find the coordinates of the points A and B.

b) Calculate the total cost of repainting the facing at £3 per square foot.

(Higher Mathematics)

Q56:

In the diagram, a winding river has been modelled by the curve $y = x^3 - x^2 - 6x - 2$ and a road has been modelled by the straight line AB. The road is a tangent to the river at the point A(1,-8).



a) Find the equation of the tangent at A and hence find the coordinates of B.

b) Find the area of the shaded part which represents the land bounded by the river and the road.

(Higher Mathematics)

2.16 Set review exercise

Set review exercise in basic integration

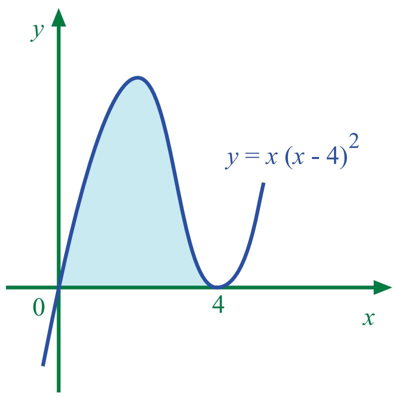
There is an on-line exercise at this point which you might find helpful.

15 min

Q57: Find $\int \frac{12}{x^3} dx$

Q58:

Calculate the shaded area shown in the diagram.

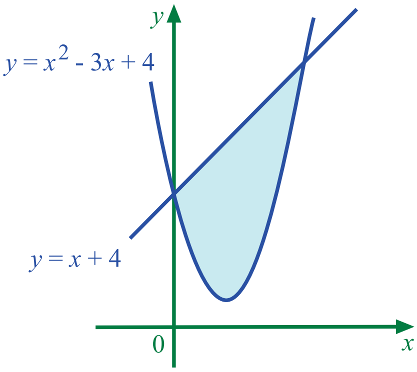


Q59:

The diagram shows the line with equation y = x + 4 and the curve with equation

$y = x^2 - 3x + 4$

Write down the integral which represents the shaded area.
Do **not** carry out the integration.



Topic 3

Trigonometric formulae

Contents

Learning Objectives

- *Solve trigonometric equations and apply trigonometric formulae*

Minimum performance criteria:

- *Solve a trigonometric equation in a given interval*
- *Apply a trigonometric formula (addition formula) in the solution of a geometric problem*
- *Solve a trigonometric equation involving an addition formula in a given interval*

Prerequisites

A sound knowledge of the following subjects is required for this unit:

- Exact values of trigonometric ratios
- The graphs of sine, cosine and tangent and the related properties
- Geometric properties of triangles

3.1 Revision exercise

15 min

Revision exercise

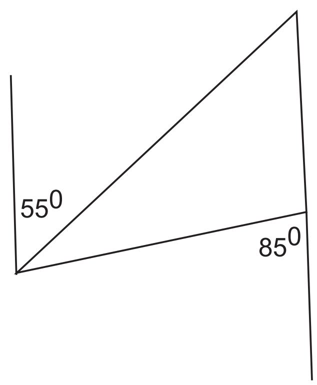
Q1: State the exact value of the following ratios:

a) $\cos 210°$
b) $\tan 135°$
c) $\sin {}^{2\pi}/_{3}$
d) $\cos {}^{\pi}/_{4}$
e) $\tan 330°$
f) $\sin 225°$

Q2: State the period in degrees and the amplitude where applicable, for the graphs of the following functions:

a) $\cos 2x$
b) $-2 \sin x$
c) $\tan 3x$
d) $\tan (-x)$
e) ${}^{1}/_{3} \cos 4x$
f) $\sin ({}^{2x}/_{3})$

Q3: Find the angles in the triangle:



Q4: If $\cos x = {}^{3}/_{5}$ in a right angled triangle, what is $\sin x$?

3.2 Solving trigonometric equations graphically

Learning Objective

Solve trigonometric equations graphically

In the trigonometric graphs section of topic 2 there were many examples of trigonometric functions with their corresponding graphs.

Techniques were explained that make exact angle calculations and the sketching of graphs of more complicated trigonometric functions such as 2 cos $(3x - 45)^\circ$ easier.

Trigonometric graphs

10 min

There are two animations on the web which will serve as a reminder on sketching trigonometric graph functions.

These demonstrations show the graphs of expressions. There are of course related equations. This was discussed in topic 5 on quadratic theory.

Recall that, for example, 2x - 3 is an expression but 2x - 3 = 6 is an equation.

The following are examples of trigonometric equations.

2 cos 3x = 0.6

tan (x + π) = 2

sin $(4x - 60)^\circ$ = -0.3

The graphs of the expressions 2 cos 3x, tan (x + π) and sin $(4x - 60)^\circ$ are straightforward to sketch and can be used to solve the equations.

For example, graphs can be used to solve 2 cos 3x = 0.6

The technique is to arrange the equation so that the trig function is on one side and the constant term is on the other (as in all the examples shown earlier).

Having done this, both sides of the equation can be sketched independently and any point of intersection between the two is a solution of the equation.

The first example demonstrates the general technique and this is followed by more specific examples.

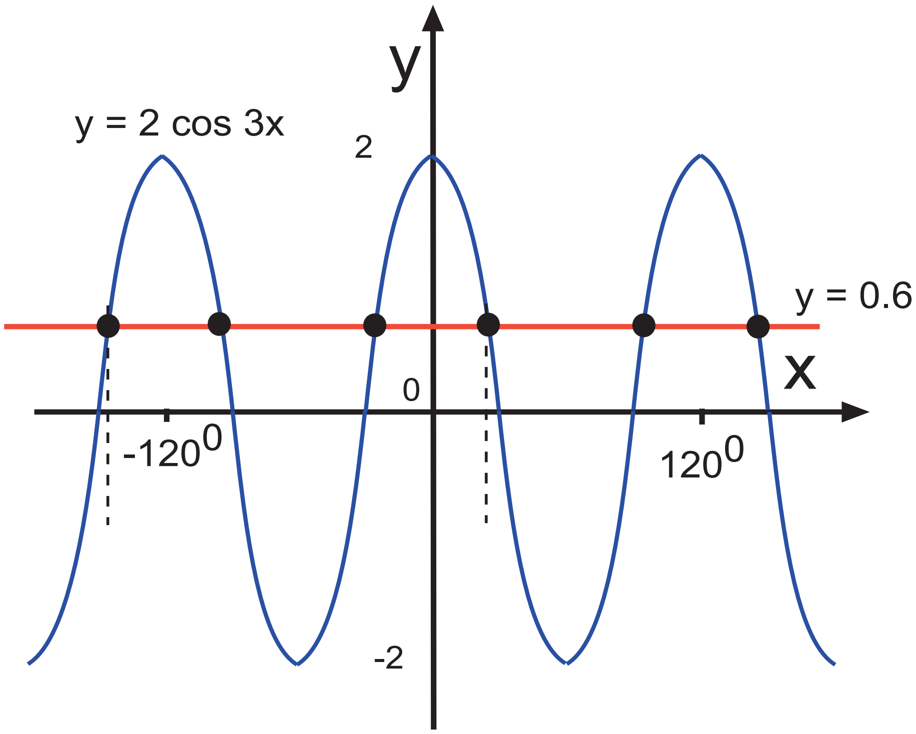
Examples

1. Solving by graphical methods

Solve the trigonometric equation 2 cos 3x = 0.6 giving the answer in degrees.

Answer:

Sketch the graphs of y = 2 cos 3x and y = 0.6 on the same axes.



The x-coordinate value of the points of intersection of the line and the cos curve give the solutions to the equation.

Two of the vertical lines from intersection points are drawn in to show the values on the x-axis. All the x-coordinates of the points shown by black dots represent solutions to the equation.

However, the cos curve has not been defined within a specific interval in this example which means that there are infinite solutions. In such a case, a general solution can be given but this will be explained later.

2. Solving by graphical methods on a given interval

Solve the equation 3 sin 2x - 1.5 = 0 ($0 \leq x < 90°$)

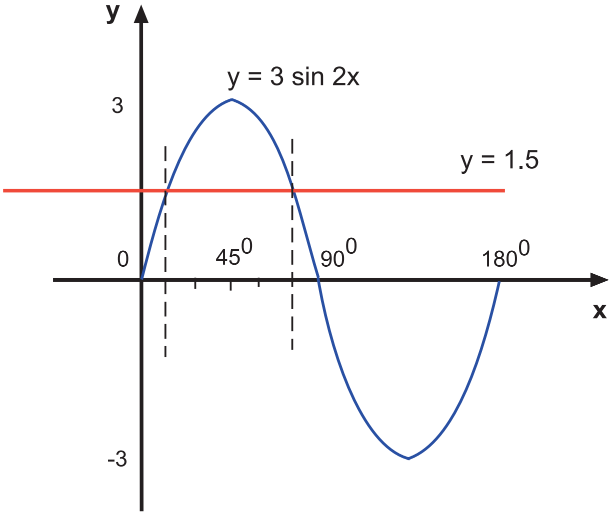
Answer:

In this case an interval is given and the solutions for x will lie in this interval. The angle in the equation is 2x

When $0 \leq x < 90°$ then $0 \leq 2x < 180°$ and the graph should be drawn over this interval.

Rearrange the equation to read 3 sin 2x = 1.5 so that the trig. function is on one side and the constant is on the other.

Draw the graphs of the equations y = 3 sin 2x and y = 1.5 on the same graph.



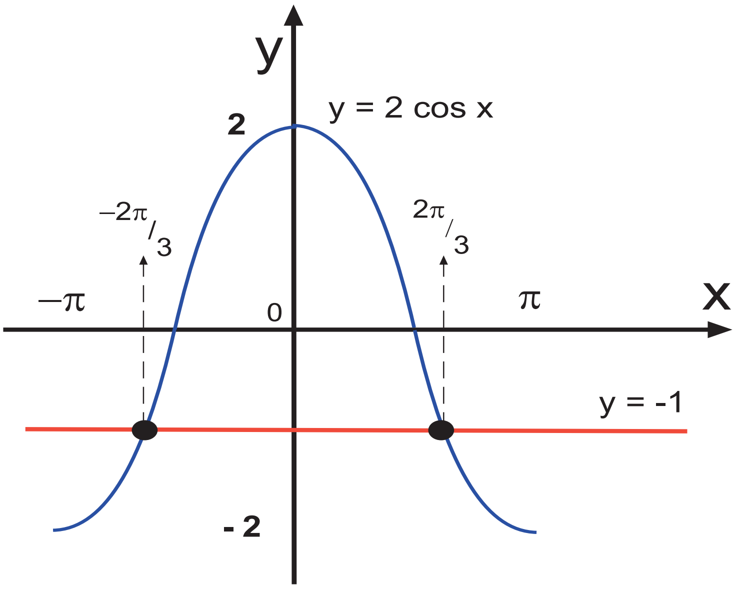
The intersections giving the answers are easily read from the graph as 15° and 75°

3. Solve 2 cos x + 1 = 0 ($-\pi \leq x < \pi$)

Answer:

Rearrange the equation to give 2 cos x = -1

Sketch the graphs of y = -1 and y = 2 cos x between $-\pi$ and π



The intersections are shown and give the two solutions in the interval required.

Solving by graphing examples

There are animated examples of these on the web.

10 min

Solving using graphs exercise

15 min

There is an alternative exercise on the web.

Use graphical methods to find the answers to the following questions within the interval given.

Q5: Solve 2 sin 3x = -1 (-90° $\leq$ x $<$ 90°)

Q6: Solve 3 cos 2x - 1 = 0 (-180° $\leq$ x $<$ 180°)

Q7: Solve 4 sin 5x = 3 (-π $\leq$ x $<$ π)

Q8: Solve 5 sin 2x + 3 = 0 (0 $\leq$ x $<$ 2π)

Q9: Solve $\sin^2$x = 1 (0 $\leq$ x $<$ 2π)

Graphic calculators can make the solving of complicated trig. equations very straightforward. The equations demonstrated in this section have been relatively simple. The following tasks give an indication of the number of solutions which can occur in trig. equations.

Extra Help: Simple trigonometry

An online exercise is provided to help you if you require additional assistance with this material, or would like to revise this subject.

3.3 Solving trigonometric equations algebraically

Learning Objective

Solve trigonometric equations algebraically

Recall the following example that was given in the previous section on solving trig. equations graphically.

Example Solve the equation 3 sin 2x - 1.5 = 0 (0 $\leq$ x $<$ 90°)

Answer:

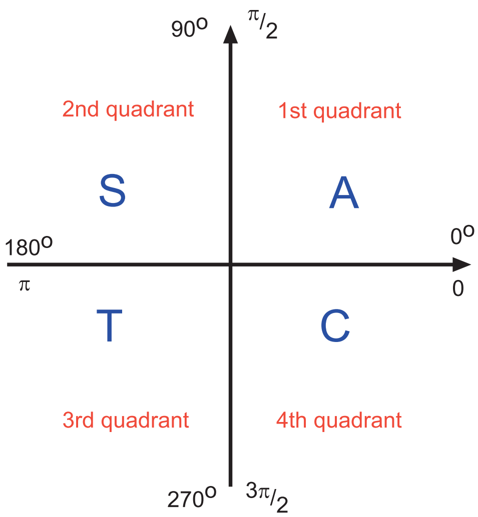
x = 15° and x = 75°

The answers were found by drawing the graphs of y = sin 2x and y = 0.5 and finding the points of intersection.

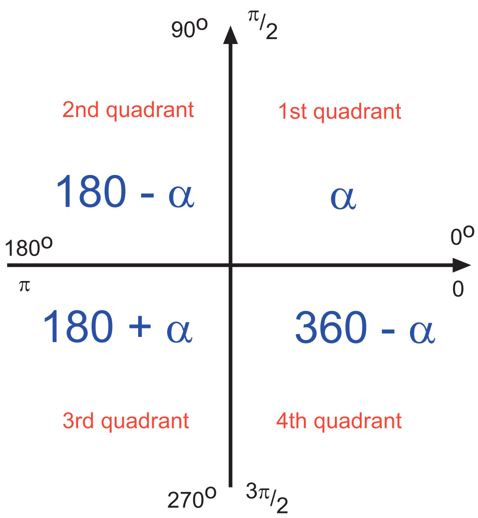
The solutions can also be found algebraically.

However, first of all it is appropriate to revise the diagrams which relate the trig. ratios to the quadrants and the first quadrant angles to associated angles.

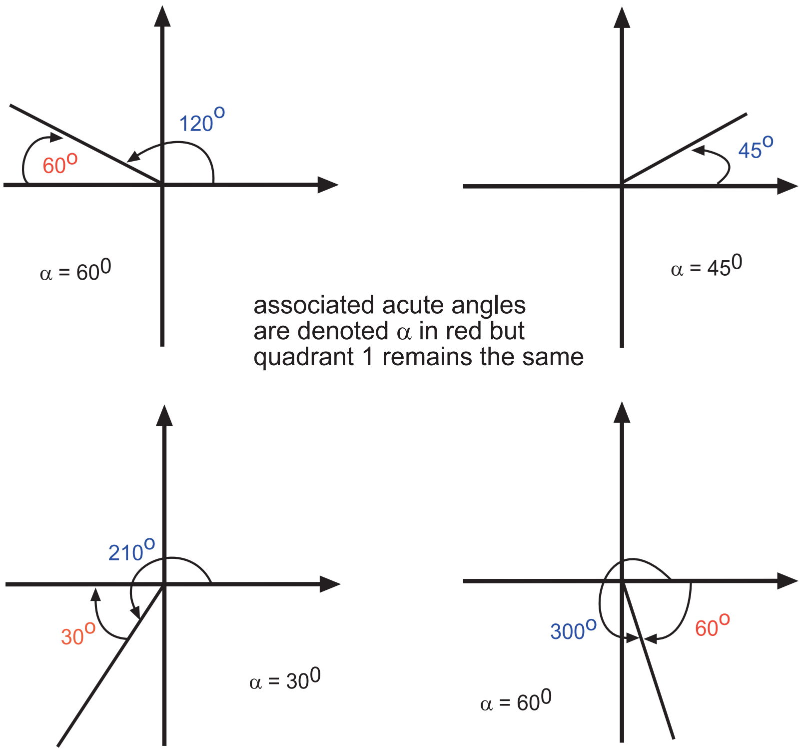
The following diagram is the quadrant diagram for trig. ratio signs. Recall that the letters indicate the ratios which are positive in each quadrant. The quadrants can be remembered by the saying: All Students Talk Constantly.



The next diagram is the quadrant diagram for the associated angles. When the angle α in the first quadrant is known (or calculated) the related angles in the other three quadrants can be obtained from the expressions shown in the diagram.



Here are some examples of how the angles relate to each other using the associated angle quadrant diagram.



The use of α aids the understanding of the following techniques and will be used to indicate the angle within the first quadrant only. With α known, any other associated angle in the required quadrant can be found.

Returning to the example of 3 sin 2x = 1.5

This can be rearranged to give sin 2x = 0.5

Solve for the first quadrant. That is solve the equation sin α = + 0.5

This gives α = $\sin^{-1}$ (+ 0.5) and so α = 30°

Since sin 2x had a positive value and the sine is positive in quadrants 1 and 2, there are two answers.

The angle 2x = 30° in quadrant 1 and
in quadrant 2 the associated angle is 2x = 180 - α = 150°

However the range given is $0 \leq x < 90°$. The angle in the question is 2x which gives a range for this angle of $0 \leq 2x < 180°$

If other solutions exist, these are found by adding or subtracting 360° until the range is covered.

In this case there are no further solutions in the range.

Thus x = 15° and x = 75° within the interval given.

This technique is more efficient and less time consuming.

Here are two more examples.

Examples

1. Solve 5 sin x = -1 for x where $0 \leq x < 360°$. Give your answer to 2 decimal places.

Answer:

5 sin x = -1 $\Rightarrow$ sin x = - 0.2

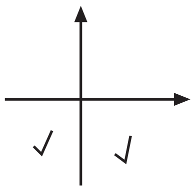
Solve for the positive value to give the first quadrant angle. This is the value of α in the angle quadrant diagrams.

That is instead of sin x = - 0.2, solve sin α = 0.2 to give $\alpha = \sin^{-1}(0.2)$

So α = 11.54°

At this point determine the quadrants in which the angle x lies by referring to the trig. ratio quadrant diagram.

Sine is negative in the third and fourth quadrants.



Use the angle quadrant diagram to obtain the solutions to the equation.

The solutions are x = -α and x = 180 + α

The solutions x = - 11.54° and x = 191.54°

Since the range given is $0 \leq x < 360°$ check the values of x + 360° to find any other values within the range.

The gives 348.46 and 551.54

Examining the four solutions -11.54° and 551.54° can be excluded as they are outwith the range.

The two solutions are: x = 191.54° and 348.46°

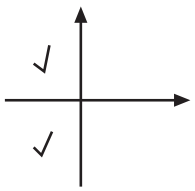
2. Solve 2 cos x + 1 = 0 for $0 \leq x < 2\pi$

Answer:

2 cos x + 1 = 0 $\Rightarrow$ 2 cos x = -1 $\Rightarrow$ cos x = $-{}^{1}/_{2}$

Thus cos $\alpha = {}^{1}/_{2} \Rightarrow \alpha = \frac{\pi}{3}$

Cos is negative in the second and third quadrants.



Therefore x = $\pi - \alpha$ and x = $\pi + \alpha$

That is, x = $\frac{2\pi}{3}$ and x = $\frac{4\pi}{3}$

For the range $0 \leq x < 2\pi$, $x \pm 2\pi$ gives no further solutions.

3. Solve 2 tan 2x = 3 for x where -45° $\leq$ x $<$ 45°. Give your answer to 2 decimal places.

Answer:

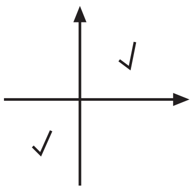
The range is -45° $\leq$ x $<$ 45° and the angle in the question is 2x

Thus -90° $\leq$ 2x $<$ 90°

2 tan 2x = 3 $\Rightarrow$ tan 2x = 1.5

In the first quadrant, tan α = + 1.5 $\Rightarrow$ α = $\tan^{-1}$ 1.5 $\Rightarrow$ α = 56.3099

The tan ratio is positive in the first and third quadrants



and so 2x = α and 2x = 180 + α

That is, 2x = 56.3099° and 2x = 236.3099°

But only the first quadrant answer lies in the interval -90° $<$ 2x $<$ 90° and no other values can be obtained by x $\pm$ 360°

Thus 2x = 56.3099° and the solution is x = 28.15°

In the last example, the angle in the question is 2x which is a function of x. It is important to find the associated first quadrant values *for the angle*α within the modified range before solving for x

Strategy for solving trig. equations

1. Adjust the range to fit the actual angle.
2. Isolate the trig ratio at one side of the equation.
3. Solve for the first quadrant angle α
4. Refer to the quadrant diagrams to relate the actual angle to α
5. Check that the values lie within the interval of the angle given and adjust if necessary.
6. Check for further values within the interval of the angle given by the addition or subtraction of 360° (or 2π) when necessary.
7. Solve each for x

Solving algebraically exercise

There is also a web exercise on this topic.

25 min

Q10: Solve the following for x giving the answer in degrees to two decimal places where appropriate.

a) $\sin 2x = 0$, $-360° \leq x < 360°$

b) $1 - 2 \tan 2x = -5$, $0° \leq x < 180°$

c) $\sin 5x + 1 = 0$, $0° \leq x < 90°$

d) $(\cos (2x - 1))^2 = 1$, $-90° \leq x < 0°$

Q11: Solve the following for x giving the answers in terms of π

a) $1 - 2 \sin 3x = 0$, $0 \leq x < \pi$

b) $2 \cos 4x + 1 = 0$, $-\pi \leq x < 0$

c) $3 \tan^2 x = 1$, $0 \leq x < \pi$

Q12: Solve the following for x giving the answers in terms of π

a) $2 \sin^2 x - 1 = 0$, $0 \leq x < \pi$

b) $2 \cos^2 x - \cos x = 0$, $0 \leq x < \pi$

c) $\cos^2 2x - 3 \cos 2x + 2 = 0$, $0 \leq x < \pi$

Q13: Solve the following for x giving the answer in degrees to two decimal places.

a) $2 \cos^2 x - 3 \cos x + 1 = 0$, $-180° \leq x < 180°$

b) $\tan^2 3x = 25$, $-45° \leq x < 45°$

c) $\sin^3 x + 1 = 0$, $-90° \leq x < 0°$

3.4 Three-dimensional calculations

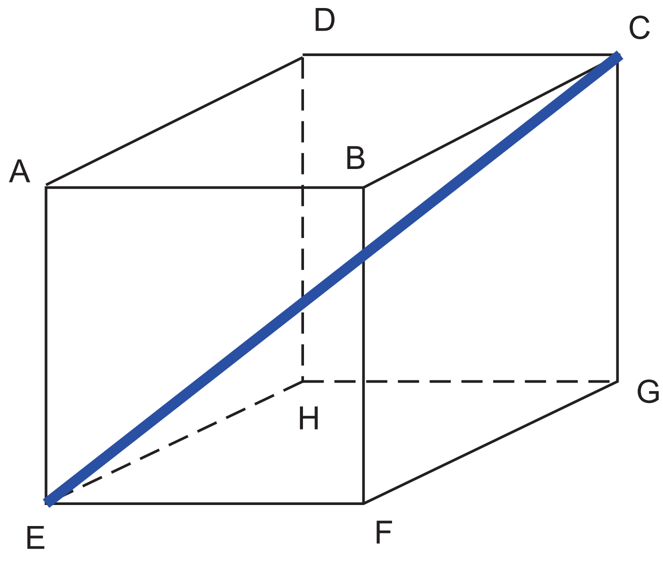
Learning Objective

Use the geometry of 3D shapes to solve trig. problems.

