SCHOLAR Study Guide

SQA Higher
Mathematics Unit 3

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SCHOLAR Study Guide Unit 3: Mathematics

1. Mathematics

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Contents

Topic 1

Vectors in three dimensions

Contents

Learning Objectives

- *Use vectors in three dimensions*

Minimum performance criteria:

- *Determine whether three points with given coordinates are collinear*
- *Determine the coordinates of the point which divides the join of two given points internally in a given numerical ratio*
- *Use the scalar product*

Prerequisites

A sound knowledge of the following subjects is required for this unit:

- 2D coordinates and plotting graphs
- Finding the midpoint of a line
- Finding the distance between two points
- Finding the angle in a right angled triangle using a trig. ratio

1.1 Revision exercise

Revision exercise

30 min

There is a web exercise if you prefer it.

Q1: Find the distance between the two points A (-3, 2) and B (3, 4) correct to two decimal places.

Q2: What is the midpoint of the line AB where A = (-4, 1) and B = (2, 5)

Q3: In a right angled triangle, ABC, with the right angle at B, the side AB = 4cm and AC = 7 cm. Find the size of angle BAC.

1.2 Definitions and terminology

Learning Objective

Find vectors in component form

Quantities that have magnitude only are called **scalars**.

Weights, areas and volumes are all examples of scalars. But there are many quantities which are not sufficiently defined by their magnitudes alone. For example, a movement or displacement from a point P to a point Q requires both the distance between the points and the direction from P to Q.

Example If Q is 45 km North West of P then the distance is 45 km and the direction is North West.

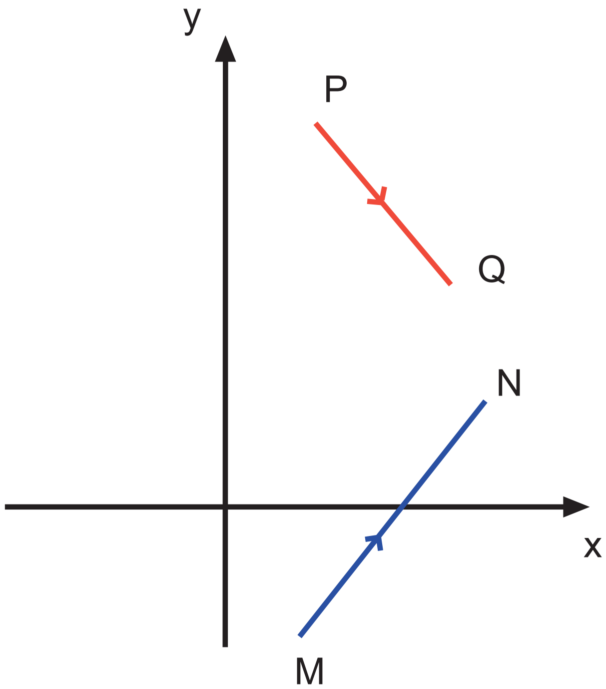
There is a special type of quantity to cope with this. It is called a vector quantity.

Vector quantity