To be able to make full use of trigonometric formulae, an understanding of three dimensional shapes is required. Most real life problems will in fact involve 3 dimensional figures and this work can help with calculations in subjects such as civil and mechanical engineering.

Although angles and lines along the surface area of shapes such as a cube are obvious, it is important to look further and envisage the lines, planes and angles which can be formed within a shape.

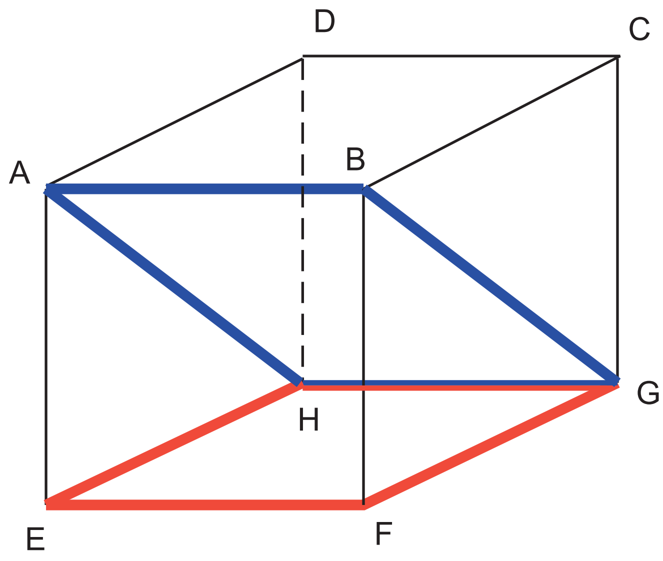
Take a look at the following examples.



The line EC is a diagonal through this cuboid.

Q14: Name the other diagonals through this cuboid.

Q15: Name a right angled triangle with CE as the hypotenuse.



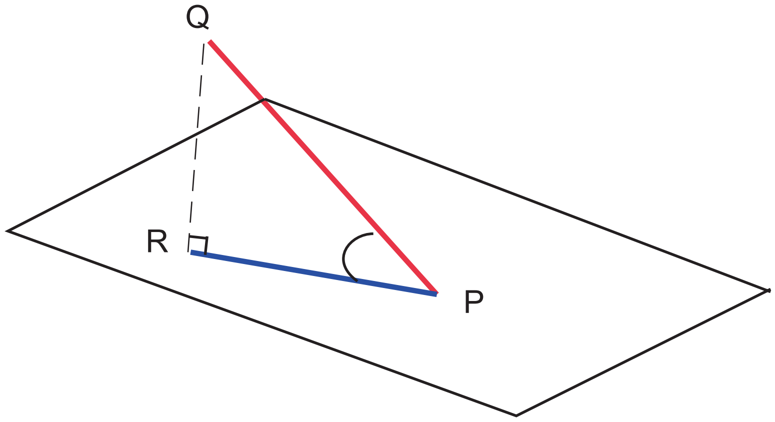
The planes EFGH and ABGH intersect at the line GH

Q16: Which other pairs of planes intersect in the line GH?

Q17: Which three pairs of planes intersect in the line BC?

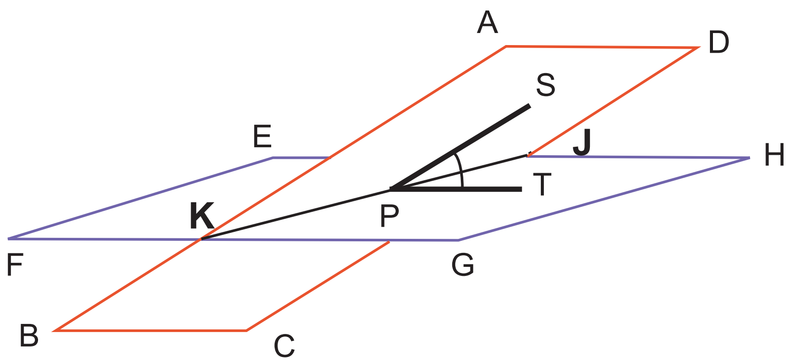
In cuboids and even more so in cubes, the intersections and the required angles are apparent. In three dimensional surfaces in general, the intersection of a line and a plane or of two planes is formed in a particular way.

When a line meets a plane, the angle between the line and plane is the angle formed by the line and its projection in the plane.



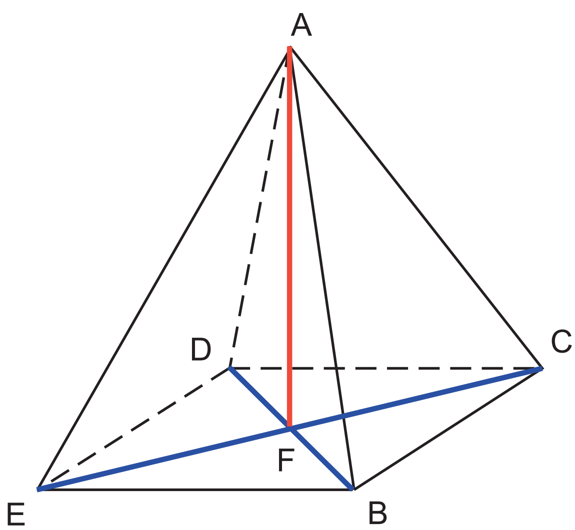
The line QR is at right angles to the plane and the angle between the line and plane is angle QPR

When two planes intersect the angle between them at a point P is formed by two lines, one in each plane at right angles to the line of intersection.



In the previous diagram the two planes ABCD and EFGH intersect in the line KJ. The point P is any point on KJ. The line PS is at right angles to KJ and lies in the plane ABCD. The line PT is at right angles to KJ and lies in the plane EFGH. The angle required is angle SPT

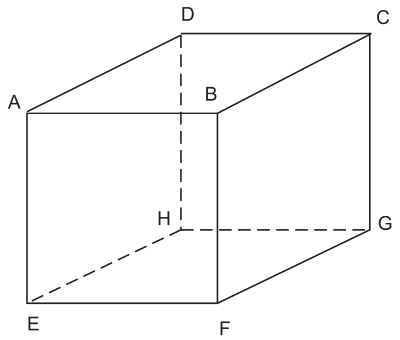
Building on the work done on cuboids other shapes can be treated in a similar fashion. Take for example a right square pyramid.



The diagonals of the base of a square pyramid intersect at right angles and the intersection is shown here at point F. The line joining point F to the apex of the pyramid at point A is at right angles to the base BCDE

Examples

1. The diagram shows a cuboid where BC = 4 cm and HG = FB = 3 cm

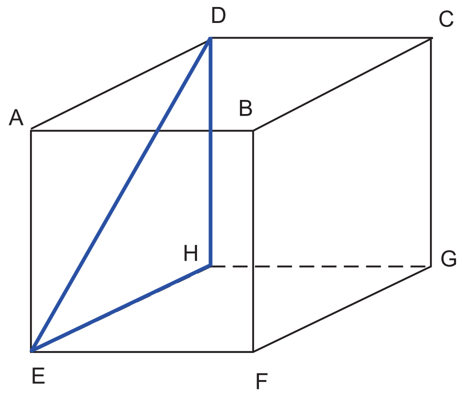


Find the following:

a) The length of DE
b) The angle AGE correct to 2 d.p.
c) The angle between the planes BDHF and ADHE

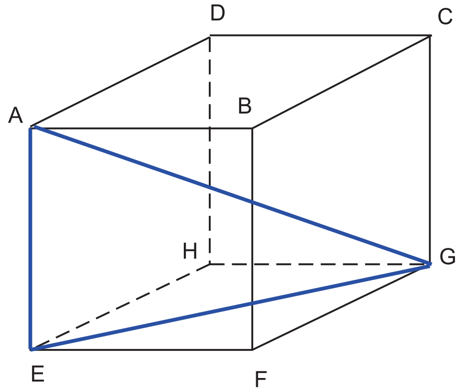
Answer:

a) By the symmetry of the cuboid BC = FG = EH = AD = 4 cm
Also BF = CG = DH = AE = HG = EF = AB = DC = 3 cm
DE is the hypotenuse of triangle DEH



By Pythagoras $DE^2 = EH^2 + DH^2 \Rightarrow DE^2 = 25$ and so DE = 5 cm

b) The angle AGE is in triangle AGE and AE = 3 cm



EG is also part of triangle EGF where EF = 3 cm and FG = 4 cm

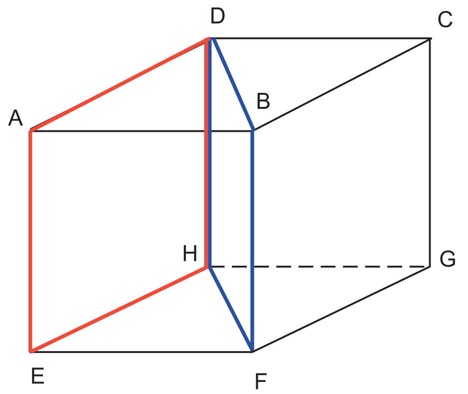
By Pythagoras in triangle EGF, EG = 5 cm

Returning to triangle AGE, this is a right angled triangle.

tan AGE = opposite ÷ adjacent = $^3/_5$ = 0.6

AGE = tan $^{-1}$ 0.6 = 30.96°

c) The intersection of the two planes is on the line DH



By definition take any point on the line and the angle required is formed by two lines, one in each plane at right angles to the line DH

In this case, it is easiest to choose either D or H

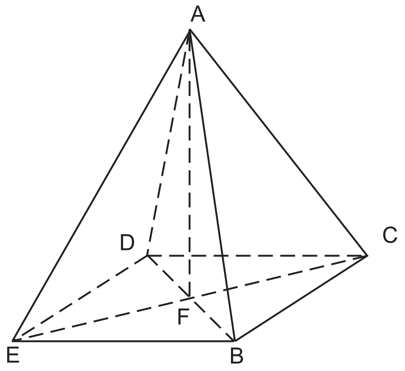
Using the point D then AD is a line in one plane at right angles to DH and the similar line in the second plane is DB

The required angle is given by ADB

AD = 4 cm and AB = 3 cm thus tan ADB = $^3/_4$ = 0.75

The angle ADB which is the angle between the two planes is 36.87°correct to 2 d.p.

2.



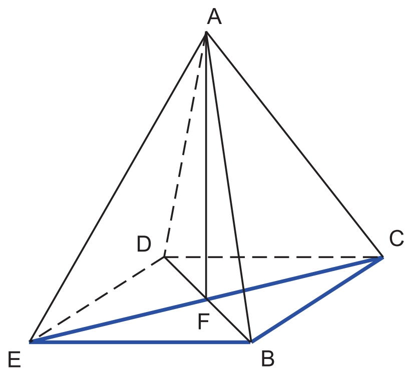
If the base of the right square pyramid has side length 4 cm and the height of the pyramid AF is 6 cm find:

a) The length of AC

b) The angle between the line AB and the plane in which triangle ACE lies.

Answer:

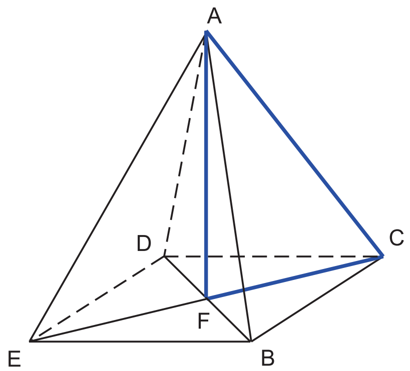
a) Since the base is square, in triangle EBC, by Pythagoras,



$EC^2 = EB^2 + BC^2 = 32 \Rightarrow EC = 5.6569$

By the symmetry of the pyramid, $FC = 0.5 \times EC = 2.8284$ cm

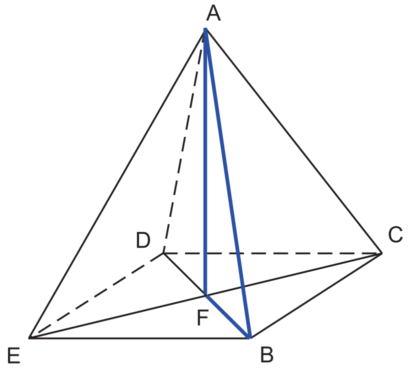
In triangle ACF,



by Pythagoras, $AC^2 = AF^2 + FC^2 = 36 + 8 = 44 \Rightarrow AC = 6.63$ cm correct to 2 d.p.

b) The required angle is angle BAF

By the symmetry of the pyramid, AB = AC = 6.63 cm and FB = FC = 2.8284 cm



In triangle ABF, sin BAF = $^{FB}/_{AB}$ = 0.427

angle BAF = 25.28° correct to 2 d.p.

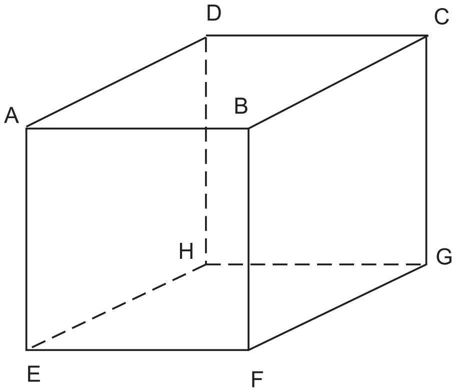
3D exercise

There is a web exercise if you prefer it.

15 min

Q18:

ABCDEFGH is a cube.

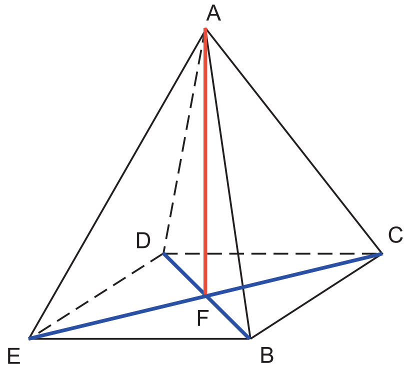


Using the diagram name the point or line of intersection of:

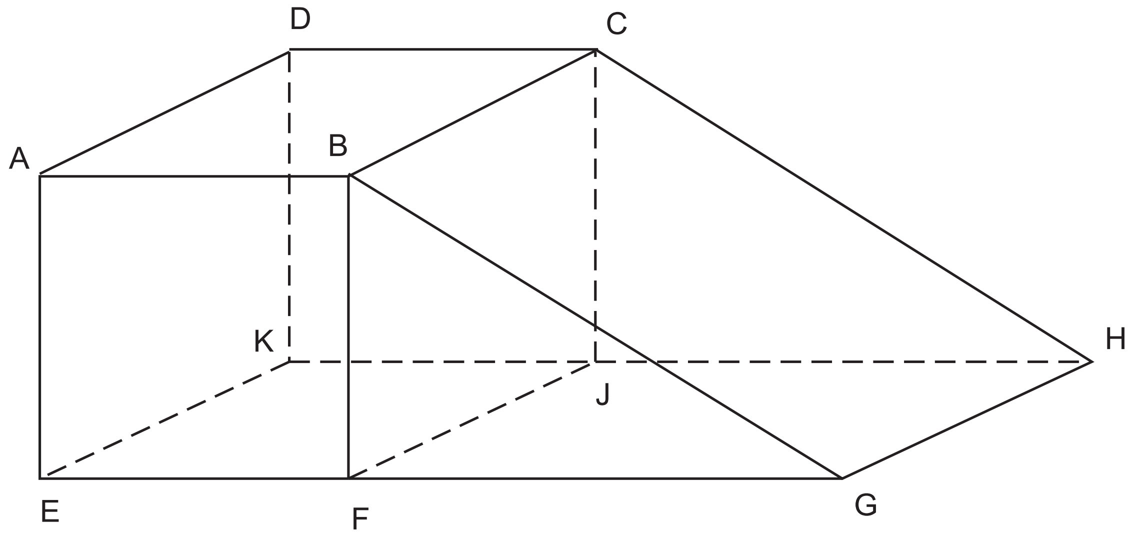
a) The planes DCGH and EFGH
b) The line BF and the plane ADGF
c) The planes ACGE and ABCD

Q19: If ABCDE is a right square pyramid whose height AF = 5 cm and whose base diagonals EC and BD = 7 cm find the angles:

a) FCA
b) DAB



Q20:



A skateboarding ramp is constructed from a cube of side length 50 m and a right angle prism making an angle with the ground of 40° at the edge GH

Calculate:

a) The length of DG

b) The angle which DG makes with the plane through EFJK

3.5 Addition and double angle formulae

Learning Objective

Use the appropriate formulae in solving trigonometric problems.

In order to understand the derivation of the new formulae in this section, the following facts should be remembered.

- sin (-x) = -sin x
- cos (-x) = cos x
- sin (90° - x) = cos x

- cos (90° - x) = sin x
- tan x = $\frac{\sin x}{\cos x}$
- $\sin^2 x + \cos^2 x = 1$

3.5.1 Addition Formulae

There are four formulae, commonly called *the addition formulae*, which can be used in trigonometric calculations.

sin (A + B) =	sin A cos B + cos A sin B
sin (A - B) =	sin A cos B - cos A sin B
cos (A + B) =	cos A cos B - sin A sin B
cos (A - B) =	cos A cos B + sin A sin B

The proofs of these formulae are given in the section headed Proofs near the end of this topic.

The right hand side entries in the table are often called the expansions of the formulae.

Example Expand:

a) cos (2x + y)

b) sin (x - 45)°

Answer:

a) cos 2x cos y - sin 2x sin y

b) sin x cos 45 - cos x sin 45 = $\frac{1}{\sqrt{2}}$ (sin x - cos x)

The formulae can also be used to find the exact values of trig. functions which are not easily seen. This is done by determining combinations of two known angles to obtain the required angle.

Example Find the exact value of cos 15°

Answer:

cos 15° = cos (45 - 30)° = cos 45 cos 30 + sin 45 sin 30

$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \times \frac{1}{2} = \frac{\sqrt{3}+1}{2\sqrt{2}}$

20 min

Addition formulae exercise

There is an alternative exercise on the web.

Q21: Using these new formulae expand:

a) sin (3x - 2y)
b) cos (30 - y)
c) sin (π + a)
d) cos (x + 2π)

Q22: Find the exact value for:

a) sin 105°
b) sin 15°
c) cos 75°
d) cos 135°

3.5.2 Double angle formulae

The expressions sin 2A and cos 2A are easier to deal with when expanded using the double angle formulae derived from the normal trig. ratios of sin and cos (in this case of the angle A).

The formulae are:

sin 2A =	2 sin A cos A
cos 2A =	$\cos^2$ A - $\sin^2$ A OR
	2 $\cos^2$A - 1 OR
	1 - 2 $\sin^2$A

The three versions of cos 2A can be useful in different situations. If cos A and sin A appear in the equation, the first version (cos 2A = $\cos^2$A - $\sin^2$ A) is appropriate.
If only cos A appears in the equation to be solved then cos 2A = 2 $\cos^2$A - 1 is probably the best to use.
Finally if the equation contains sin A then try cos 2A = 1 - 2 $\sin^2$A first.

The derivations/proofs of these formulae are given in the section headed Proofs near the end of this topic.

Examples

1. Write down a formula for sin 4x in terms of 2x

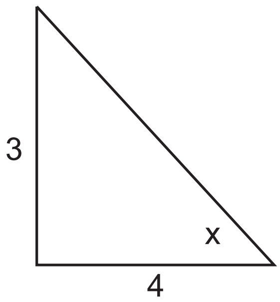
Answer:

sin 4x = sin 2(2x) = 2 sin 2x cos 2x

2. If tan x = $^3/_4$ in a right angled triangle, give the exact values for:

a) sin 2x
b) cos 2x
c) tan 2x

Answer



The hypotenuse, by Pythagoras is 5

sin x = $^3/_5$ and cos x = $^4/_5$

a) sin 2x = 2 sin x cos x = 2 $\times$ $^3/_5$ $\times$ $^4/_5$ = $^{24}/_{25}$
b) cos 2x = $\cos^2$ x - $\sin^2$ x = $^7/_{25}$
c) tan 2x = sin 2x $\div$ cos 2x = $^{24}/_7$

3. Without using a calculator evaluate $\cos^2 15^\circ - \sin^2 15^\circ$

Answer:

$\cos^2 15^\circ - \sin^2 15^\circ = \cos(2 \times 15)^\circ = \cos 30^\circ = \frac{\sqrt{3}}{2}$

4. Find the value of cos 2x if $2\cos^2 x - 4\sin^2 x + 1 = 3$

Answer:

$2\cos^2 x - 4\sin^2 x + 1 = 3 \Rightarrow$

$(2\cos^2 x - 1) + 2(1 - 2\sin^2 x) = 3 \Rightarrow$

$\cos 2x + 2\cos 2x = 3 \Rightarrow 3\cos 2x = 3 \Rightarrow \cos 2x = 1$

Double angle exercise

There is a web exercise if you prefer it.

20 min

Q23: Find the value of cos 2A if sin A = $^1/_2$

Q24: If cos A = $^9/_{15}$ find the value of sin 2A

Q25: Show that sin 2x = 0 when $(\sin x - \cos x)^2 = 1$

Q26: Evaluate $1 - 2\sin^2 15°$ by using the double angle formulae.

Q27: Simplify the following:

a) $2\sin\frac{\pi}{4}\cos\frac{\pi}{4}$

b) $2\cos^2\frac{3\pi}{4} - 1$

c) $\cos 170° - 2\cos^2 85°$

3.6 Application of formulae

Learning Objective

Use the appropriate trigonometric formulae to solve a problem.

This section may use any sin, cos or tan relationships which have been dealt with in this topic or have been studied previously. This means that it would be possible for the sine and cosine ratios and Pythagoras to appear as well as the addition formulae, double angle formulae and the common trig. relationships such as $\cos^2 x + \sin^2 x = 1$

Examples

1. If $(\cos x + \sin x)^2 = {}^5/_3$ find sin 2x

Answer:

$(\cos x + \sin x)^2 = \cos^2 x + 2\sin x\cos x + \sin^2 x = 1 + 2\sin x\cos x = 1 + \sin 2x$

So $1 + \sin 2x = {}^5/_3 \Rightarrow \sin 2x = {}^2/_3$

This example has used straight algebraic manipulation of the terms, a trig. relationship and a double angle formula.

2. Show that $\frac{2\tan A}{1 + \tan^2 A} = \sin 2A$

Answer:

$$\frac{2\tan A}{1 + \tan^2 A} = \frac{\frac{2\sin A}{\cos A}}{\frac{\cos^2 A + \sin^2 A}{\cos^2 A}} = \frac{2\sin A}{\cos A} \times \frac{\cos^2 A}{\cos^2 A + \sin^2 A} = 2\sin A\cos A = \sin 2A$$

Here the tan relationship was used, followed by some algebraic manipulation and a final conversion with a double angle formula.

3. Solve $\cos 2x = 5\cos x - 4$ for $0 \le x < 90°$

Answer:

$\cos 2x - 5\cos x + 4 = 0 \Rightarrow 2\cos^2 x - 5\cos x + 3 = 0 \Rightarrow$

$(2\cos x - 3)(\cos x - 1) = 0$

So cos x = 1 only as cos x = $^3/_2$ is not possible.

Solving for the first quadrant gives α = 0°

So since cos is positive in quadrants 1 and 4

2x = 0° (or 360° which is 0°)

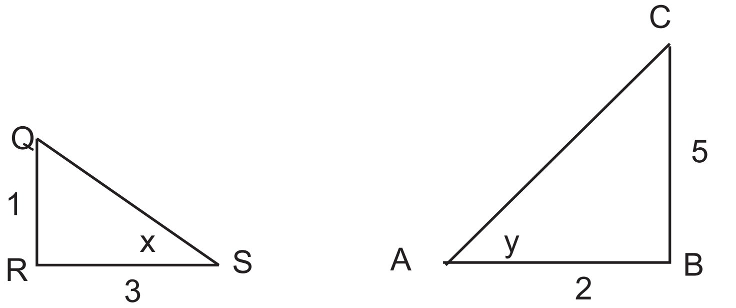
The range $0 \leq x < 90° \Rightarrow 0 \leq 2x < 180°$

x = 0° is the only solution.

Use of the double angle formula, algebraic manipulation and the technique for solving trig. equations gives the answer.

4.

Triangles ABC and QRS are right angled as shown.



Find the value of cos (x + y)

Answer:

From triangle QRS,

$\sin x = \frac{1}{\sqrt{10}}$ and $\cos x = \frac{3}{\sqrt{10}}$

From triangle ABC,

$\sin y = \frac{5}{\sqrt{29}}$ and $\cos y = \frac{2}{\sqrt{29}}$

cos (x + y) = cos x cos y - sin x sin y

$= \frac{1}{\sqrt{290}}$

Mixed formulae exercise

20 min

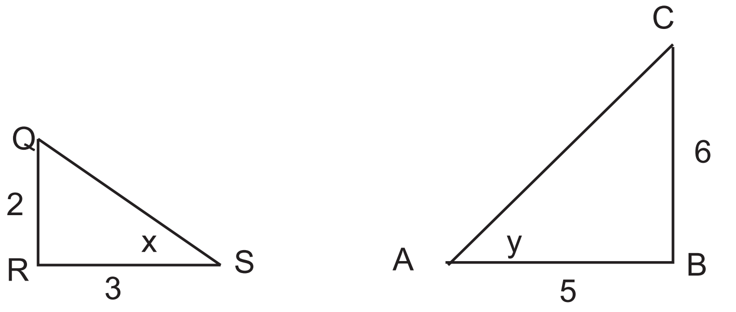
There is another exercise on the web if you prefer it.

Q28: Show that $\cos 2A = \frac{1 - \tan^2 A}{1 + \tan^2 A}$

Q29: Simplify $(\sin x - \cos x)^2$ and show that

$\frac{1 - \sin 2x}{\sin x - \cos x} = \sin x - \cos x$

Q30: The triangles ABC and QRS are right angled.

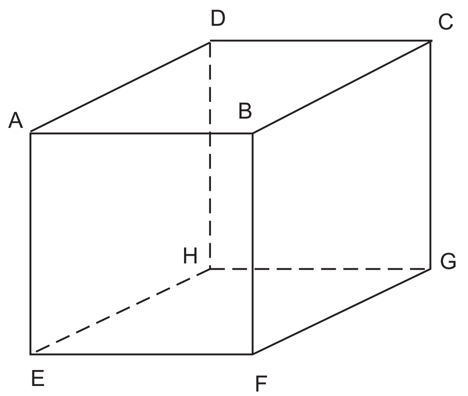


Find the values of:

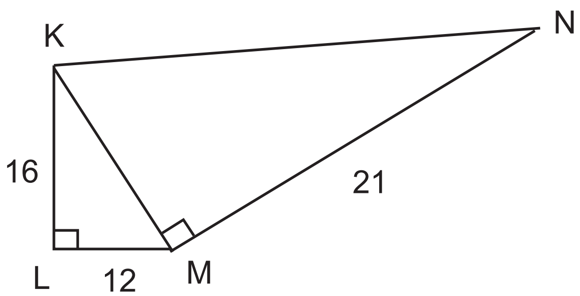
a) sin 2y
b) sin (x - y)
c) cos 4x

Q31:

ABCDEFGH is a cube of side length 3 cm find the angle which BH makes with the plane ABCD



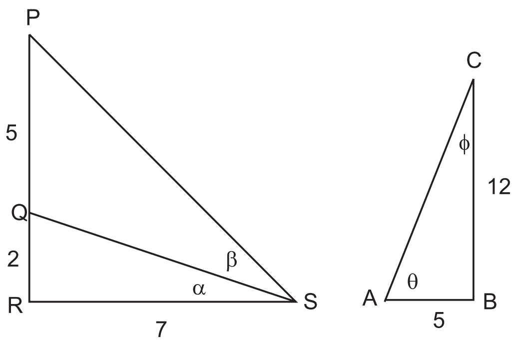
Q32:



Find the exact value of cos LKN

Q33:

Using the diagram below find the following:



a) Angle RPS in terms of α and β

b) cos (α + θ) as a surd

c) sin 2β:
1) in terms of sin α and cos α
2) in terms of cos 2α
3) evaluate sin 2β

d) sin (β - α)

e) cos 2ϕ

Further solutions exercise

20 min

Q34: Solve each of the following graphically and algebraically. Using a graphics calculator determine the solutions in the range -1440° $\leq$ x $<$ -1080.
Algebraically find the solutions in the range 0 $\leq$ x $<$ 2π

a) sin 2x - 3 sin x = 0

b) cos 2x - cos x = -1

Q35: Using the information from the last question,
give the general solution of sin 2x - 3 sin x = 0

Extra Help: Solving multiple angle equations

An online exercise is provided to help you if you require additional assistance with this material, or would like to revise this subject.

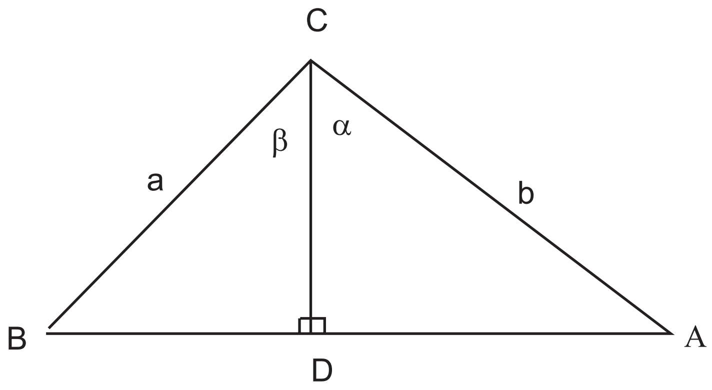
3.7 Summary

The following points and techniques should be familiar after studying this topic:

- Solve trigonometric equations graphically.
- Solve trigonometric equations algebraically in a given interval.
- Apply the addition and double angle formulae to 2 and 3 dimensional problems.
- Solve problems using the addition and double angle formulae.

3.8 Proofs

Proof 1: sin (A + B) = sin A cos B + cos A sin B



In triangle ACD, CD = b cos α

Thus the area of triangle BCD = $^1/_2$ (a b cos α sin β)

But in triangle BCD, CD = a cos β

Thus the area of triangle ACD = $^1/_2$ (b a cos β sin α)

However the area of triangle ABC = $^1/_2$ (ab sin (α + β)

But the area of triangle ABC = area of triangle ACD + area of triangle BCD

Thus $^1/_2$ (ab sin (α + β) = $^1/_2$ (b a cos β sin α) + $^1/_2$ (a b cos α sin β)

This simplifies to give

sin (α + β) = sin α cos β + cos α sin β or

sin (A + B) = sin A cos B + cos A sin B

Proof 2: sin (A - B) = sin A cos B - cos A sin B

Since sin (-x) = - sin x and cos (-x) = cos x

sin (A - B) = sin (A + (-B)) = sin A cos (-B) + cos A sin (-B) by proof 1

= sin A cos B - cos A sin B using the identities mentioned

That is, sin (A - B) = sin A cos B - cos A sin B

Proof 3: cos (A + B) = cos A cos B - sin A sin B

This proof uses the relationships: cos x = sin (90 - x)$^\circ$ and sin x = cos (90 - x)$^\circ$

cos (A + B) = sin (90 - (A + B)) = sin ((90 - A) - B) from the relationship mentioned.

= sin (90 - A) cos B - cos (90 - A) sin B using proof 2 result.

= cos A cos B - sin A sin B using the relationships mentioned.

That is, cos (A + B) = cos A cos B - sin A sin B

Proof 4: cos (A - B) = cos A cos B + sin A sin B

Since sin (-x) = - sin x and cos (-x) = cos x

cos (A - B) = cos (A + (-B))

= cos A cos (-B) - sin A sin (-B) by proof 3.

= cos A cos B + sin A sin B using the relationships mentioned.

That is, cos (A - B) = cos A cos B + sin A sin B

Proof 5: sin 2A = 2 sin A cos A

sin 2A = sin (A + A) = sin A cos A + cos A sin A by proof 1

That is, sin 2A = 2 sin A cos A

Proof 6: cos 2A = cos^2 A - sin^2 A

cos 2A = cos (A + A) = cos A cos A - sin A sin A

That is, cos 2A = cos^2 A - sin^2 A

Proof 7: cos 2A = 2cos^2 A - 1

Since cos^2 A + sin^2 A = 1 then

sin^2 A = 1 - cos^2 A

From proof 6, cos 2A = cos^2 A - sin^2 A

= cos^2 A - (1 - cos^2 A) = 2 cos^2 A - 1

That is, cos 2A = 2 cos^2 A - 1

Proof 8: cos 2A = 1 - 2sin^2 A

Since cos^2 A + sin^2 A = 1 then

cos^2 A = 1 - sin^2 A

From proof 6, cos 2A = cos^2 A - sin^2 A

= 1 - sin^2 A - sin^2 A (substituting for cos^{2}A)

That is, cos 2A = 1 - 2 sin^2 A

3.9 Extended Information

There are links on the web which give a selection of interesting sites to visit. Browsing the web under 'trigonometry' will lead to many other good sites which cover this topic.

The St. Andrews web site at
http://www-groups.dcs.st-and.ac.uk/~history/HistTopics/Trigonometric_functions.html
gives a comprehensive history of trigonometry.

Hipparchus

This Greek mathematician and astronomer greatly influenced the thinking of his time. He is generally believed to be the founder of trigonometry.

Ptolemy

He built on the work of Hipparchus and although the terminology was yet to be introduced, he discovered some of the relationships between ratios.

Aryabhata

The origin of the word 'sine' lies in the work of the Hindu mathematician and astronomer, Aryabhata who lived in the sixth century.

Fincke

It was much later that the term 'tangent' was used by the Dane, Thomas Fincke, in 1583.

Gunther

Edmund Gunther completed the trio of terms by adopting the term 'cosine' in 1620.

Pitiscus

Many astronomers and mathematicians over the centuries contributed to the work in trigonometry but the word 'trigonometry' first appeared in print in a treatise by Bartholomaeus Pitiscus in 1595. He also discovered the double angle formulae although some were known to the ancient Greek astronomers in a different format.

De Moivre and Euler

De Moivre and Euler, in the eighteenth century continued to explore trigonometry and this led to the study of trigonometry of complex variables. De Moivre's theorem and Euler's formula are well known in current day studies of complex numbers.

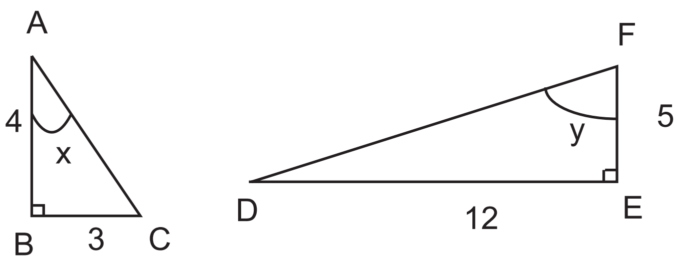
3.10 Review exercise

Review exercise

There is another exercise on the web if you prefer it.

25 min

Q36: The diagram shows two right angled triangles ABC and DEF



a) Write down the values of cos x° and sin y°

b) By expanding cos (x - y)° show that the exact value of cos (x - y)° is $\frac{56}{65}$

Q37: Express sin 75 cos A - cos 75 sin A in the form sin (x - y)°
Hence use the result to solve sin 75 cos A - cos 75 sin A = 0.4 for $0 \leq A < 180°$

Q38: Solve 3 cos 2x + 2 = 0 for $0 \leq x < \pi$

Q39: Show that cos 2x + 2 $\sin^2$ x = 1

3.11 Advanced review exercise

Advanced review exercise

There is another exercise on the web if you prefer it.

25 min

Q40: Solve cos 2x - cos x = 0 for $0 \leq x < 360°$

Q41: Using the formulae and relationships given in this topic find cos 3A in terms of cos A and powers of cos A only.

Q42: Solve 6 sin (x - 10)° cos (x - 10)° = 2, -180° $\leq$ x $<$ 0

Q43: Show that (cos x + sin x)(cos y + sin y) = cos (x - y) + sin (x + y) and hence that (cos x + sin x)(cos y + sin y) = 1 + 2 sin x cos x if x = y

3.12 Set review exercise

Set review exercise

20 min

Work out your answers and then access the web to input your answers in the online test called set review exercise.

Q44: If x° is an acute angle such that tan x = 2

show that the exact value of cos (x + 45)° is $-\dfrac{1}{\sqrt{10}}$

Q45: If f (x) = π - x and g (x) = cos x show that g (f (x)) = - cos x

Q46: Solve cos 2x = - cos x for -π $\leq$ x $<$ 0

Q47: By use of double angle formulae show that A = $\dfrac{5 + \sqrt{2}}{2}$ when

cos 45(2 cos 45 + 1) + 1 = -2 sin 15 cos 15 + A

Topic 4

The equation of a circle

Contents

Learning Objectives

- *Use the equation of the circle*

Minimum performance criteria:

- *Given the centre (a, b) and radius r,*
  find the equation of the circle in the form $(x - a)^2 + (y - b)^2 = r^2$
- *Find the radius and centre of a circle given the equation in the form*
  $x^2 + y^2 + 2gx + 2fy + c = 0$
- *Determine whether a given line is a tangent to a given circle*
- *Determine the equation of the tangent to a circle given the point of contact*

Prerequisites

A sound knowledge of the following subjects is required for this unit:

- Equation of a straight line
- Distance between two points
- Algebraic manipulation
- 2D coordinates and plotting graphs

4.1 Revision exercise

Revision exercise

30 min

There is a web exercise if you prefer it.

Q1: Find the equation of a straight line in the form y = mx + c with gradient -2 and passing through the point (3, 4)

Q2: Find the equation of the straight line perpendicular to the line y = 3x - 4 and passing through the point (0, 1)

Q3: Find the distance between the two points (-2, 5) and (3, -4) correct to one decimal place.

Q4: Expand and simplify $(x - 3)^2 + (y - 2)^2 - x^2 + 3(2x - 5) + 4y - 2$

Q5: The point P (-3, 4) is reflected in the line y = x to give point Q. Give the coordinates of point Q and the distance between the points P and Q correct to one decimal place.

4.2 The equations of a circle

Learning Objective

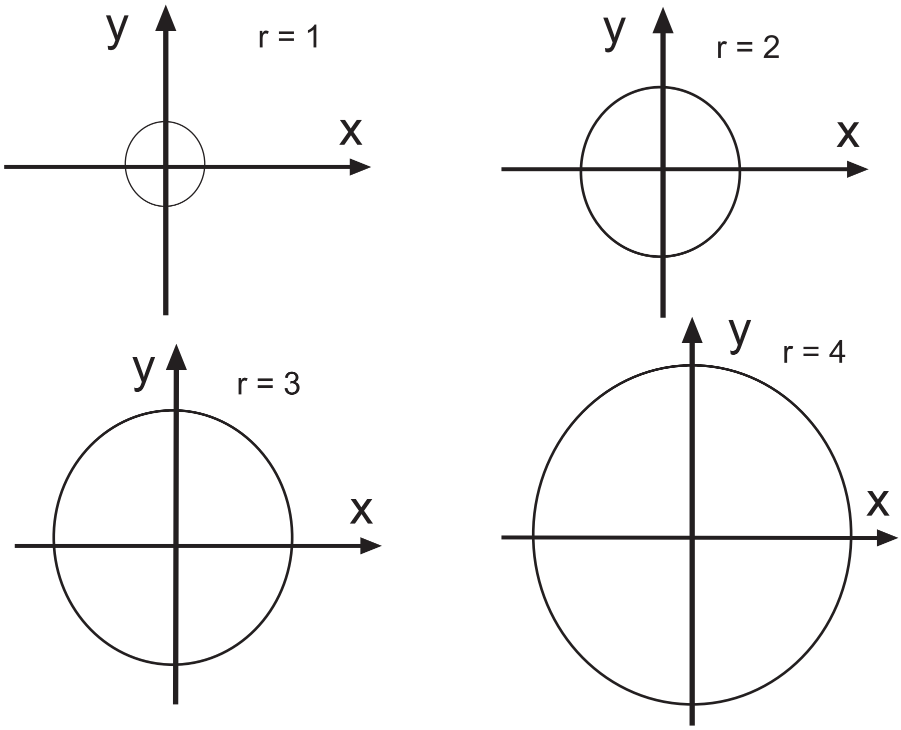
Determine the different forms of the equation of a circle. Find the radius and centre of a circle.

Circle

A circle is defined as the set of points P (x, y) that are at a given (constant) distance r (the radius) from a given point C (a, b)(the centre).

4.2.1 Circles with centre at the origin

The easiest circle to construct is a circle with centre O, the origin. Here are some examples.



Equation of a circle with centre, the origin
The equation of a circle C with centre O (0, 0) and radius r is
$x^2 + y^2 = r^2$

The proof of this equation is given as proof 1 in the section headed proofs near the end of this topic.

Examples

1. What is the radius of a circle with equation $x^2 + y^2 = 9$?

Answer:

Since $x^2 + y^2 = r^2$

Then $r^2 = 9$

The radius is 3

2. $x^2 + y^2 = 25$ is the equation of the circle C and the points P and Q are on the circumference with x-coordinate of 4. Find the y-coordinates of P and Q

Answer:

$x^2 + y^2 = 25$

Since P is on the circle, substitute x = 4 into the equation.

$16 + y^2 = 25 \Rightarrow y^2 = 9 \Rightarrow y = \pm 3$

3. What is the radius and centre of the circle with equation $4x^2 + 4y^2 - 36 = 0$?

Answer:

Since this is a circle, rearrange into the correct form.

$4x^2 + 4y^2 = 36 \Rightarrow x^2 + y^2 = 9$

This is in the correct form for a circle with radius 3 units and centre, the origin.

4. Find the equation of a circle passing through the point (5, 12) with centre at the origin.

Answer:

Since the equation is $x^2 + y^2 = r^2$

Substitute x = 5 and y = 12 into the equation to find r

$25 + 144 = r^2 \Rightarrow 169 = r^2 \Rightarrow r = 13$

The equation is $x^2 + y^2 = 169$

Note that for any circle calculations, the equation must be in the exact form for the equation of a circle with x^2 and y^2 having coefficients equal to one.

Circles at the origin exercise

10 min

There is a web exercise with randomised parameters if you prefer it.

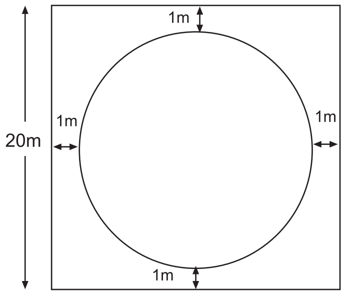
Q6: Find the radii of the following circles with centre, the origin:

a) $x^2 + y^2 = 144$
b) $x^2 + y^2 - 4 = 0$
c) $3x^2 + 3y^2 - 20 = 7$
d) $x^2 = - y^2 + 8$
e) $169 = x^2 + y^2$
f) $-2x^2 = 2y^2 - 8$

Q7: Find the equation of the following circles with centre, the origin:

a) The circle with radius 2
b) The circle passing through the point (20, 21)
c) The circle with radius $3\sqrt{5}$
d) The circle passing through the point (4, 0)
e) The circle with diameter 10
f) The circle passing through the point on the line y = 2x - 3 where x = 2

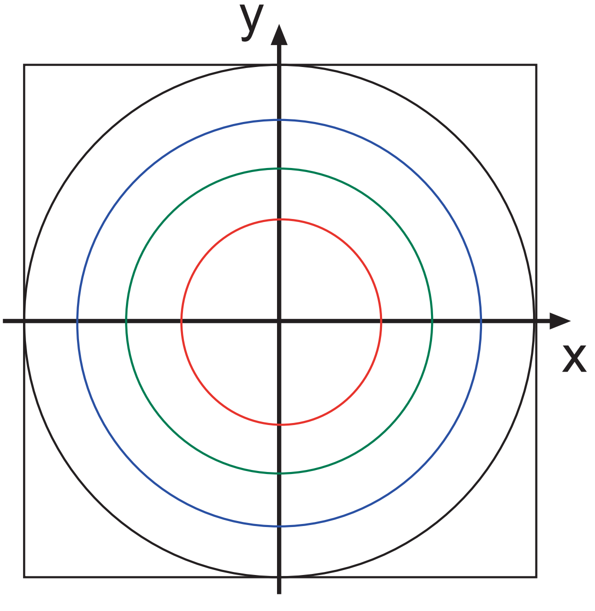
Q8: An exercise ring for the stable's horses is circular and located in a square field of side length 20 m. Assuming a clearance of 1 m outside the ring for safety, find the equation of the ring taking its centre to be the origin.



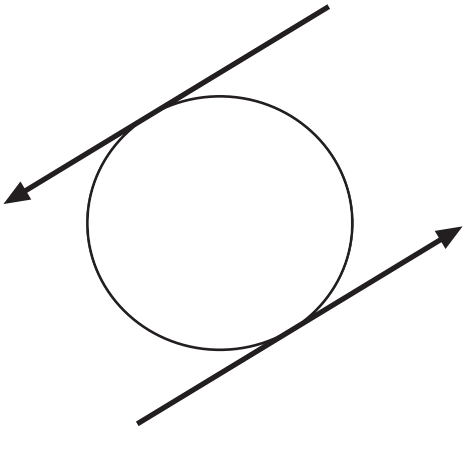
Q9: A square emblem has a circle at its centre with equation $x^2 + y^2 = 4$ cm

The pattern repeats to the edge of the emblem with circles on the same centre but with a radius of 1 cm more each time.

The emblem side is 10 cm what is the equation of the largest circle on the emblem?



Q10: A sweatshirt badge consists of two parallel arrows and a circle as shown.



If the centre of the circle is the origin and the arrows touch the circle at the points (-1, 5) and (1, -5) find the equation of the circle.

4.2.2 Circles with centres other than the origin

Calculator activity

10 min

Using a graphics calculator plot the pairs of values given and investigate the relationship by looking at the centre. On the TI83 calculator, the circles are drawn using ’2nd”prgm’ to access the ’draw’ menu and then select option 9.

a) Circle centre (0, 0) radius 2 and centre (0, 1) radius 2

b) Circle centre (0, 0) radius 4 and centre (-3, 0) radius 4

c) Circle centre (0, 0) radius 5 and centre (2, -3) radius 5

The activity clearly demonstrates that the second circle in each case is a copy of the first circle that has been moved along, down or up to the new centre.

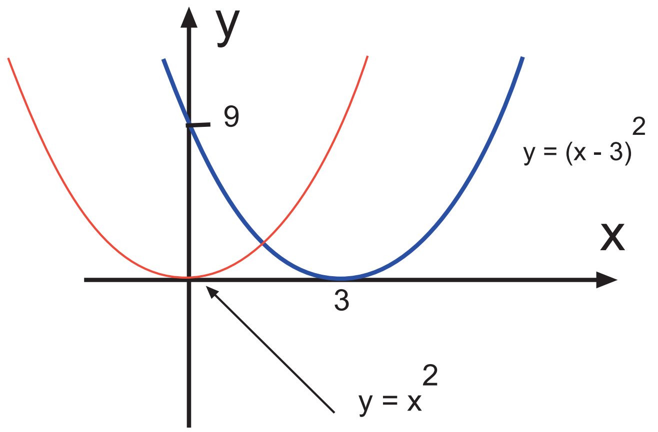
Recall that in the functions and graphs topic of this course, the relationships between functions were explored and that certain rules could be applied to find the formula of one function given the other.

Here is one example.

Example Sketch the graph of $y = (x - 3)^2$

This is of the form $f(x + k)$ where $k = -3$ and $f(x) = x^2$

The effect is a sideways shift of the graph.



A rule was given which enabled the graphing of a function related to f (x) and conversely this rule was useful in finding the equation of the related function from a graph. The rule is given below.

To obtain $y = f(x + k)$ take $y = f(x)$ and:

- For $k > 0$ slide the graph to the *left* by k units.
- For $k < 0$ slide the graph to the *right* by k units.

This rule and the initial activity in this section give an insight into the form of the equation of a circle where the centre is not at the origin.

General equation of a circle with centre other than the origin

The equation of a circle C with centre (a, b) and radius r where a, b $\neq$ 0 is

$$(x - a)^2 + (y - b)^2 = r^2$$

The proof of this equation is given as proof 2 in the section headed proofs near the end of this topic.

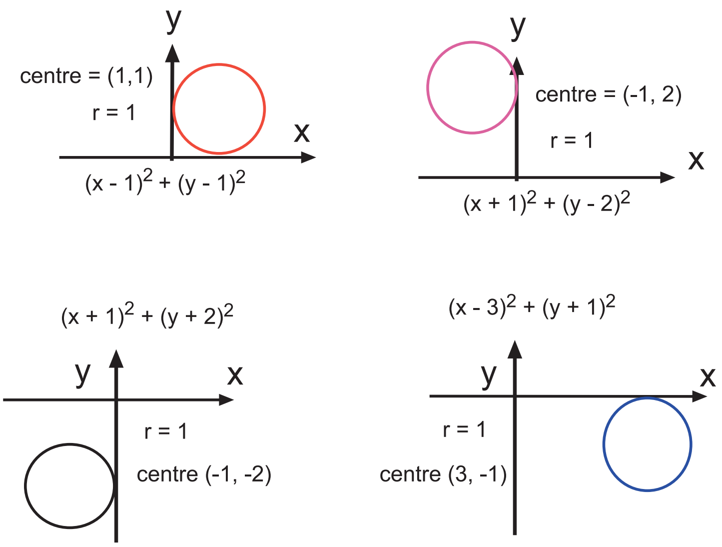
Thus the circle with equation $(x - a)^2 + (y - b)^2 = r^2$ can be found by taking the circle with equation $x^2 + y^2 = r^2$ and

For a, b positive: moving it to the right by a units and upwards by b units

For a, b negative: moving it to the left by a units and downwards by b units

Other variations follow from that rule.

Here are some examples.



Examples

1. What is the radius and the centre of a circle with equation $(x - 2)^2 + (y - 5)^2 = 9$?

Answer:

The radius is 3 and the centre is (2, 5)

2. $(x - 1)^2 + (y + 3)^2 = 36$ is the equation of the circle C and the points P and Q lie on the circumference . If the line through P, Q and C is parallel to the y-axis find the coordinates of P and Q if P lies above Q in a diagram.

Answer:

$(x - 1)^2 + (y + 3)^2 = 36$

The centre of the circle is (1, -3)

So P and Q have x-coordinate of 1

But the radius is 6 and so P has y-coordinate of -3 + 6 = 3 and Q has a y-coordinate of -3 - 6 = -9

P is the point (1, 3) and Q is the point (1, -9)

3. What is the radius and centre of the circle with equation $3(x + 2)^2 + 3(y - 1)^2 - 15 = 0$?

Answer:

Rearrange into the correct form.

$3(x + 2)^2 + 3(y - 1)^2 = 15 \Rightarrow (x + 2)^2 + (y - 1)^2 = 5$

This is in the correct form for a circle with radius $\sqrt{5}$ units and centre with coordinates (-2, 1)

4. Find the equation of a circle passing through the point P (6, 4) with centre at C (-3, 2)

Answer:

The general equation is $(x - a)^2 + (y - b)^2 = r^2$

The distance between the points P and C is the radius.

$\sqrt{(6 + 3)^2 + (4 - 2)^2} = \sqrt{85} = 9.2$

The equation is $(x + 3)^2 + (y - 2)^2 = 85$

Note that in some instances (see the last example) it is not necessary to calculate the square root as the equation of the circle uses the value of the square of the radius.

Points to remember when problem solving:

1. Use the exact formulae for the circle
2. Be aware of the geometry by making a sketch
3. Use the formula for the distance between two points
4. Consider the techniques of straight line equations where appropriate

General equation circles exercise

15 min

There is an alternative exercise on the web with randomised questions.

Q11: Find the radii and centres of the following circles:

a) $(x + 2)^2 + (y - 1)^2 = 49$
b) $x^2 + (y - 2)^2 - 9 = 0$
c) $3(x + 1)^2 + 3(y + 1)^2 - 2 = 1$
d) $(x + 6)^2 = - y^2 + 12$
e) $-6 = -x^2 - (y - 1)^2 - 3$
f) $-2(x - 1)^2 = 2(y + 4)^2 - 72$

Q12: Find the equation of the following circles:

a) The circle with radius 4 and centre (-3, 4)
b) The circle passing through the point (4, -5) with centre (2, 2)

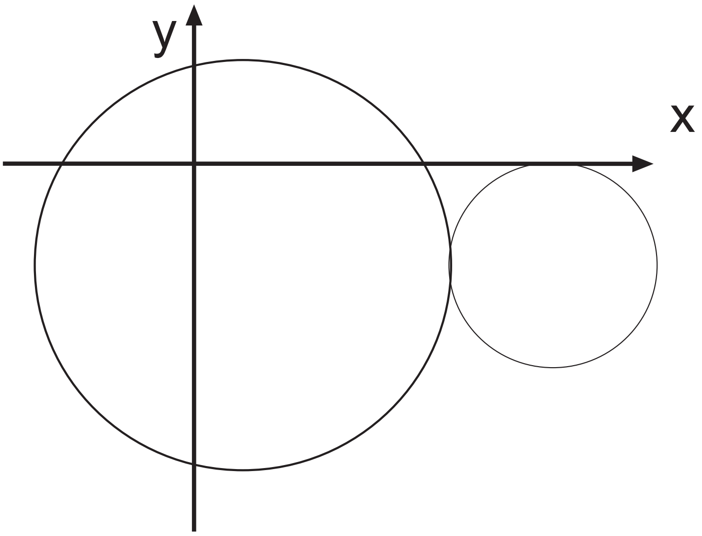
c) The circle with radius $4\sqrt{2}$ and centre on the positive x-axis such that the origin is on the circumference.

d) The circle passing through the point (4, 0) and with centre (-3, -1)

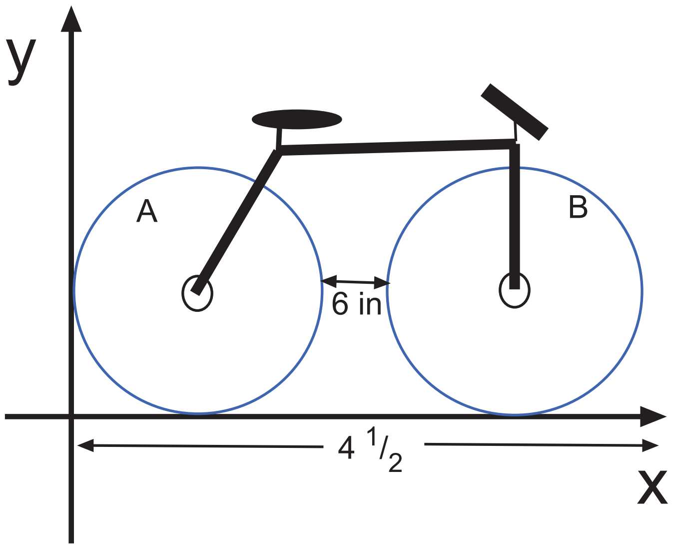
e) The circle with diameter 6 and centre midway between the points A (2, 4) and B (-6, 2)

f) The circle with diameter PQ where P is the point (8, -4) and Q is the point (2, -12)

Q13: A gearing system has two wheels, one large and one small. The line of centres of the two wheels is parallel to the x-axis and the equation of the larger wheel is $(x - 2)^2 + (y + 4)^2 = 64$. Find the equation of the smaller wheel which lies to the right of the larger wheel and has a radius of half of that of the larger wheel.



Q14: A child's bicycle has two identical wheels with a clearance between them of 6 inches. When held against a wall (represented by the y-axis with the ground as the x-axis) the bike measures $4^1/_2$ feet lengthwise. Find the equations of the two circles which represent the wheels.



4.2.3 Extended equation of the circle with centre (a, b): a, b $\neq$ 0

The general form of the equation of a circle can be expanded algebraically.

$(x - a)^2 + (y - b)^2 = r^2 \Rightarrow$

$x^2 - 2ax + a^2 + y^2 - 2by + b^2 - r^2 = 0 \Rightarrow$

$x^2 + y^2 + 2gx + 2fy + c = 0$ where $g = -a$, $f = -b$ and $c = a^2 + b^2 - r^2$

This form avoids confusion with a and b by using g and f and also uses addition in the equation instead of subtraction.

Extended equation of the circle

The extended equation of the circle where (-g, -f) is the centre and $\sqrt{(g^2 + f^2 - c)}$ is the radius (provided $g^2 + f^2 - c > 0$) is

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

The derivation is also included in the section on proofs for completeness as proof 3.

Examples

1. Find the centre and radius of the circle represented by the equation

$x^2 + y^2 + 10x + 6y - 2 = 0$

Answer:

This is in the extended form of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$
where g = 5 and f = 3

The centre is (-5, -3)

$r^2 = 25 + 9 + 2 = 36$ so $r = 6$

The radius is 6

2. In extended form give the equation of the circle with radius of 4 and centre (-6, 2)

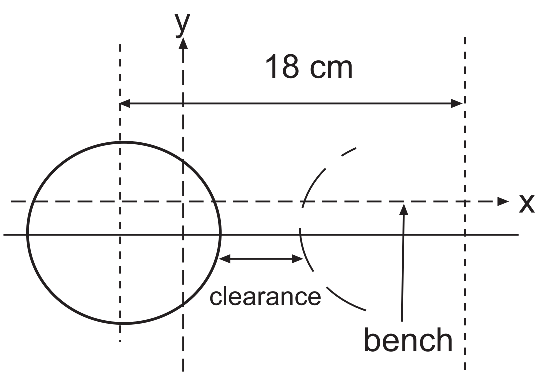
Answer:

The centre is represented by (-g, -f) $\Rightarrow$ g = 6 and f = -2

$r^2 = g^2 + f^2 - c \Rightarrow c = 36 + 4 - 16 = 24$

The equation is $x^2 + y^2 + 12x - 4y + 24 = 0$

3. The centres of two grinding wheels have to be at least 18 cm apart for safety. The equation of one of the wheels is $x^2 + y^2 + 4x + 2y - 4 = 0$. If the clearance between the wheels must be at least 1 cm find the range of values which the radius of the second wheel can take.



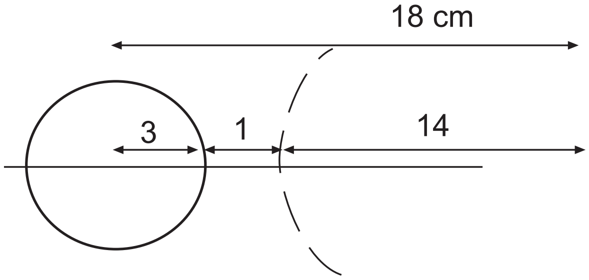
If the clearance is set at 11 cm find the equation of the second wheel given that the centres of the two wheels lie on a horizontal bench.

Answer:

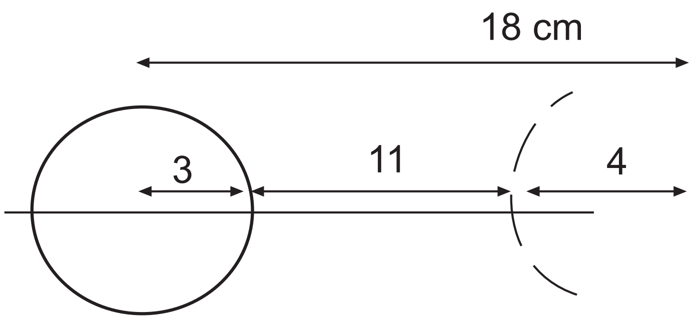
From the equation g = 2, f = 1 and $r^2 = g^2 + f^2 - c = 9 \Rightarrow$ the radius is 3 cm

With a clearance of at least 1 cm the edge of the second wheel is at least 4 cm away; but the centres are at least 18 cm apart $\Rightarrow$ the second wheel radius must be *at the most* 14 cm

The second wheel radius has a range of $0 < r \leq 14$



If the clearance is then set at 11 cm, the second wheel has a radius of 4 cm



The x-coordinate of the centre will be -2 + 18 = 16

The y-coordinate is the same for both wheels at y = -1

The equation for the second wheel is

$x^2 + y^2 - 32x + 2y + 241 = 0$

4. Find the equation of the circle through the three points
A (-2, 1), B (1, 4) and D (-2, 7)

Answer:

(This is an alternative way to some textbooks)

Let the centre of the circle be C (a, b), then AC = BC = DC = radius

Using the distance between two points gives

$AC^2 = (a + 2)^2 + (b - 1)^2$

$AB^2 = (a - 1)^2 + (b - 4)^2$

$DC^2 = (a + 2)^2 + (b - 7)^2$

Equate AC^2 with AB^2 and simplify to give the equation b = -a + 2: call this equation 1

Equate AC^2 with DC^2 and simplify to give the equation b = 4: call this equation 2

Solving equations 1 and 2 gives a = -2 and b = 4

The centre is (-2, 4) and the radius is found by substituting, say in AC^2, to give
$AC^2 = 9$. That is r = 3. The equation is $x^2 + y^2 + 4x - 8y + 11 = 0$

The last example demonstrates that a good grounding in geometry will help with the work on circles. In fact this last example could easily be solved in other ways. For example, the fact that perpendicular bisectors of chords pass through the centre of a circle could be used. The equations of the perpendicular bisectors of the two chords AB and BD can be equated to give the centre of the circle.

15 min

Geometry in circles investigation

Take the last example which was: Find the equation of the circle through the three points A (-2, 1), B (1, 4) and D (-2, 7)

Use the perpendicular bisector method to check the result. What other geometric facts can be ascertained about these three points? Does this suggest another approach to solving this problem?

15 min

Extended equation of a circle exercise

There is a web exercise if you prefer it.

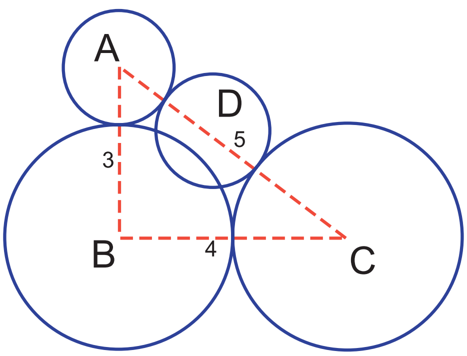
Q15: Find the radii and centres of the following circles:

a) The circle with equation $x^2 + y^2 - 4x - 2y - 4 = 0$
b) The circle with equation $x^2 + 6x - 1 = - y^2 + 4y + 2$
c) The circle with equation $x^2 + y^2 + 8y - 9 = 0$
d) The circle with equation $x^2 + y^2 - 10x + 21 = 0$

Q16: Find the equation in extended form of the following circles:

a) The circle with radius 3 and centre (-2, 1)
b) The circle with diameter 8 and centre (0, 5)
c) The circle with diameter through the points (2, 3) and (8, -3)
d) The circle C with its centre on the circumference of the circle A at x = -2 and with the same radius. Circle A has equation $x^2 + y^2 -10x + 8y - 8 = 0$

Q17: A company logo consists of three circles with centres at each vertex of a right angled triangle with sides of 3 cm, 4 cm and 5 cm as shown.



Circle A has equation $x^2 + y^2 - 2x + 4y + 4 = 0$. All circles are only touching (not intersecting). Find the equations of the remaining circles A and C. The logo is shown as it hangs with AB vertical. What is the radius and centre of circle D?

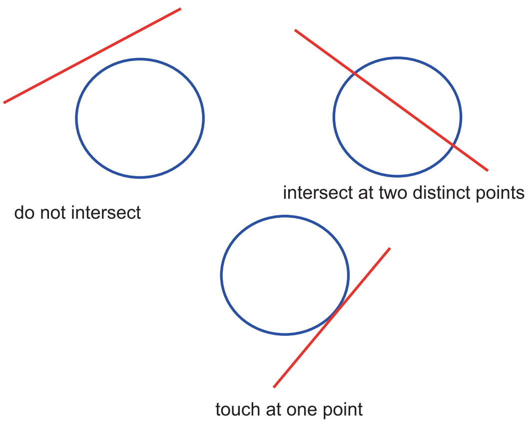
4.3 Circle intersections and tangents

Learning Objective

Solve problems including intersections with circles and lines

The problems considered previously were based solely on circles and the relationship with other circles. Many problems however, are concerned with the relationship between lines and circles. Geometrically, a circle and line can:

1. not intersect
2. intersect at two distinct points
3. touch at one point (in effect this is intersection at two equal points)



When a line meets a circle at one point, it is called a tangent to the circle. Tangents and circles have many geometric properties, some of which will be used in this section.

Examples

1. Intersection of a line and a circle

Find where the line y = -2x - 3 meets the circle with equation $x^2 + y^2 - 4x + 2y - 4 = 0$

Answer:

Substitute the value y = -2x - 3 into the equation of the circle to give

$x^2 + (-2x - 3)^2 - 4x + 2(-2x - 3) - 4 = 0$

$x^2 + 4x^2 + 12x + 9 - 4x - 4x - 6 - 4 = 0$

$5x^2 + 4x - 1 = 0$

$(5x - 1)(x + 1) = 0 \Rightarrow x = {}^1/_5$ or $x = -1$

When $x = {}^1/_5$, $y = -2 \times {}^1/_5 - 3 = {}^{-17}/_5$

When $x = -1$, $y = -2 \times -1 - 3 = -1$

The two points are $({}^1/_5, {}^{-17}/_5)$ and $(-1, -1)$

2. Tangent to a circle

Find the equation of the tangent to the circle $x^2 + y^2 = 25$ at the point P (3, 4)

Answer:

The circle centre is the O, the origin. The gradient of OP = ${}^{4}/_{3}$

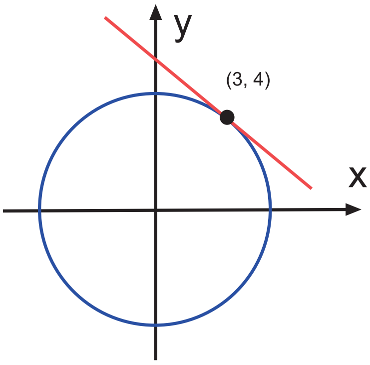
The geometry of tangent shows that the gradient of a tangent at a point is at right angles to the gradient of the radius to that point.

Therefore the gradient of the tangent at P is ${}^{-3}/_{4}$

The tangent is a straight line.

Thus the equation of the tangent is $(y - 4) = {}^{-3}/_{4}(x - 3)$

That is $4y = -3x + 25$



3. Does the line $y = 2x + 5$ intersect the circle with equation $(x - 2)^2 + (y + 1)^2 = 4$?

Answer:

Substitute $y = 2x + 5$ into the equation of the circle.

$x^2 - 4x + 4 + (2x + 6)^2 = 4$

$x^2 - 4x + 4 + 4x^2 + 24x + 36 - 4 = 0$

$5x^2 + 20x + 36 = 0$

Look at the discriminant D ($= b^2 - 4ac$).

$D = 400 - 720 = -320$

Since the discriminant is negative there is no solution and the line and circle do not meet.

4. Intersection of circle and tangent

Find the point at which the circle with equation $x^2 + y^2 + 6x - 8y - 7 = 0$ and the tangent $y = x - 1$ meet.

Answer:

Substitute $y = x - 1$ into the equation of the circle to give

$x^2 + x^2 - 2x + 1 + 6x - 8x + 8 - 7 = 0$

$2x^2 - 4x + 2 = 0$

$(2x - 2)(x - 1) = 0 \Rightarrow x = 1$ twice. (therefore it is a tangent!).

When $x = 1$, $y = x - 1 = 0$

The point at which the circle and tangent meet is (1, 0)

5. Find the two values of k such that the line $y = -2x + k$ is a tangent to the circle $x^2 + y^2 - 8x - 2y + 12 = 0$

Substitute for y in the equation of the circle to give

$x^2 + (-2x + k)^2 - 8x - 2(-2x + k) + 12 = 0$

$5x^2 + x(-4 - 4k) + k^2 - 2k + 12 = 0$

For the line to be a tangent, the discriminant is zero

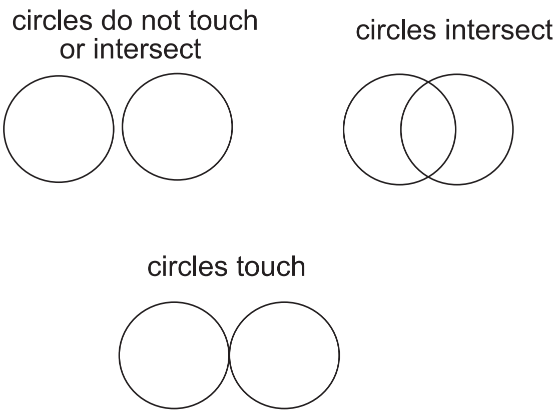
$(-4 - 4k)^2 - 4 \times 5 \times (k^2 - 2k + 12) = 0$

$-4k^2 + 72k - 224 = 0$

$k^2 - 18k + 56 = 0$

$(k - 4)(k - 14) = 0 \Rightarrow k = 4$ or $k = 14$

It is also possible to determine if two circles touch or intersect.



Again much of the technique is dictated by the particular problem. The next example relies on the distance formula.

Example Determine if the two circles with equations $(x - 2)^2 + (y - 3)^2 = 16$ and $(x + 1)^2 + (y - 3)^2 = 25$ touch, intersect or avoid each other.

Answer:

This is one way in which it may be possible to decide without reverting to looking for intersection points.

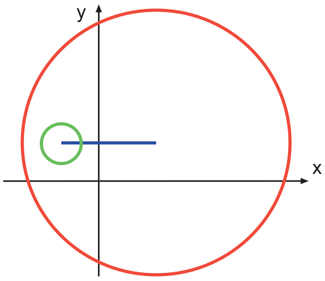
Let C be the circle with equation $(x - 2)^2 + (y - 3)^2 = 16$
and D be the circle with equation $(x + 1)^2 + (y - 2)^2 = 25$

C has centre (2, 3) and radius 4 and D has centre (-1, 2) and radius 5

The distance between the centres is found by using the distance formula and is $\sqrt{10} = 3.2$ to 1 d.p.

The two radii are both greater than this distance and so the circles intersect.

Beware: there are times where one circle could lie completely within another circle and so does not intersect. Using the approach shown in the last example works in some cases only. It will not work if one circle has a radius less than the distance between the centres when the second circle has a radius greater than this distance plus the radius of the smaller circle. Here is an example.



Circles within circles

20 min

Investigate circles within circles by varying the distance between the centres for different radii. Try to confirm the statement given in the last paragraph, viz, circles do not intersect if one circle has a radius less than the distance between the centres when the second circle has a radius greater than this distance plus the radius of the smaller circle. Find another condition when they do not intersect.

Returning to the example the following method will provide the coordinates of the points of intersection of the two circles.

Example : Intersecting circles

Find the intersection points of the two circles with equations

$(x - 2)^2 + (y - 3)^2 = 16$ and $(x + 1)^2 + (y - 3)^2 = 25$

The intersection points can be found by solving the two equations simultaneously.

$(x - 2)^2 + (y - 3)^2 - 16 = 0$: equation 1

$(x + 1)^2 + (y - 3)^2 - 25 = 0$: equation 2

equation 1 - equation 2 gives

$(x - 2)^2 - 16 - (x + 1)^2 + 25 = 0$

$x^2 - 4x + 4 - 16 - x^2 - 2x - 1 + 25 = 0$

$-6x + 12 = 0 \Rightarrow x = 2$

Substitute in one equation, say, $(x - 2)^2 + (y - 3)^2 = 16$ to give:

$(y - 3)^2 - 16 = 0 \Rightarrow$

$y^2 - 6y - 7 = 0 \Rightarrow$

$(y - 7)(y + 1) = 0$

$y = 7$ or $y = -1$

The two points of intersection are (2, 7) and (2, -1)

If the two circles touch instead of intersect, the quadratic equation will give two equal values on substitution.

Q18: What condition will ensure that two circles touch but do not intersect?

This can in fact be used to show that circles touch.

Intersection exercise

There is a web exercise if you prefer it.

30 min

Q19: When y = 0, there are two tangents to the circle $x^2 + y^2 + 6x - 8y - 7 = 0$. Find the equations of these tangents.

Q20: Find the equation of the tangent to the circle $x^2 + y^2 - 2x + 4y - 4 = 0$ at the point P (2, -5)

Q21: Find the points of intersection of the line $y = 2x - 4$ and the circle with equation $x^2 + y^2 - 5x - 2y - 54 = 0$

Q22: Determine whether the line $y = x + 3$ is a tangent to the circle with equation $x^2 + y^2 + 4x + 2y + 3 = 0$. Consider the line $y = -x + 3$: does this line touch, intersect or avoid the circle?

Q23: Find the intersection points of the circles with equations $x^2 + y^2 + 4x - 6y - 3 = 0$ and $x^2 + y^2 - 2x - 6y - 9 = 0$

Q24: Find k such that the line $y = -x + k$ is a tangent to the circle with equation $x^2 + y^2 - 10x - 2y + 18 = 0$

Q25: Find k such that the line $y = -2x + k$ is a tangent to the circle with equation $x^2 + y^2 - 4x - 4y + 3 = 0$

4.4 Summary

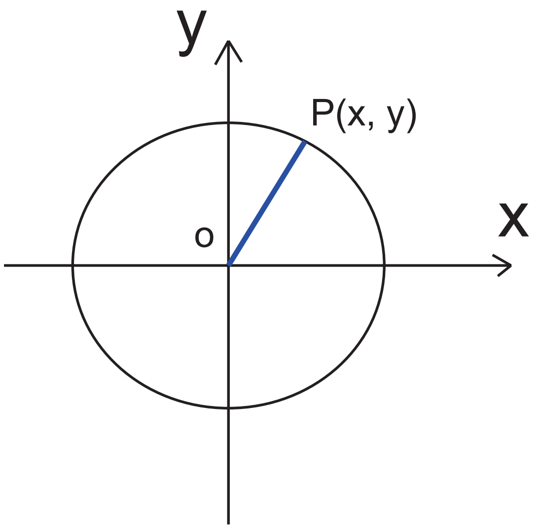
The following points and techniques should be familiar after studying this topic:

- The equation of the circle in the form $(x - a)^2 + (y - b)^2 = r^2$.
- The equation in the form $x^2 + y^2 + 2gx + 2fy + c = 0$.
- The points of intersection of lines and circles or two circles.
- The equations of tangents to a circle and points of contact of tangents.

4.5 Proofs

Proof 1:

$x^2 + y^2 = r$ is a circle centre the origin.



Let P be any point (x, y) on the circumference of the circle.

Then OP = radius, r

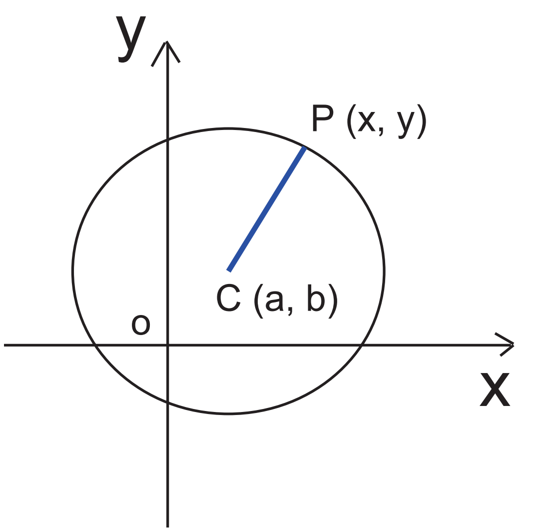
By the formula for the distance between two points,

$r = \sqrt{(x - 0)^2 + (y - 0)^2} = \sqrt{x^2 + y^2}$

It follows that $x^2 + y^2 = r^2$

Proof 2:

$(x - a)^2 + (y - b)^2 = r^2$ is a circle centre (a, b) and radius r



Let the circle have a centre at the point C (a, b) for any a, b $\neq$ 0

Let P be any point (x, y) on the circumference of the circle.

By definition, r is the radius and is the distance CP but by the distance formula, the length of CP is

$\sqrt{(x - a)^2 + (y - b)^2}$

Thus $(x - a)^2 + (y - b)^2 = r^2$

Proof 3:

$x^2 + y^2 + 2gx + 2fy + c = 0$ is a circle with centre (-g, -f) and radius $\sqrt{}(g^2 + f^2 - c)$

This is not a proof. It is a derivation of another formula from the one shown at proof 2.

The equation $(x - a)^2 + (y - b)^2 = r^2$ represents a circle with centre (a, b) and radius r

$(x - a)^2 + (y - b)^2 = r^2 \Rightarrow$

$x^2 - 2ax + a^2 + y^2 - 2by + b^2 - r^2 = 0$

Let -g = a, -f = b then the centre of the circle (a, b) is expressed as (-g, -f)

$x^2 + 2gx + g^2 + y^2 + 2fy + f^2 - r^2 = 0$

Let $c = g^2 + f^2 - r^2$

Then the equation becomes $x^2 + 2gx + y^2 + 2fy + c = 0$

Rearranging $c = g^2 + f^2 - r^2$ gives $r = \sqrt{}(g^2 + f^2 - c)$ as required.

4.6 Extended Information

Learning Objective

Find resources on the web for circle study

There are links on the web which give a selection of interesting sites to visit. Browsing the web under 'circle' will lead to many other good sites which cover this topic.

The study of circles is one of the more ancient topics with discoveries going back into the distant centuries. Many of the properties were discovered by the Egyptians and Greeks.

The history of the Mathematicians who studied circles is naturally complex and combined with studies of curves and other geometric shapes. The mathematicians detailed here give an insight into the vast amount of work and study undertaken.

In this topic it is best for the reader to search for topics which are of interest, such as squaring the circle, investigations on spirals and ellipses. On the web, the St Andrews University site provides an excellent link to some of the Mathematicians.

Euclid

Euclid solved all his problems using logical reasoning and concentrated on the geometry of circles and straight lines.

Thales

This Greek mathematician formulated some of the earliest theorems on the circle in the seventh century BC.

Hippocrates

One of the most famous investigations is that of 'squaring a circle'. Hippocrates spent much of his life working on this problem.

Ramanujan

Mathematicians through the centuries continue to wrestle with the problem but Ramanujan gave a construction in the early 1900s which is accurate to 6 d.p.

Plato

Plato was a great astronomer who believed that the planets were spherical. He was fascinated by geometric constructions and much of his work laid the foundations for others to follow.

4.7 Review exercise

Review exercise

30 min

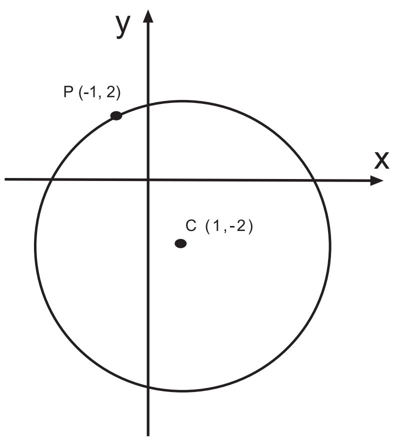
There is another exercise on the web if you prefer it.

Q26: A circle has radius 3 units and centre (-4, -2). Write down the equation of the circle in general form.

Q27: A circle has equation $x^2 + y^2 - 2x + 4y - 11 = 0$. Write down the coordinates of the centre of the circle and the length of the radius.

Q28: Show that the straight line with equation $y = 2x + 1$ is a tangent to the circle with equation $x^2 + y^2 - 10x - 12y + 56 = 0$

Q29: The point P (-1, 2) lies on a circle with centre C (1, -2).



Find the equation of the circle and of the tangent at P

Q30: A circle has equation $x^2 + y^2 - 8x + 2y - 8 = 0$. Write down the coordinates of the centre of the circle and the length of the radius.

Q31: The point P (3, -2) lies on a circle with centre C (9, -5). Find the equation of the tangent at P

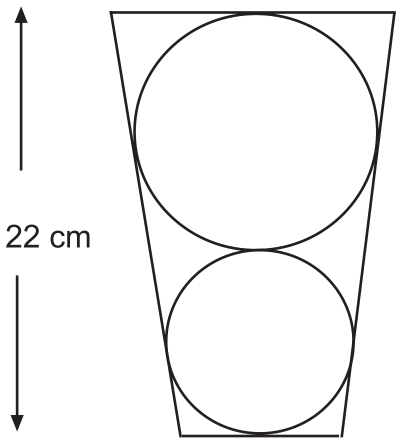
4.8 Advanced review exercise

Advanced review exercise

There is another exercise on the web if you prefer it.

30 min

Q32: A new company sign has a logo in the shape of two circles, one sitting on top of the other as shown and 22 cm high.



The equation of the smaller circle is $x^2 + y^2 - 12x - 26y + 189 = 0$ and the line of centres is parallel to the y-axis. Find the equation of the larger circle.

Q33: The company decides that the logo is better with the large circle on the bottom. If the equation of the smaller circle remains the same, what is the new equation of the larger circle?

Q34: Three circular cogs in a machine are set so that the largest (and central) cog A has equation $x^2 + y^2 - 2x + 4y - 4 = 0$. The two smaller cogs B and C have radii of 1 and 2 units respectively and are set so that AB is horizontal and BC is perpendicular to AB. Cog B touches cog A and cog C. If the cog B lies to the right of cog A and C lies above B, find the equations of the two circles which represent cogs B and C in the general form of $(x - a)^2 + (y - b)^2 = r^2$ and state with reasons whether cogs A and C touch, intersect or avoid contact with each other.

Q35: The circles A and B with centres at (0, -1) and (-1, -1) intersect. If the radius of A is $\sqrt{40}$ and the radius of B is $\sqrt{45}$, find the point (with positive coefficients) at which the circles cross. Give the equation of the lines through each centre to this point.

4.9 Set review exercise

Set review exercise

Work out your answers and then access the web to input your answers in the online test called set review exercise.

30 min

Q36: A circle has radius 5 units and centre (-2, 4). Write down the equation of the circle.

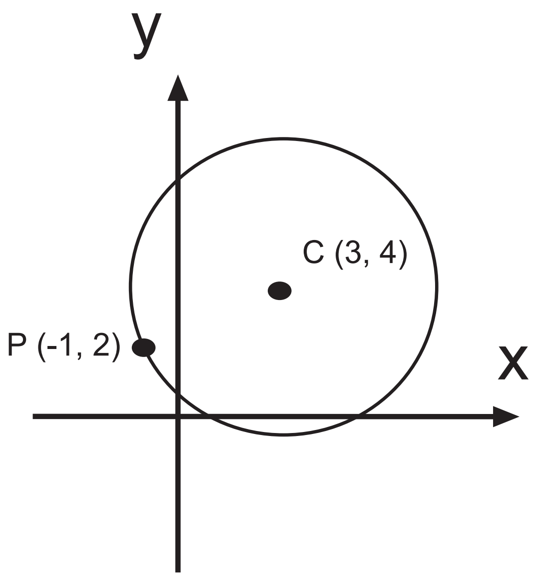
Q37: A circle has equation $x^2 + y^2 + 8x - 2y - 8 = 0$. Write down the coordinates of the centre of the circle and the length of the radius.

Q38: Show that the straight line with equation $y = 2x - 3$ is a tangent to the circle with equation $x^2 + y^2 - 2x - 8y + 12 = 0$

Q39: A circle has equation $x^2 + y^2 - 2x + 6y - 6 = 0$. Write down the coordinates of the centre of the circle and the length of the radius.

Q40: The point P (-4, -1) lies on a circle with centre C (-2, 2). Find the equation of the tangent at P

Q41: The point P (-1, 2) lies on a circle with centre C (3, 4).



Find the equation of the tangent at P

Glossary

Circle

A circle is defined as the set of points P (x, y) that are at a given (constant) distance r (the radius) from a given point C (a, b)(the centre).

constant of integration

When $\int f(x)\,dx = F(x) + C$ then C is the **constant of integration**.

definite integral

A **definite integral** is one which has a numerical value thus when F (x) is the anti-derivative of f (x) then

$\int_a^b f(x)\,dx = F(b) - F(a) \quad (a \leqslant x \leqslant b)$

Degree of a polynomial

The degree of a polynomial is the value of the highest power of x in the expression.

Differential equation

A **differential equation** is an equation involving an unknown function and its derivatives.

Discriminant

The discriminant of the quadratic equation $ax^2 + bx + c = 0$ is $b^2 - 4ac$

Dividend

The dividend in a long division calculation is the expression which is being divided. As a fraction it is the numerator.

Divisor

The divisor is the expression which is doing the dividing.
It is the expression outside the division sign. As a fraction it is the denominator.

Equation of a circle with centre, the origin

The equation of a circle C with centre O (0, 0) and radius r is

$$x^2 + y^2 = r^2$$

Extended equation of the circle

The extended equation of the circle where (-g, -f) is the centre and $\sqrt{(g^2 + f^2 - c)}$ is the radius (provided $g^2 + f^2 - c > 0$) is

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

Factor Theorem

The **Factor theorem** states that if p (h) = 0 then (x - h) is a factor of p (x)

and if (x - h) is a factor of p (x) then p (h) = 0

fundamental theorem of calculus

The **fundamental theorem of calculus** states that if F (x) is an anti-derivative of f (x) then

$$\int_a^b f(x)\,dx = (F(b) + C) - (F(a) + C)$$

$$= F(b) - F(a) \qquad (a \leqslant x \leqslant b)$$

General equation of a circle with centre other than the origin

The equation of a circle C with centre (a, b) and radius r where a, b $\neq$ 0 is

$$(x - a)^2 + (y - b)^2 = r^2$$

general solution

The **general solution** of a differential equation contains an arbitrary constant and gives infinitely many solutions that all satisy the differential equation.

initial condition

For a differential equation an **initial condition** is additional information required to determine a particular solution. This could be a coordinate on a curve, a velocity at t = 0, the amount of money in a bank account on 1st January 2000, etc.

integral

When $\int f(x)\,dx = F(x) + C$ then F (x) is the **integral**.

integrand

When $\int f(x)\,dx = F(x) + C$ then f (x),which is the function to be integrated, is called the **integrand**.

integration

Integration is the process of finding anti-derivatives.

lower limit of integration

a is the **lower limit of integration** in the definite integral $\int_a^b f(x)\,dx$

particular solution

The **particular solution** of a differential equation is a solution which is often obtained from the general solution when an initial condition is known.

Polynomial of degree n.

An expression of the form $a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \ldots + a_2 x^2 + a_1 x^1 + a_0$ where $a_0, \ldots, a_n$ are constants with $a_n \neq 0$ is called a polynomial of degree n.

Quadratic formula

The roots of the quadratic $ax^2 + bx + c$ are given by the formula

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Quotient

The quotient is the answer to the division **but not including** the remainder.

Remainder theorem

If a polynomial p (x) is divided by (x - h) the remainder is p (h)

Root of a polynomial

A root of a polynomial p (x) is a solution to the equation p (x) = 0

upper limit of integration

b is the **upper limit of integration** in the definite integral $\int_a^b f(x)\,dx$

x-coordinate of the turning point of a quadratic graph

The x-coordinate of the turning point of the graph of the quadratic $ax^2 + bx + c$ is given by $^{-b}/_{2a}$

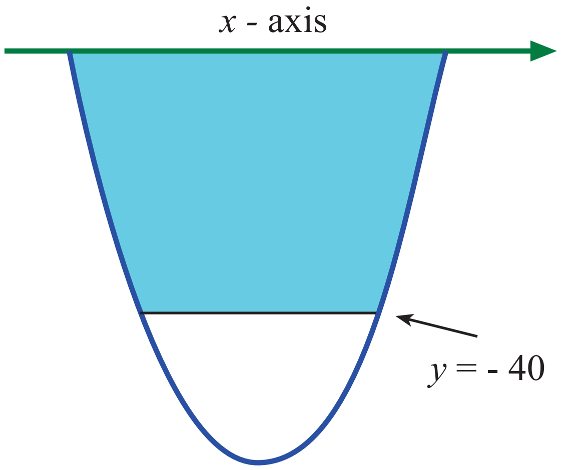
Hints for activities

Topic 2: Basic Integration

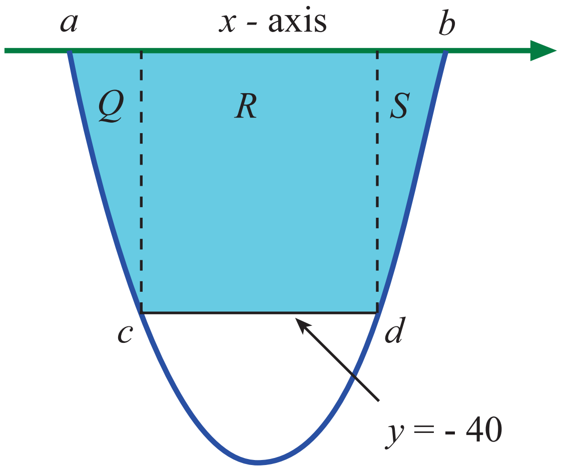
Float the Boat

Hint 1:

Volume of water = cross-section shaded area x length of dry dock.



Hint 2: To calculate the area of the cross section you will need to calculate the x coordinates for a,b,c and d as indicated in the following diagram.

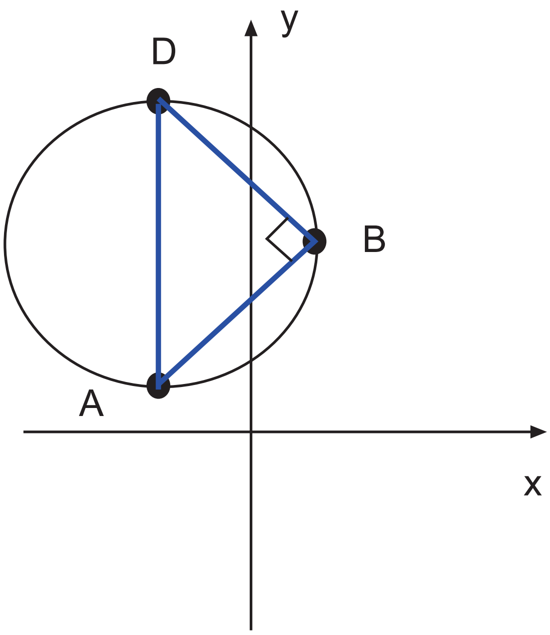


The shaded area can be calculated in three parts. Notice that R is a rectangle and that Q = S

Topic 4: The equation of a circle

Geometry in circles investigation

Hint 1: The three points ABD form a right angled triangle with the right angle at B . This only occurs if AD is the diameter of the circle. Thus if this fact is known or can be easily ascertained then the radius and centre can be calculated in an easier way than either of the previous methods. However, this is only the case where ABD is a right angled triangle. In other examples AD could be a chord.


y
D
B
A
x

Answers to questions and activities

1 Factor/Remainder theorem and quadratic theory

Revision exercise (page 2)

Q1: (3x - 1)(x - 1)

Q2: $\frac{b}{3d} - \frac{c}{2a}$

Q3: This evaluates to 8

Q4: It crosses x at x = 3, that is at the point (3, 0)

It crosses y at y = 6, that is at the point (0, 6)

Q5:

a) $6\sqrt{2}$
b) $\sqrt{2}$

Terminology exercise (page 3)

Q6:

a) This is an equation and so it has solutions. The expression is of degree 2
The solutions are x = - 2 or x = 2
b) This is a polynomial of degree 1 and has a root. The root is x = 1
c) This is a polynomial of degree 2 and has roots. The roots are x = $^3/_2$ and x = -3

Q7: p (1) = 3

p (2) = 19

p (3) = 55

p (-1) = 7

Finding coefficients exercise (page 5)

Q8:

a) k = -1
b) k = -3
c) k = -16
d) k = -3

Q9:

a) c = -2 and d = -1
b) c = 1 and d = -4
c) c = 0 and d = -7
d) c = 1 and d = 12

Division of a polynomial exercise (page 10)

Q10:

a) $(x + 1)(x - 1)(x - 2)(x + 3)$

b) $(x + 2)(x^2 + 3x + 5)$

c) $(x - 1)(x + 1)(x - 2)(x + 2)$

Q11:

a) $(2x + 1)(4x + 2)(x + 3) = 2(2x + 1)^2(x + 3)$

b) $(3x - 1)(2x + 3)(5x - 1)$

c) $(2x + 3)(3x - 1)(2x - 1)$

Quadratic formula exercise (page 11)

Q12:

a) $\frac{-5}{6} + \frac{\sqrt{37}}{6}$ and $\frac{-5}{6} - \frac{\sqrt{37}}{6}$

b) $-1 + \frac{\sqrt{7}}{2}$ and $-1 - \frac{\sqrt{7}}{2}$

c) $2 + \sqrt{3}$ and $2 - \sqrt{3}$

d) $\frac{-3}{2} + \frac{\sqrt{13}}{2}$ and $\frac{-3}{2} - \frac{\sqrt{13}}{2}$

Calculator exercise (page 12)

The roots are x = - 0.18 and 1.85

Calculator program (page 12)

One possible program is:

```
: Prompt {A, B, C}
: √(B^2 - 4 * A * C) = D
: (-B + D) / (2 * A) = P
: (-B - D) / (2 * A) = M
: Disp P, M
Disp D
```

Note that this version also displays the discriminant.

Discriminant exercise (page 14)

Q13:

a) -24 and the quadratic has no real roots.
b) 4 and the quadratic has distinct real roots.
c) 0 and the quadratic has equal roots.
d) 9 and the quadratic has distinct real roots.
e) -44 and the quadratic has no real roots
f) 16 and the quadratic has distinct real roots. Note that b = 0 in this case.

Q14:

a) $-4 \leq q < 4$
b) $q < {}^{25}/_{8}$
c) $q = \pm 12$

Q15:

a) The graph crosses the x-axis
b) The graph crosses the x-axis
c) The graph avoids contact with the x-axis

Intersection exercise (page 16)

Q16: The tangent has the formula y = 2x + c

At intersection, $2x + c = 2x^2 - 2x - 3$

$\Rightarrow 2x^2 - 4x - 3 - c = 0$

The discriminant condition for a tangent is $b^2 - 4ac = 0$

$\Rightarrow 16 - 8(-3 - c) = 0 \Rightarrow 40 + 8c = 0 \Rightarrow c = -5$

The equation of the tangent is y = 2x - 5

Q17: The tangent has the formula y = x + c

At intersection, $x + c = -3x^2 - 5x + 5$

$\Rightarrow -3x^2 - 6x + 5 - c = 0$

The discriminant condition for a tangent is $b^2 - 4ac = 0$

$\Rightarrow 36 + 12(5 - c) = 0 \Rightarrow 96 - 12c = 0 \Rightarrow c = 8$

The equation of the tangent is y = x + 8

Q18:

a) If they intersect $2x - 3 = x^2 - 4x + 2$
$\Rightarrow x^2 - 6x + 5 = 0$
The discriminant is $b^2 - 4ac = 36 - 20 = 16$
$\Rightarrow b^2 - 4ac > 0$ and the curve and line do intersect.

The quadratic formula gives x = 1 or x = 5
Substitution of these values in the line y = 2x - 3 gives y = -1 or y = 7
The intersection points are (1, -1) and (5, 7)

b) If they intersect $-2x + 1 = 2x^2 - 5x - 1$
$\Rightarrow 2x^2 - 3x - 2 = 0$
The discriminant is $b^2 - 4ac = 9 + 16 = 25$
$\Rightarrow b^2 - 4ac > 0$ and the curve and line do intersect.
The quadratic formula gives x = -0.5 or x = 2
Substitution of these values in the line y = -2x + 1 gives y = 2 or y = -3
The intersection points are (-0.5, 2) and (2, -3)

c) If they intersect $-4x + 5 = -x^2 - 3x + 2$
$\Rightarrow -x^2 + x - 3 = 0$
The discriminant is $b^2 - 4ac = 1 - 12 = -11$
$\Rightarrow b^2 - 4ac < 0$ and the curve and line do not intersect.

d) If they intersect $5x - 3 = x^2 + 3x - 2$
$\Rightarrow x^2 - 2x + 1 = 0$
The discriminant is $b^2 - 4ac = 4 - 4 = 0$
$\Rightarrow b^2 - 4ac = 0$ and the curve and line do intersect by touching.
The quadratic formula gives x = 1
Substitution of this value in the line y = 5x - 3 gives y = 2
The intersection point is (1, 2) (The line is a tangent to the curve.)

Inequalities exercise (page 18)

Q19:

a) $x < -2$ and $x > 3$
b) $-1 < x < 2$
c) $1 < x < 2$
d) x = 2 and x = -5
e) $x \leq -4$ and $x \geq 6$

Q20:

a) $x < -2.61$ and $x > 4.61$
b) $x \leq -1.62$ and $x \geq 0.62$
c) $x \leq -2.77$ and $x \geq 1.27$
d) x = 1.37 and x = -4.37
e) $-4.61 \leq x \leq 2.61$

Approximate roots exercise (page 20)

Q21: -0.4

Q22: 0.2

Q23: p (7) = -2 and p (8) = 6

The root to 1 d.p. is 7.3

Q24: p (0) = -2 and p (1) = 1

The root to 2 d.p. is 0.87

Review exercise (page 22)

Q25: p (2) = 0 and so x - 2 is a factor

p (x) = (x - 2)(x - 1)(x + 4)

Q26: There are no real roots since the discriminant value is negative.

Q27: By trial p (1) = 0 and so x - 1 is one factor. Division gives the quotient as $2x^2 - x - 1$ which subsequently factorises to give (x - 1)(2x + 1)

The roots are (x - 1) twice and (2x + 1)

Q28: The roots are equal roots of x - 3

Advanced review exercise (page 23)

Q29: $\frac{(x - 5)^2}{x^2 + 5} = k \Rightarrow$

$(x - 5)^2 = k (x^2 + 5) \Rightarrow$

$x^2 - 10x + 25 = kx^2 + 5k \Rightarrow$

$(k - 1)x^2 + 10x - 25 + 5k = 0$

For equal roots then

$b^2 - 4ac = 0 \Rightarrow 100 - 4(k -1)(5k - 25) = 0 \Rightarrow$

$-20k^2 + 120k = 0 \Rightarrow$

k (k - 6) = 0 (by dividing through first by -20) $\Rightarrow$ k = 0 or k = 6

Q30: $f (3) = 3^3 + p \times 3^2 - 4 \times 3 + q = 0$

9p + q = -15 $\Rightarrow$ 5p = -15 since q = -4p

p = - 3 and so q = 12

Q31: $b^2 - 4ac = 4 - 12 = -8$

The discriminant is negative and there are no real roots.

Q32: If they intersect, $x^2 - 4x + 1 = -x^2 - 5x - 6$

$\Rightarrow 2x^2 + x + 7 = 0$

$b^2 - 4ac = 1 - 56 = -55$

The discriminant is negative and the two quadratics do not intersect.

The discriminant of $x^2 - 4x + 1$ is 12 and since this is positive, the quadratic has two distinct real roots.

The discriminant of $-x^2 - 5x - 6$ is 1 and since this is positive, the quadratic has two distinct real roots.

Set review exercise (page 23)

Q33: This answer is only available on the web.

Q34: This answer is only available on the web.

Q35: This answer is only available on the web.

Q36: This answer is only available on the web.

2 Basic Integration

Exercise 1 (page 28)

Q1: 27

Q2: 125

Q3: 216

Q4: $A(t) = t^2$

Q5: $A(t) = t^4$

Q6: $A(t) = t^5$

Q7:

f(t)	2t	$3t^2$	$4t^3$	$5t^4$	$6t^5$	...	$100t^{99}$
Area, A(t)	t^2	t^3	t^4	t^5	t^6	...	t^{100}

Q8: $A(t) = t^{n+1}$

Exercise 2 (page 30)

Q9:

a) $f(x) = x^9$
b) $f(x) = x^{15}$
c) $f(x) = x^{20}$
d) $f(x) = x^2$

Q10:

a) $f(x) = x^{-7}$
b) $f(x) = x^{-21}$
c) $f(x) = x^{-9}$
d) $f(x) = x^{-1}$

Q11:

a) $f(x) = x^{3/2}$
b) $f(x) = x^{5/4}$
c) $f(x) = x^{3/4}$
d) $f(x) = x^{2/5}$
e) $f(x) = x^{-1/4}$
f) $f(x) = x^{-5/6}$

Exercise 3 (page 33)

Q12:

a) $\frac{1}{5}x^5 + C$

b) $\frac{1}{10}x^{10} + C$

c) $\frac{1}{12}t^{12} + C$

d) $\frac{1}{2}x^2 + C$

e) $\frac{1}{15}y^{15} + C$

f) $\frac{1}{22}s^{22} + C$

Q13:

a) $-\frac{1}{2x^2} + C$

b) $-\frac{1}{4p^4} + C$

c) $-\frac{1}{17x^{17}} + C$

d) $-\frac{1}{5u^5} + C$

Q14:

a) $\frac{3}{5}x^{5/3} + C$

b) $\frac{4}{9}x^{9/4} + C$

c) $3x^{1/3} + C$

d) $-\frac{5}{2x^{2/5}} + C$

e) $\frac{6}{5}x^{5/6} + C$

f) $-\frac{6}{r^{1/6}} + C$

Exercise 4 (page 35)

Q15:

a) $\frac{5}{3}x^3 + C$

b) $2x^4 + C$

c) $-\frac{3}{2t^6} + C$

d) $8x + C$

e) $8x^{3/2} + C$

f) $\frac{7}{2v^2} + C$

g) $-3y + C$

h) $12a^{1/3} + C$

i) $-\frac{20}{x^{2/5}} + C$

j) $\frac{8}{x^{3/4}} + C$

Q16:

a) $2x^3 + \frac{5}{2}x^2 + C$

b) $\frac{1}{2}x^6 - 2x^3 + 7x + C$

c) $2t^{5/2} - t + C$

d) $-\frac{7}{3y^3} + y^2 + 8y + C$

e) $12x^{1/3} + \frac{3}{5}x^{5/3} + C$

f) $\frac{12}{x^{5/6}} - \frac{7}{2x^2} + 3x + C$

Exercise 5 (page 36)

Q17:

a) $\frac{4}{5}x^{5/4} + C$

b) $\frac{3}{5}x^{5/3} + C$

c) $2x^{7/2} + C$

d) $-\frac{1}{2x^2} + C$

e) $-\frac{1}{3x} + C$

f) $\frac{1}{8x^4} + C$

g) $15x^{1/3} + C$

h) $-\frac{2}{5x^{1/2}} + C$

Q18:

a) $\frac{1}{3}x^3 - 5x^2 + 25x + C$

b) $\frac{4}{3}t^3 + 2t^2 + t + C$

c) $\frac{1}{4}x^4 + 2x + C$

d) $\frac{2}{7}u^{7/2} - \frac{8}{3}u^{3/2} + C$

e) $\frac{1}{3}v^2 + 2v - \frac{1}{v} + C$

f) $\frac{3}{4}p^4 + \frac{5}{3}p^3 - p^2 + C$

Q19:

a) $\frac{3}{4}x^4 + \frac{1}{3}x^3 + C$

b) $-\frac{5}{2t^2} - \frac{1}{4}t^4 + C$

c) $\frac{2}{3}x^{3/2} + 2x^{1/2} + C$

d) $\frac{2}{5}v^{5/2} + \frac{2}{3}v^{3/2} - 8v^{1/2} + C$

e) $\frac{1}{3}a^3 - 6a - \frac{9}{a} + C$

f) $\frac{1}{3}t^3 + 2t - \frac{1}{t} + C$

Exercise 6 (page 39)

Q20:

a) $\int_2^6 (x + 2)\ dx$

b) $\int_0^7 (7 - x)\ dx$

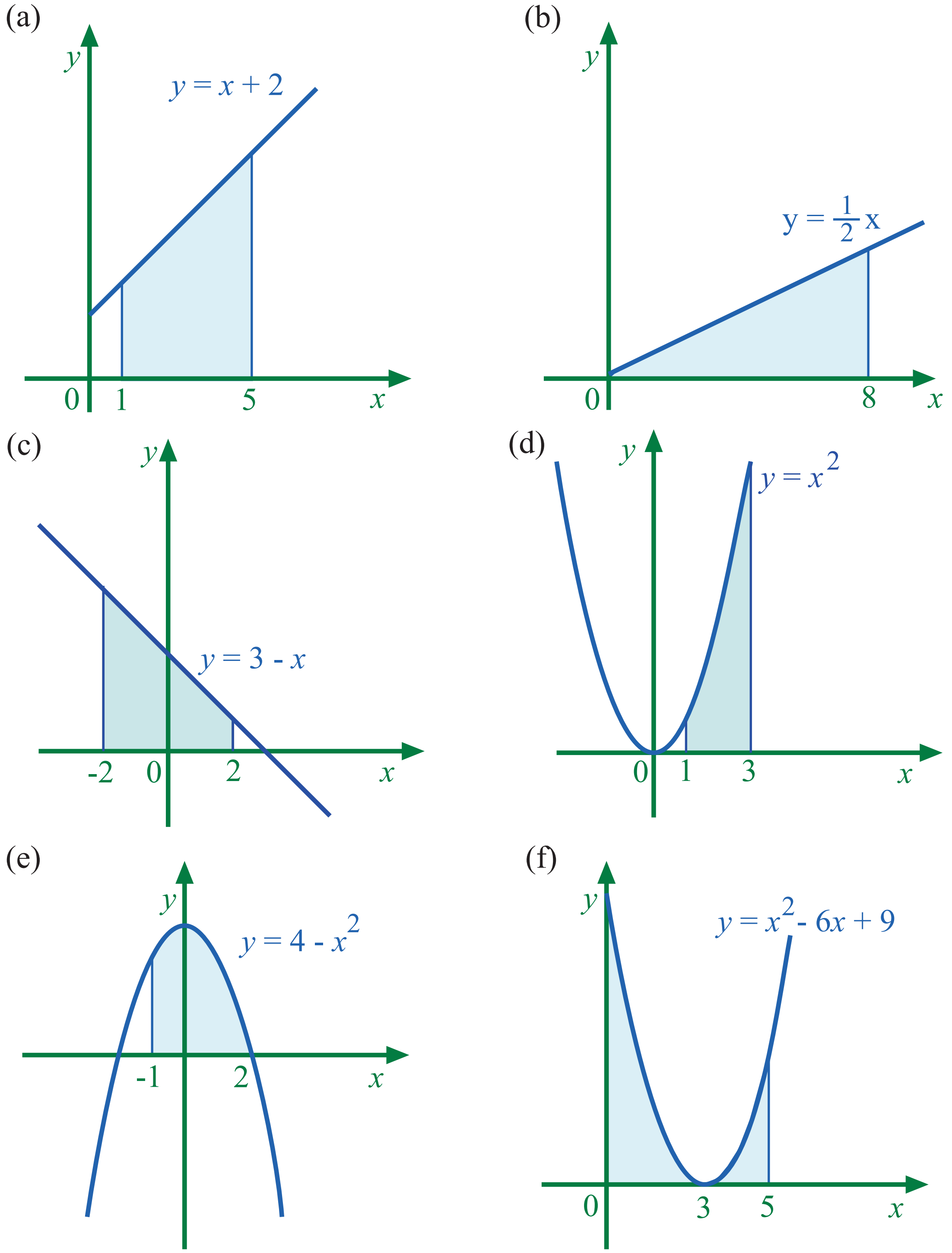
c) $\int_1^5 (x - 5)\ dx$

d) $\int_3^6 \left(x^2 - 6x + 9\right)\ dx$

e) $\int_{-5}^{1} \left(5 - 4x - x^2\right)\ dx$

f) $\int_2^8 \frac{8}{x} dx$

Q21:



Exercise 7 (page 43)

Q22:

a) 9

b) 21

c) 80
d) 45
e) 42
f) $14\frac{1}{4}$

Q23:

a) 6
b) 52
c) $\dfrac{3}{10}$
d) $4\frac{1}{2}$
e) 78
f) 3744

Q24:

a) 40
b) $345\frac{1}{3}$
c) $5\frac{1}{3}$
d) $17\frac{1}{2}$
e) $15\frac{3}{4}$
f) 408

Q25:

a)

$$\begin{aligned}\int_0^a (3\mathrm{x}-5)\,\mathrm{dx} = 4 \Rightarrow \left[\tfrac{3}{2}\mathrm{x}^2 - 5\mathrm{x}\right]_0^a &= 4\\ \left(\tfrac{3}{2}\mathrm{a}^2 - 5\mathrm{a}\right) - \left(\tfrac{3}{2}\times 0^2 - 5\times 0\right) &= 4\\ \tfrac{3}{2}\mathrm{a}^2 - 5\mathrm{a} &= 4\\ \tfrac{3}{2}\mathrm{a}^2 - 5\mathrm{a} - 4 &= 0\\ 3\mathrm{a}^2 - 10\mathrm{a} - 8 &= 0\\ (3\mathrm{a}+2)\,(\mathrm{a}-4) &= 0\\ \mathrm{a} = -\frac{2}{3} \text{ or } \mathrm{a} &= 4\end{aligned}$$

Since we are given that a $>$ 0 the solution is a = 4 b)

$$\int_0^a \sqrt{x}dx = 18 \Rightarrow \left[\frac{2x^{3/2}}{3}\right]_0^a = 18$$

$$\left(\frac{2a^{3/2}}{3}\right) - \left(\frac{2 \times 0^{3/2}}{3}\right) = 18$$

$$\frac{2a^{3/2}}{3} = 18$$

$$2a^{3/2} = 54$$

$$a^{3/2} = 27$$

$$\sqrt{a}^3 = 27$$

$$a = 9$$

Answers from page 44.

Q26: -4

Q27: 4

Answers from page 45.

Q28: 0

Q29: The answer is 0 because the negative value below the axis between x = 1 and x = 3 cancels out the positive value between x = 3 and x = 4

Q30: 8

Exercise 8 (page 46)

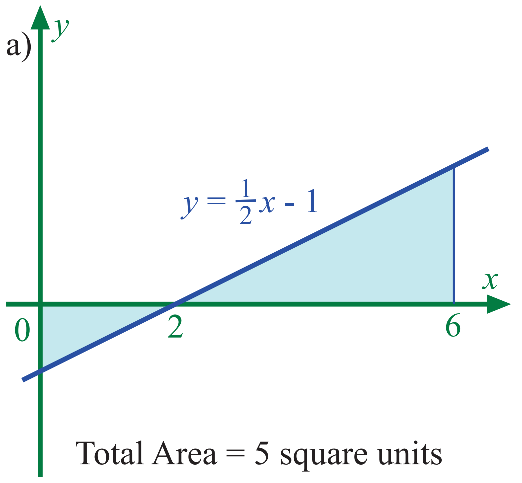
Q31:

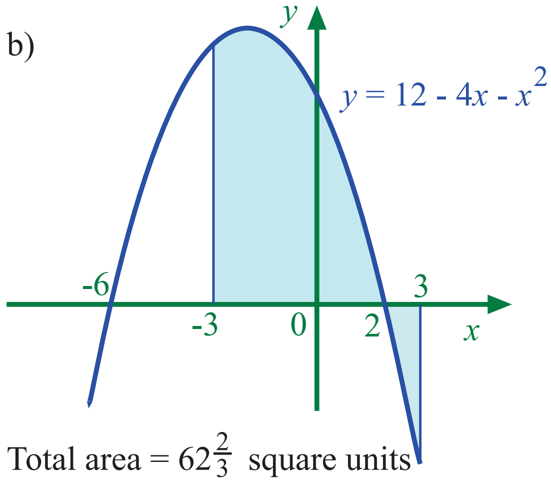
a) 20 square units
b) 68 square units
c) $21\frac{1}{3}$ square units
d) $41\frac{1}{3}$ square units

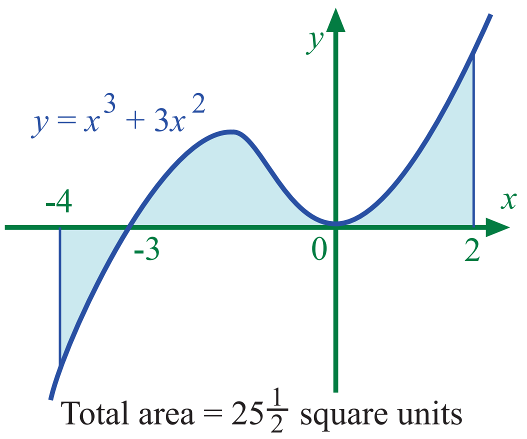
Q32:

a) Line cuts the x-axis at x = 1
Shaded area is 10 square units
b) Curve cuts the x-axis at x = 1
Shaded area is $9\frac{1}{2}$ square units
c) Curve cuts the x-axis at x = 1 and x = 5
Shaded area is 13 square units
d) Curve cuts the x-axis at x = -4, x = 0 and x = 2
Shaded area is $49\frac{1}{3}$ square units

Q33:







Q34:

a) 36 square units

b) $1\frac{1}{3}$ square units

Q35:

a) $x^3 - 3x - 2 = (x + 1)^2(x - 2)$ hence the curve cuts the axes at x = -1 and x = 2

b) Maximum stationary point at (-1, 0)
Minimum stationary point at (1, - 4)

c) Total area is $44\frac{3}{4}$ square units

Exercise 9 (page 50)

Q36:

a) $85\frac{1}{3}$ square units

b) $4\frac{1}{2}$ square units

Q37:

a) $20\frac{5}{6}$ square units
b) $41\frac{2}{3}$ square units

Q38:

a) $10\frac{2}{3}$ square units
b) $4\frac{1}{2}$ square units
c) $21\frac{1}{3}$ square units
d) $62\frac{1}{2}$ square units

Q39:

a) $20\frac{1}{4}$ square units
b) $20\frac{1}{4}$ square units
c) $40\frac{1}{2}$square units
d) The areas should be calculated separately because the upper graph is different in each case.

Q40:

a) (-2, 4) (0, -2) (2, 0)
b) 8 square units

Q41:

$$\begin{aligned}\text{Area under parabola} &= \int_{-4}^{4} \left(8 - \tfrac{1}{2}x^2\right) dx \\ &= 42\tfrac{2}{3}\ \text{cm}^2 \\ \text{Surface area of silver} &= 240 - 2 \times \left(42\tfrac{2}{3}\right) \\ &= 154\tfrac{2}{3}\ \text{cm}^2\end{aligned}$$

Q42:

a) $266\frac{2}{3}$ m, $166\frac{2}{3}$ m

b) Let s = distance then $s = \int v\, dt$

The particles have travelled the same distance when:

$\int \left(12t - t^2\right) dt = \int 0.5t^2 dt$

$\Rightarrow t = 12$ seconds

Q43:

When $y = 2$ then $x = \pm 4$

When $y = 14$ then $x = \pm 8$

$$\begin{aligned}\text{Total area = Rectangle } (8 \times 12) &+ \int_{-8}^{-4} \left(14 - \left(\tfrac{1}{4}x^2 - 2\right)\right) dx \\ &+ \int_{4}^{8} \left(14 - \left(\tfrac{1}{4}x^2 - 2\right)\right) dx \\ &= 96 + 2 \times 26\tfrac{2}{3} \\ &= 149\tfrac{1}{3}\ \text{m}^2\end{aligned}$$

Float the Boat (page 53)

Cross section area = $1633\frac{1}{3}$ m^2
Volume of water = $1633\frac{1}{3} \times$ length of dry dock

Remember to round your answer to the nearest 100 m^3

Exercise 10 (page 55)

Q44:

a) $y = 2x^2 + C$
b) $s = 2t^3 - 5t + C$
c) $y = 3x - \frac{1}{2}x^2 + C$
d) $s = \frac{1}{4}t^2 + t^3 + C$

Q45:

a) $s = \frac{5}{2}t^2$
b) $y = x^3 + \frac{1}{2}x^2 - 5x + 4$
c) $s = - t^3 + \frac{1}{2}t^2 + 6$
d) $y = 6x^{3/2} + 2$

Q46: $y = x^2 + 4x - 3$

Q47: $f(x) = \frac{1}{4}x^2 - 2x + 3$

Q48: 22 m/s

Q49: 17.5 m^3

Review exercise in basic integration (page 58)

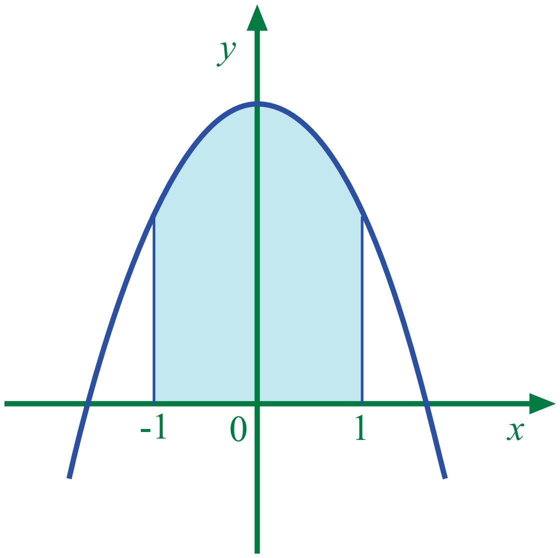
Q50: $-\frac{2}{x^3} + C$

Q51: $6\frac{3}{4}$ square units

Q52: $\int_0^5 \left(5x - x^2\right) dx$

Advanced review exercise in basic integration (page 59)

Q53: $\int_{-1}^{1} \left(10 - 3x^2\right) dx = 18$



Q54: $f(x) = 2x^2 + x - 6$

Q55:

a) A(-12,0), B(12,0)
b) £816

Q56:

a) Equation of tangent at A is 5x + y + 3 = 0, B(-1,2)
b) Area $= \frac{4}{3}$ square units.

Set review exercise in basic integration (page 61)

Q57: This answer is only available on the course web site.

Q58: This answer is only available on the course web site.

Q59: This answer is only available on the course web site.

3 Trigonometric formulae

Revision exercise (page 64)

Q1:

a) $-\frac{\sqrt{3}}{2}$

b) -1

c) $\frac{\sqrt{3}}{2}$

d) $\frac{1}{\sqrt{2}}$

e) $-\frac{1}{\sqrt{3}}$

f) $-\frac{1}{\sqrt{2}}$

Q2:

a) period of 180° and amplitude of 1

b) period of 360° and amplitude of 2

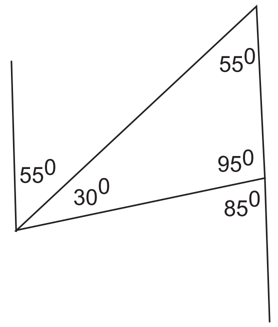
c) period of 60° and there is no amplitude for a tan graph.

d) period of 180° and there is no amplitude for a tan graph.

e) period of 90° and an amplitude of $^1/_3$

f) period of 540° and an amplitude of 1

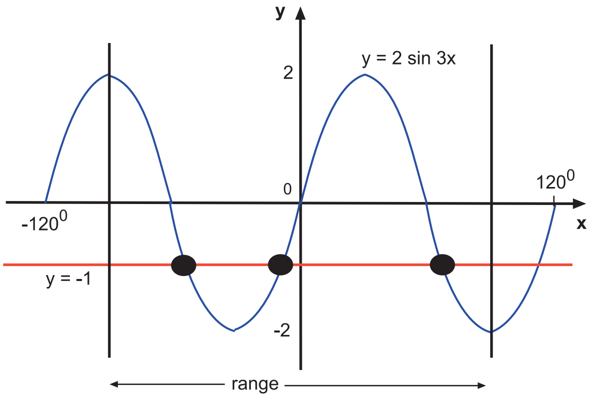
Q3: The angles are shown in the diagram:



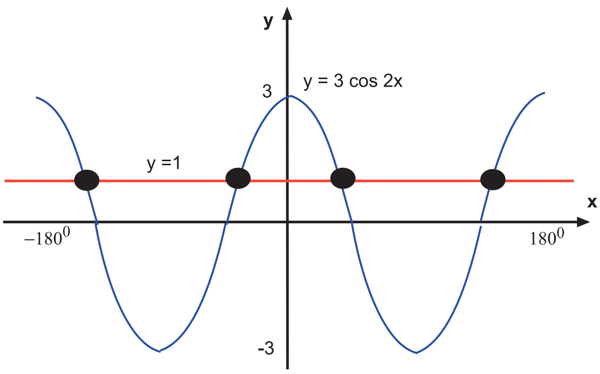
Q4: sin x = $^4/_5$

Solving using graphs exercise (page 67)

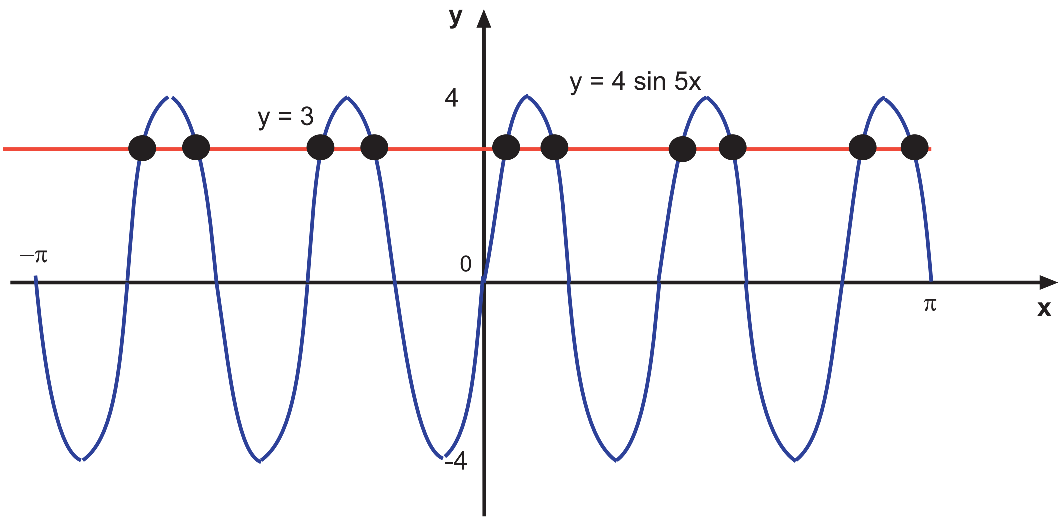
Q5: x = -50°, -10° and 70°



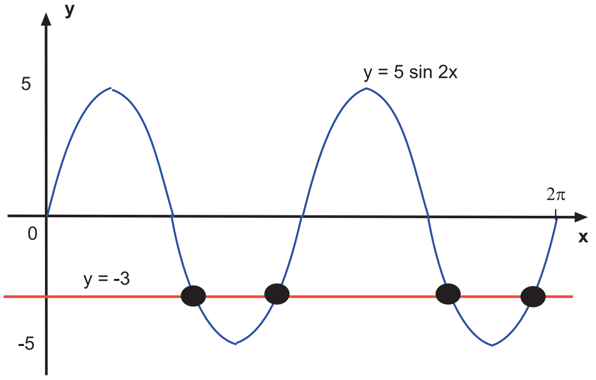
Q6: x = -144.75°, -35.25°, 35.25° and 144.75°



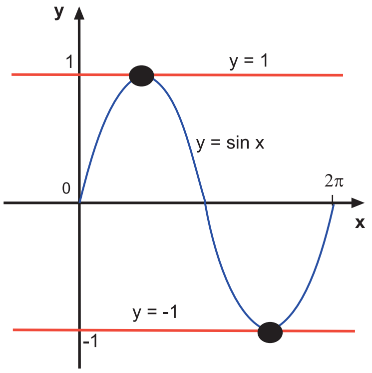
Q7: x = -2.3, -2.0, -1.1, -0.8, 0.2, 0.5, 1.4, 1.7, 2.7, 3.0



Q8: x = 1.9, 2.8, 5.0 and 6.0



Q9: x = π/2, 3π/2



Solving algebraically exercise (page 73)

Q10:

a) -360° $\leq$ x $<$ 360° and so -720° $\leq$ 2x $<$ 720°

sin α = 1 $\Rightarrow$ α = sin $^{-1}$1

α = 0°

Sin is positive in the first and second quadrants

Thus 2x = α or 2x = 180 - α°

Giving 2x = 0° or 180°

The interval is -720° $\leq$ 2x $<$ 720° so subtract and add 360° to each of the values

2x = 0 - 360° = -360°,

2x = 180 - 360° = -180°,

2x = 0 + 360° = 360°and

2x = 180 + 360° = 540°

But this does not yet cover the range so continue to add and subtract 360°

2x = -360° - 360° = -720°,

2x = -180 - 360° = -540°,

2x = 360 + 360° = 720° (too large -discard)

2x = 540 + 360° = 900° (too large- discard)

Within the interval the solutions are x = -360°, -270°, -180°, -90°, 0°, 90°, 180°, 270°

(of course -360° = 0° but the interval makes specific mention of these bounds.)

b) $0° \leq x < 180°$ and so $0° \leq 2x < 360°$
1 - 2 tan 2x = -5 $\Rightarrow$ 2 tan 2x = 6 $\Rightarrow$ tan 2x = 3
tan α = 3 $\Rightarrow$ α = 71.565°
Tan is positive in quadrants 1 and 3
2x = α° = 71.565° and 2x = 180 + α° = 251.565°
Check that there are no further values in the range $0° \leq 2x < 360°$
Therefore x = 35.78°, 125.78° to 2 d.p.

c) x = 54°

d) $-90° \leq x < 0°$ and so $-181° \leq 2x - 1 < -1°$
$(\cos(2x - 1))^2 = 1$
Take the square root of both sides to give cos (2x - 1) = -1 OR cos (2x - 1) = 1
Thus cos α = 1 $\Rightarrow$ α = 0
In this case the cos can be either negative or positive and so all quadrants apply.
2x - 1 = α° = 0°,
2x - 1 = 180 - α° = 180°,
2x - 1 = 180 + α° = 180° and
2x - 1 = -α° = 0°
Check the range $-181° \leq 2x - 1 < -1°$ for further values or ones to discard.
2x - 1 = 180 - 360° = -180° is in the range.
Discard 2x - 1 = 0° and 2x - 1 = 180°
Thus 2x - 1 = -180° which gives x = -89.5°

Q11:

a) $0 \leq x < \pi$ gives $0 \leq 3x < 3\pi$
2 sin 3x = 1 $\Rightarrow$ sin 3x = $^1/_2$
In the first quadrant sin α = $^1/_2$ which gives
$\alpha = \frac{\pi}{6}$
Sin is positive and this occurs in quadrants 1 and 2 thus
$3x = \alpha = \frac{\pi}{6}$ and $3x = \pi - \alpha = \frac{5\pi}{6}$
Further values are given within the range by adding 2π to these values to give
$3x = \frac{\pi}{6} + 2\pi = \frac{13\pi}{6}$ and $3x = \frac{5\pi}{6} + 2\pi = \frac{17\pi}{6}$
The four values of 3x solve to give
$x = \frac{\pi}{18}, \frac{5\pi}{18}, \frac{13\pi}{18}, \frac{17\pi}{18}$

b) $\alpha = \frac{\pi}{3}$ and cos is negative in quadrants 2 and 3
Therefore $4x = \alpha = \frac{2\pi}{3}$ or $\alpha = \frac{4\pi}{3}$ but within the range given the values are found by subtracting 2π
The solutions within the range are then $4x = -\frac{2\pi}{3}, -\frac{4\pi}{3}, -\frac{8\pi}{3}, -\frac{10\pi}{3}$
which gives $x = -\frac{\pi}{6}, -\frac{\pi}{3}, -\frac{2\pi}{3}, -\frac{5\pi}{6}$

c) $\alpha = \frac{\pi}{6}$
Since tan can be positive or negative, all quadrants apply.
Within the range given the solutions for x are $\frac{\pi}{6}$ and $\frac{5\pi}{6}$

Q12:

a) $\alpha = \frac{\pi}{4}$
Since sin can be positive or negative, all quadrants apply.
Within the range given $0 \leq x < \pi$
$x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

b) $2\cos^2 x - \cos x = 0 \Rightarrow$
$\cos x (2\cos x - 1) = 0 \Rightarrow$
cos x = 0 OR 2 cos x = 1
Within the range given the solutions are
$x = \frac{\pi}{3}, \frac{\pi}{2}$

c) x = 0

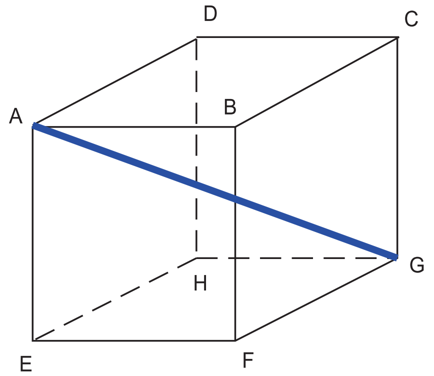
Q13:

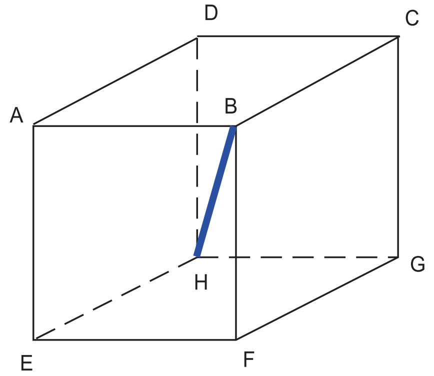
a) $x = 0°$ and $x = \pm 60°$
b) Within the range $x = -26.23,°, 26.23°, -33.77°, 33.77°$
c) $x = -90°$

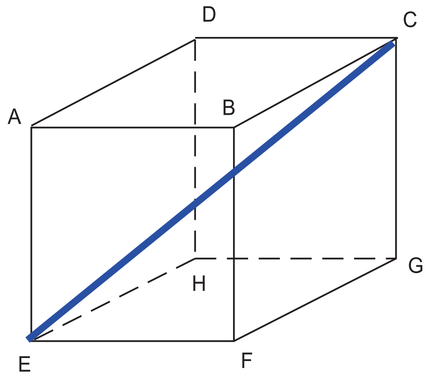
Answers from page 74.

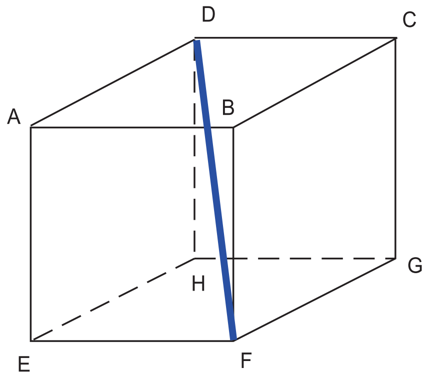
Q14: AG, BH and DF

Here are all four.



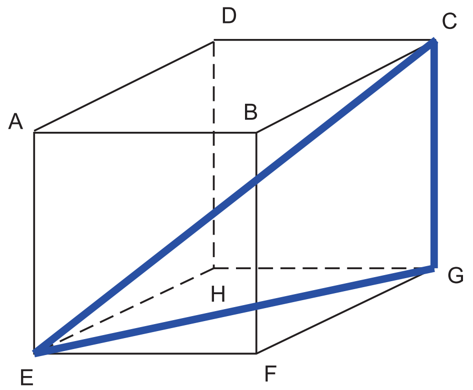






Answers from page 74.

Q15: Triangle CEG where the angle CGE is the right angle.



Answers from page 74.

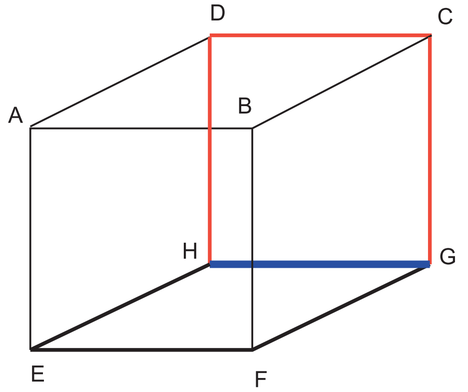
Q16: ABGH and DCGH, EFGH and DCGH

Q17: ABCD and BCGF, ABCD and BCHE, BCHE and BCGF

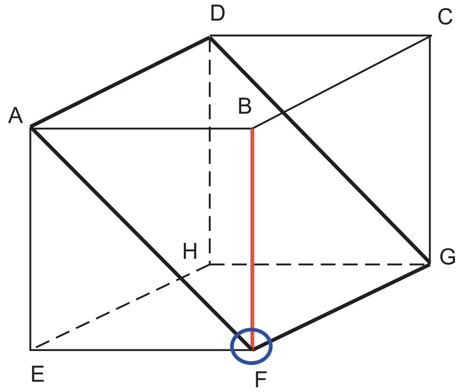
3D exercise (page 79)

Q18:

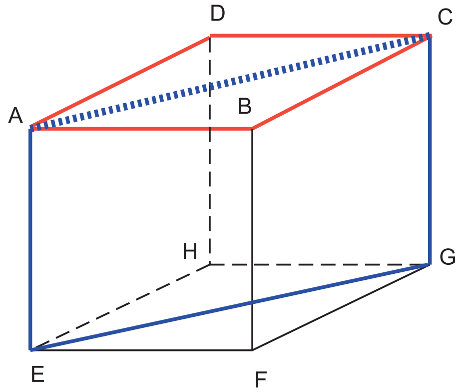
a) The line GH



b) The point F



c) The line AC



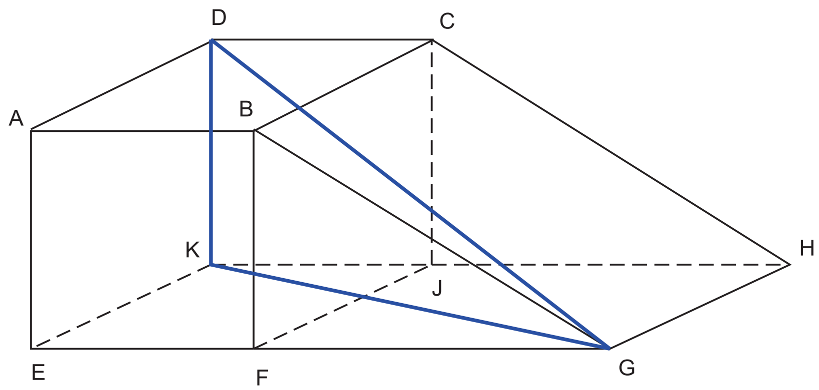
Q19:

a) Angle FCA = 55° to the nearest degree.

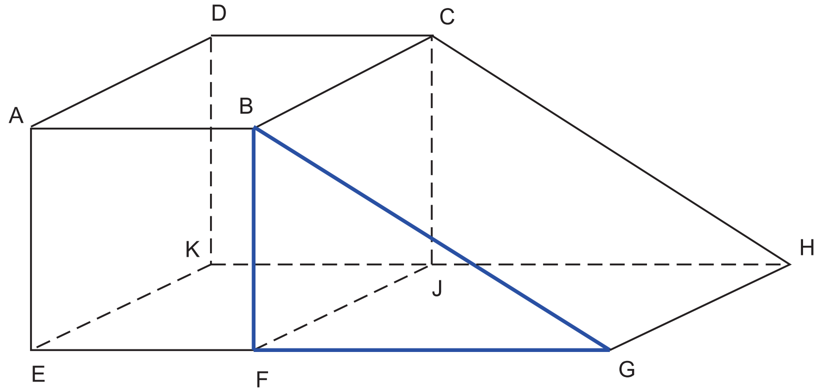
b) Angle DAB = 70° to the nearest degree.

Q20:

a) DG lies in triangle DKG. KG lies in triangle KEG

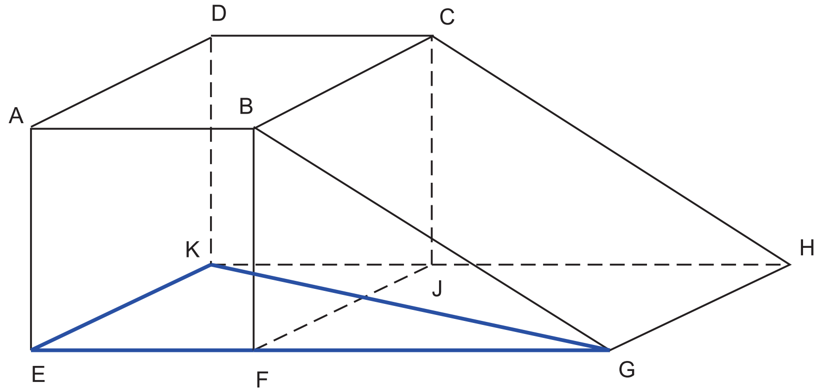


AE = 50m and EG = 50 + FG but FG lies in triangle BFG where angle BGF = 40°



Thus since tan 40° = $^{50}/_{FG}$ then FG = 59.59 m

EG = 50 + 59.59 = 109.59 m and so in triangle KEG,



$KG^2 = EG^2 + EK^2 = 2500 + 12009.97 = 14509.97$

which gives KG = 120.46 m

In triangle DKG, $DG^2 = DK^2 + KG^2 = 2500 + 14509.97 = 17009.97$

DG = 130.42 m correct to 2 d.p.

b) The angle required is DGK

tan DGK = $^{DK}/_{KG}$ = $^{50}/_{120.46}$ = 0.415

Thus angle DGK = 22.54° correct to two d.p.

Addition formulae exercise (page 82)

Q21:

a) sin 3x cos 2y - cos 3x sin 2y

b) $\cos 30 \cos y + \sin 30 \sin y = {}^1/_2(\sqrt{}3 \cos y + \sin y)$

c) $\sin \pi \cos a + \cos \pi \sin a = -\sin a$

d) $\cos x \cos 2\pi - \sin x \sin 2\pi = \cos x$

Q22:

a) Use $\sin (90 + 45)^\circ = \sin 90 \cos 45 + \cos 90 \sin 45 = \frac{1}{\sqrt{2}}$

b) Use $\sin (45 - 30) = \sin 45 \cos 30 - \cos 45 \sin 30 =$
$\frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \times \frac{1}{2} = \frac{\sqrt{3} - 1}{2\sqrt{2}}$

c) Use $\cos (30 + 45)^\circ = \cos 30 \cos 45 - \sin 30 \sin 45 =$
$\frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \times \frac{1}{2} = \frac{\sqrt{3} - 1}{2\sqrt{2}}$

d) Use $\cos (90 + 45) = \cos 90 \cos 45 - \sin 90 \sin 45 = -\frac{1}{\sqrt{2}}$

Double angle exercise (page 83)

Q23: The third side of the triangle is $\sqrt{}3$

$\cos 2A = \cos^2 A - \sin^2 A = {}^1/_2$

Q24: The third side of the triangle is 12

$\sin 2A = 2 \sin A \cos A = {}^{216}/_{225}$

Q25: $(\sin x - \cos x)^2 = 1 \Rightarrow$

$\sin^2 x - 2 \sin x \cos x + \cos^2 x = 1 \Rightarrow$

$1 - 2 \sin x \cos x = 1 \Rightarrow -2 \sin x \cos x = 0 \Rightarrow -\sin 2x = 0$

That is, $\sin 2x = 0$

Q26: $1 - 2 \sin^2 15^\circ = \cos 30^\circ = \frac{\sqrt{3}}{2}$

Q27:

a) Since $\sin 2x = 2 \sin x \cos x$
$2 \sin \frac{\pi}{4} \cos \frac{\pi}{4} = \sin \frac{\pi}{2} = 0$

b) Since $\cos 2x = 2 \cos^2 x - 1$
$2 \cos^2 \frac{3\pi}{4} - 1 = \cos \frac{3\pi}{2} = 0$

c) $\cos 170^\circ - 2 \cos^2 85^\circ =$
$2 \cos^2 85^\circ - 1 - 2 \cos^2 85^\circ = -1$

Mixed formulae exercise (page 85)

Q28: $\dfrac{1 - \tan^2 A}{1 + \tan^2 A} = \dfrac{\dfrac{\cos^2 A - \sin^2 A}{\cos^2 A}}{\dfrac{\cos^2 A + \sin^2 A}{\cos^2 A}} = \cos^2 A - \sin^2 A = \cos 2A$

Q29: $(\sin x - \cos x)^2 = \sin^2 x - 2 \sin x \cos x + \cos^2 x = 1 - 2 \sin x \cos x = 1 - \sin 2A$

Since $(\sin x - \cos x)^2 = 1 - \sin 2A$ then dividing through by $\sin x - \cos x$ gives

$\dfrac{1 - \sin 2x}{\sin x - \cos x} = \sin x - \cos x$

Q30:

a) $\sin 2y = 2 \sin y \cos y = {}^{60}/_{61}$

b) $\sin (x - y) = \sin x \cos y - \cos x \sin y = \dfrac{-8}{\sqrt{793}}$

c) $\cos 4x = \cos 2(2x) = 2 \cos^2 2x - 1$
$\cos 2x = {}^{5}/_{13}$ and so $\cos 4x = -{}^{119}/_{169}$

Q31: The angle is HBD which lies in triangle BDH with DH = 3 cm

BD is the diagonal of the square plane ABCD = $\sqrt{18}$

$\tan HBD = {}^{DH}/_{BD} = 0.707$

angle HBD = $35.26°$

Q32: In triangle KLM, by Pythagoras, KM = 20

In triangle KMN, by Pythagoras, KN = 21

cos LKN = cos (LKM + MKN) = cos LKM cos MKN - sin LKM sin MKN

$= {}^{16}/_{20} \times {}^{20}/_{29} - {}^{12}/_{20} \times {}^{21}/_{29} = {}^{17}/_{145}$

Q33:

a) Angle RPS = $90 - (\alpha + \beta)$

b) $\dfrac{11}{13\sqrt{53}}$

c) 1) $2 \sin (45 - \alpha) \cos (45 - \alpha)$(the triangle is isosceles and $\beta = (45 - \alpha)°$)
2) Using the addition formulae and the exact values of sin 45 and cos 45 reduces this expression to $(\cos^2 \alpha - \sin^2 \alpha) = \cos 2\alpha$
3) ${}^{45}/_{53}$ (easier to use $\cos^2 \alpha - \sin^2 \alpha$ to evaluate).

d) $\beta = 45 - \alpha$ so $\sin (\beta - \alpha) = \sin (45 - \alpha - \alpha) = \sin (45 - 2\alpha)$
$\sin 2\alpha = {}^{28}/_{53}$ and $\cos 2\alpha = {}^{45}/_{53}$
$\sin (\beta - \alpha) = \dfrac{17}{53\sqrt{2}}$

e) $\cos 2\phi = 2 \cos^2 \phi - 1$ but $\cos \phi = {}^{5}/_{13}$
so $\cos 2\phi = {}^{-119}/_{169}$

Further solutions exercise (page 87)

Q34:

a) By calculator the solutions are x = -1440°, -1260°
Algebraically the solutions are x = 0, π

b) By calculator the solutions are x = -1380°, -1350°, -1170°, -1140°
Algebraically the solutions are x = $\frac{\pi}{3}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{6}$

Q35: x = $\pm$n180° where n is an integer

Review exercise (page 91)

Q36:

a) cos x = $^4/_5$ and sin y = $^{12}/_{13}$

b) cos (x - y) = cos x cos y + sin x sin y =
$\frac{4}{5} \times \frac{5}{13} + \frac{3}{5} \times \frac{12}{13} = \frac{56}{65}$

Q37: $0 \leq A < 180$ so $-105 \leq 75 - A < 75$

sin 75 cos A - cos 75 sin A = sin (75 - A)°

sin (75 - A) = 0.4 $\Rightarrow$ sin α = 0.4 where α is the first quadrant angle.

α = 23.58°

Sin is positive and lies therefore in quadrants 1 or 2

So 75 - A = 23.58° or 75 - A = 156.42°

Check the range : 156.42 is outwith it. There are no further values

A = 75 - 23.58 = 51.42°

Q38: $0 \leq x < \pi \Rightarrow 0 \leq 2x < 2\pi$

3 cos 2x + 2 = 0 $\Rightarrow$ cos 2x = - 0.6667

cos α = 0.6667 and α = 0.84

cos is negative and will lie in quadrants 2 and 3

2x = π - 0.84 = 2.30 and π + 0.84 = 3.98

Also 2.3 + π = 5.44 and is in the range.

There are no further values within the range.

x = 1.15, 1.99 and 2.72

Q39: cos 2x + 2 sin^2 x = 2 cos^2 x - 1 + 2 sin^2 x

= 2 (cos^2 x + sin^2 x) - 1 = 2 - 1 = 1

Advanced review exercise (page 91)

Q40: $\cos 2x - \cos x = 0 \Rightarrow$

$2 \cos^2 x - \cos x - 1 = 0 \Rightarrow$

$(2 \cos x + 1)(\cos x - 1) = 0 \Rightarrow$

$\cos x = -{}^1/_2$ or $\cos x = 1$

By reference to the quadrants and the given range the solutions are:

$x = 0°, 120°, 240°$

Q41: $\cos 3A = \cos (2A + A) =$

$\cos 2A \cos A - \sin 2A \sin A =$

$(2 \cos^2 A - 1) \cos A - (2 \sin A \cos A) \sin A =$

$2 \cos^3 A - \cos A - 2 \sin^2 A \cos A =$

$2 \cos^3 A - \cos A - 2 (1 - \cos^2 A) \cos A =$

$2 \cos^3 A - \cos A - 2 \cos A + 2 \cos^3 A =$

$4 \cos^3 A - 3 \cos A$

Q42: $6 \sin (x - 10)° \cos (x - 10)° = 2 \Rightarrow$

$3 \sin 2 (x - 10) = 2 \Rightarrow$

$\sin (2x - 20) = {}^2/_3$

$\sin \alpha = 41.81°$

sin is positive so $2x - 20 = 41.81$ or 138.19

Checking the range $-180° \leq x < 0 \Rightarrow -380° \leq 2x - 20 < -20°$

so $2x - 20 = 41.81 - 360° = -318.19$ or $138.19 - 360° = -221.81°$

$x = -149.1°$or $-100.9°$ correct to 1 d.p.

Q43: $(\cos x + \sin x)(\cos y + \sin y) =$

$\cos x \cos y + \sin x \sin y + \cos x \sin y + \sin x \cos y =$

$\cos (x - y) + \sin (x + y)$

When $x = y$ then

$(\cos x + \sin x)(\cos y + \sin y) = \cos (x - y) + \sin (x + y) =$

$\cos 0 + \sin 2x = 1 + 2 \sin x \cos x$

Set review exercise (page 92)

Q44: This answer is only available on the web.

Q45: This answer is only available on the web.

Q46: This answer is only available on the web.

Q47: This answer is only available on the web.

4 The equation of a circle

Revision exercise (page 94)

Q1: Use the formula $(y - y_1) = m(x - x_1)$ to give

$y - 4 = -2(x - 3)$

$y = -2x + 10$

Q2: Let L1 be $y = 3x - 4$ and L2 be the line perpendicular to it and passing through the point (0, 1)

Perpendicular lines have gradients m_1 and m_2 such that $m_1 \times m_2 = -1$

The gradient of the line L2 = $-{}^1/_3$

Using the formula gives $y - 1 = -{}^1/_3(x - 0)$

$y = -{}^1/_3 x + 1$ or $3y = -x + 3$

Q3: Use the formula $\sqrt{(x - x_1)^2 + (y - y_1)^2}$

The distance is:

$\sqrt{(3 + 2)^2 + (- 4 - 5)^2} = \sqrt{25 + 81} = 10.3$

Q4: $(x - 3)^2 + (y - 2)^2 - x^2 + 3(2x - 5) + 4y - 2 =$

$x^2 - 6x + 9 + y^2 - 4y + 4 - x^2 + 6x - 15 + 4y - 2 =$

$y^2 - 4$ or $(y - 2)(y + 2)$

Q5: Draw it and see. On reflection the positive y-axis becomes the positive x-axis and the negative also switch. The point Q is therefore (4, -3)

The distance between P and Q is $\sqrt{[(4 + 3)^2(-3 - 4)^2]}$ = 9.9 units correct to one decimal place.

Circles at the origin exercise (page 96)

Q6:

a) Radius is 12
b) Radius is 2
c) $x^2 + y^2 = 9 \Rightarrow$ radius is 3
d) Radius is $\sqrt{8}$ or $2\sqrt{2}$
e) Radius is 13
f) $x^2 + y^2 = 4 \Rightarrow$ radius is 2

Q7:

a) $x^2 + y^2 = 4$
b) $20^2 + 21^2 = r^2$ so r = 29
   The equation is $x^2 + y^2 = 841$
c) $x^2 + y^2 = 45$

d) $x^2 + y^2 = 16$

e) $x^2 + y^2 = 25$

f) If x = 2, y = 1 and $2^2 + 1^2 = r^2$ so r = $\sqrt{5}$
The equation is $x^2 + y^2 = 5$

Q8: The diameter of the ring is 18 m and the radius is 9 m

The equation is $x^2 + y^2 = 81$

Q9: The radius of the first circle is 2 cm which is a diameter of 4 cm. The next circle has diameter of 6 cm, then 8 cm and so the largest circle has diameter of 10 cm and therefore a radius of 5 cm

The equation of the largest circle is $x^2 + y^2 = 25$

Q10: The length between the two points where the arrows touch the centre is the diameter. This length is calculated with the distance formula.

$r = \sqrt{(1 + 1)^2 + (- 5 - 5)^2} = \sqrt{104}$

The equation of the circle is $x^2 + y^2 = 104$

General equation circles exercise **(page 100)**

Q11:

a) Radius is 7 and the centre is (-2, 1)

b) Note that x^2 is the same as $(x - 0)^2$. Rearrange into the general form.
Radius is 3 and the centre is (0, 2)

c) $(x + 1)^2 + (y + 1)^2 = 1 \Rightarrow$ radius is 1 and the centre is (-1, -1)

d) Radius is $\sqrt{12}$ or 2 $\sqrt{3}$ and the centre is (-6, 0)

e) Rearrange to give $x^2 + (y - 1)^2 = 3$
Radius is $\sqrt{3}$ and the centre is (0, 1)

f) Rearrange to give $(x - 1)^2 + (y + 4)^2 = 36$
The radius is 6 and the centre is (1, -4)

Q12:

a) $(x + 3)^2 + (y - 4)^2 = 16$

b) The radius r is the distance between the centre and the point.
so $r^2 = (4 - 2)^2 + (-5 - 2)^2 = 53$
The equation is $(x - 2)^2 + (y - 2)^2 = 53$

c) The centre is $(4\sqrt{2}, 0)$
The equation is $(x - 4\sqrt{2})^2 + y^2 = 32$

d) $r^2 = (4 + 3)^2 + (0 + 1)^2 = 50$
The equation is $(x + 3)^2 + (y + 1)^2 = 50$

e) Radius is 3
The midway point is (-2, 3)
The equation is $(x + 2)^2 + (y - 3)^2 = 9$

f) The distance between P and Q is
$\sqrt{(8 - 2)^2 + (-4 + 12)^2} = \sqrt{100} = 10$
The radius is 5
The centre of the circle is the midway point between P and Q which is (5, -8)
The equation is $(x - 5)^2 + (y + 8)^2 = 25$

Q13: The centre of the larger wheel is (2, -4) and the radius is 8

Thus the smaller wheel has radius 4

The centre of the smaller wheel is 12 units to the right of the larger wheel. The centre is at (14, -4)

The equation of the smaller wheel is $(x - 14)^2 + (y + 4)^2 = 16$

Q14: The two wheels measure 4 feet so each is 2 feet across and therefore has a radius of 1 foot.

The first wheel A has centre at (1, 1) since it rests against the wall and the ground. It has equation $(x - 1)^2 + (y - 1)^2 = 1$

The second wheel centre is $2^1/_2$ feet away and has a centre at (3.5, 1)

It has equation $(x - 3.5)^2 + (y - 1)^2 = 1$

Extended equation of a circle exercise (page 104)

Q15:

a) The centre is given by (2, 1) and the radius is 3
b) The centre is given by (-3, 2) and the radius is 4
c) The centre is given by (0, -4) and the radius is 5
d) The centre is given by (5, 0) and the radius is 2

Q16:

a) $x^2 + y^2 + 4x - 2y - 4 = 0$
b) $x^2 + y^2 - 10y + 9 = 0$
c) $x^2 + y^2 - 10x + 7 = 0$
d) $x^2 + y^2 + 4x + 8y - 29 = 0$

Q17: Circle A has centre (1, -2) and radius 1.

Some geometry is needed.

As AB = 3 and circles A and B touch then circle B has radius 2

Circle B has centre with x-coordinate of 1 (as in A) and y -coordinate of -2 - 1(radius of A) - 2 (radius of B) = -5

So for circle B with r = 2, g = -1 and f = 5 the equation is

$x^2 + y^2 - 2x + 10y + 22 = 0$

Since BC has length 4 and circle B has radius of 2, circle C has also radius of 2

The centre of C has y -coordinate of -5 (as has B) and x-coordinate of 1 + 2 + 2 = 5

The equation of circle C is

$x^2 + y^2 - 10x + 10y + 46 = 0$

Since circle A has radius 1 and circle C has radius 2, then circle D has diameter 2 since AC = 5

It follows that circle D has radius of 1 and the centre lies along AC at a point in the ratio 2:3

The centre is $(1 + {}^{8}/_{5}, -2 - {}^{6}/_{5}) = ({}^{13}/_{5}, {}^{-16}/_{5})$

Answers from page 109.

Q18: The sum of the two radii of the circles is equal to the distance between the centres.

Intersection exercise (page 109)

Q19: When y = 0 the equation becomes

$x^2 + 6x - 7 = 0$

$(x + 7)(x - 1) = 0 \Rightarrow x = -7$ or $x = 1$

The points are P (1, 0) and Q (-7, 0)

The centre of the circle is C (-3, 4)

$m_{CP} = -1$ therefore $m_{tan} = 1$ at this point P

Using the equation of a straight line $(y - b) = m(x - a)$ gives

$y = x - 1$

$m_{CQ} = 1$ so $m_{tan} = -1$

In this case $y = - x - 7$

Q20: The circle has centre C (1, -2)

$m_{CP} = {}^{-3}/_{1} = -3$

$m_{tan} = {}^{1}/_{3}$

The equation is given by $(y - a) = m(x - a)$

$y + 5 = {}^{1}/_{3}(x - 2)$

$3y = x - 17$

Q21: Substitute $y = 2x - 4$ into the equation of the circle.

$x^2 + 4x^2 - 16x + 16 - 5x - 4x + 8 - 54 = 0$

$5x^2 - 25x - 30 = 0$

$x^2 - 5x - 6 = 0$

$(x - 6)(x + 1) = 0 \Rightarrow x = 6$ or $x = -1$

When x = 6, y = 8 giving the point (6, 8)

When x = -1, y = -6 giving the point (-1, -6)

Q22: Substitute $y = x + 3$ into the equation of the circle.

$x^2 + x^2 + 6x + 9 + 4x + 2x + 6 + 3 = 0$

$2x^2 + 12x + 18 = 0$

$x^2 + 6x + 9 = 0$

EITHER $(x + 3)(x + 3) = 0 \Rightarrow$ that the line is a tangent as the two values are equal

OR the discriminant $b^2 - 4ac = 36 - 36 = 0 \Rightarrow$ the roots are equal $\Rightarrow$ the line is a tangent.

Substituting $y = -x + 3$ into the circle equation and simplifying gives

$x^2 - 2x + 9 = 0$

But the discriminant $= 4 - 36 = -32 \Rightarrow$ no real solutions $\Rightarrow$ the line $y = -x + 3$ does not come into contact with the circle.

Q23: General or extended form can be used. Here is general form.

$x^2 + y^2 + 4x - 6y - 3 = 0$ rearranges to give

$(x + 2)^2 + (y - 3)^3 = 16$

$x^2 + y^2 - 2x - 6y - 15 = 0$ rearranges to give

$(x - 1)^2 + (y - 3)^2 = 25$

At intersection these solve simultaneously to give $(x + 2)^2 - 16 - (x - 1)^2 + 25 = 0$

$x^2 + 4x + 4 - 16 - x^2 + 2x - 1 + 25 = 0$

$6x + 12 = 0 \Rightarrow x = -2$

Substitute $x = -2$ into say, $x^2 + y^2 + 4x - 6y - 3 = 0$ to give

$4 + y^2 - 8 - 6y - 3 = 0$

$(y - 7)(y + 1) = 0 \Rightarrow y = 7$ or $y = -1$ and the two intersection points are

$(-2, 7)$ and $(-2, -1)$

Q24: Substitute for y in the circle to give

$x^2 + (-x + k)^2 - 10x - 2(-x + k) + 18 = 0$

This simplifies to $2x^2 + x(-8 - 2k) + k^2 - 2k + 18 = 0$

The discriminant gives $k^2 - 12k + 20 = 0$ for the line to be a tangent.

$k = 2$ or $k = 10$

Q25: Substitute for y in the circle to give

$x^2 + (-2x + k)^2 - 4x - 4(-2x + k) + 3 = 0$

This simplifies to $5x^2 + x(4 - 4k) + k^2 - 4k + 3 = 0$

The discriminant gives $k^2 - 12k + 11 = 0$

$k = 1$ or $k = 11$

Review exercise (page 112)

Q26: The equation takes the form $(x - a)^2 + (y - b)^2 = r^2$ where (a, b) is the centre of the circle and r is the radius.

The equation of this circle is $(x + 4)^2 + (y + 2)^2 = 9$

Q27: $x^2 + y^2 + 2gx + 2fy + c = 0$ is a general equation of a circle

This circle has g = -1, f = 2 and c = -11

The centre is given by (-g, -f) and thus the centre is (1, -2)

$r^2 = g^2 + f^2 - c = 1 + 4 + 11 = 16$

The radius is 4

Q28: Substitute y = 2x + 1 into the equation of the circle to give:

$x^2 + 4x^2 + 4x + 1 - 10x - 24x - 12 + 56 = 0 \Rightarrow$

$5x^2 - 30x + 45 = 0 \Rightarrow$

$x^2 - 6x + 9 = 0 \Rightarrow$

$(x - 3)^2 = 0 \Rightarrow x = 3$ twice $\Rightarrow$ it is a tangent and touches at this one point.

Q29: The circle has centre (1, -2)

$r^2 = 4 + 16 = 20$

The equation of the circle is

$(x - 1)^2 + (y + 2)^2 = 20$

or alternatively $x^2 + y^2 - 2x + 4y - 15 = 0$

The gradient of PC is -2 $\Rightarrow$ the gradient of the tangent is $^1/_2$

$y - 2 = {}^1/_2(x + 1)$

The equation of the tangent is 2y = x + 5

Q30: The centre is (4, -1) and the radius is 5

Q31: The gradient of PC is $-^1/_2 \Rightarrow$ gradient of the tangent is 2

Thus y + 2 = 2 (x - 3) $\Rightarrow$ the equation of the tangent is y = 2x - 8

Advanced review exercise (page 113)

Q32: Let the small circle be circle S and the large circle be circle L

The centre of circle S is (6, 13) and the line of centres has equation x = 6 since it is parallel to the y-axis.

The x-coordinate of the centre of circle L is 6

The radius of circle S = $\sqrt{(6^2 + 13^2 - 189)} = 4$

Therefore the diameter of circle L is 22 - 8 cm = 14 cm and the radius is 7 cm

The distance from circle S centre to circle L centre is 11 cm and so the y-coordinate of the centre of circle L is 24

Thus circle L has centre (6, 24) and radius 7 cm

The equation is $(x - 6)^2 + (y - 24)^2 = 49$ or
$x^2 + y^2 - 12x - 48y - 563 = 0$

Q33: The distance from circle S centre to circle L centre is still 11 cm
and the y -coordinate is 13 - 11 = 2

The circle has centre (6, 2) and radius 7 cm

The new equation is $(x - 6)^2 + (y - 2)^2 = 49$ or
$x^2 + y^2 - 12x - 4y - 9 = 0$

Q34: Cog A has centre (1, -2) and radius 3

Cog B has radius 1 and centre with x-coordinate of 1 + 3 + 1 = 5

Thus it has centre (5, -2) and the equation of cog B is

$(x - 5)^2 + (y + 2)^2 = 1$

Cog C has centre at a y-coordinate of -2 + 1 + 2 = 1 and x coordinate of 5

Since its radius is 2 units, the equation of cog C is

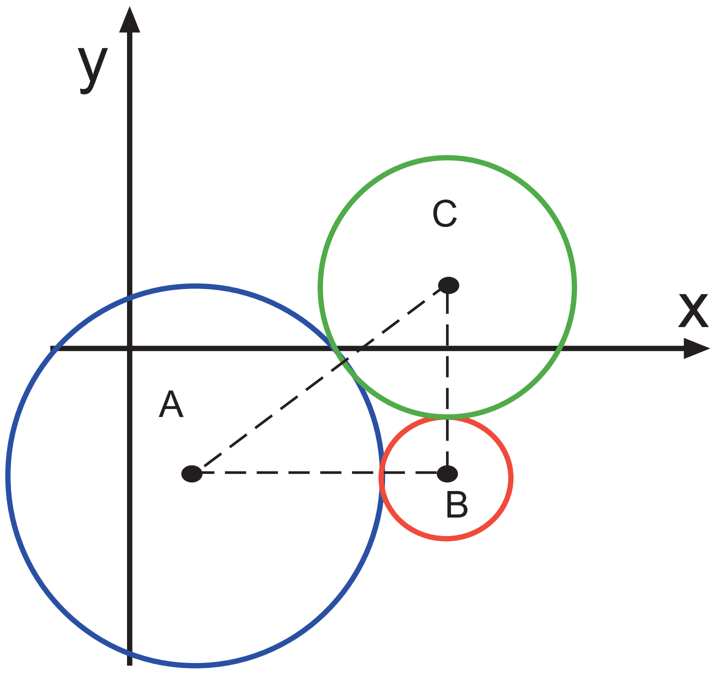
$(x - 5)^2 + (y - 1)^2 = 4$

Use the distance formula to give AC = 5 cm

But radius A + radius B = 5 cm

Therefore the two circles touch.

Alternatively a sketch will reveal that the centres form a right angled triangle and that cogs A and C do in fact touch.



Q35: Circle A has equation $x^2 + (y + 1)^2 = 40$ and

Circle B has equation $(x + 1)^2 + (y + 1)^2 = 45$

At a common point the equations will solve simultaneously to give

$x^2 - 40 - (x + 1)^2 + 45 = 0 \Rightarrow$

$2x = 4$

$x = 2$

Substitution in circle A gives $4 + (y + 1)^2 = 40 \Rightarrow (y + 7)(y - 5) = 0 \Rightarrow$

y = 5 since the tangent point required has positive coefficients.

The common tangent is at the point (2, 5)

For circle A the gradient is 3 $\Rightarrow$ y - 5 = 3 (x - 2) $\Rightarrow$ y = 3x - 1

For circle B the gradient is 2 $\Rightarrow$ y - 5 = 2 (x - 2) $\Rightarrow$ y = 2x + 1

Set review exercise (page 113)

Q36: This answer is only available on the web.

Q37: This answer is only available on the web.

Q38: This answer is only available on the web.

Q39: This answer is only available on the web.

Q40: This answer is only available on the web.

Q41: This answer is only available on the web.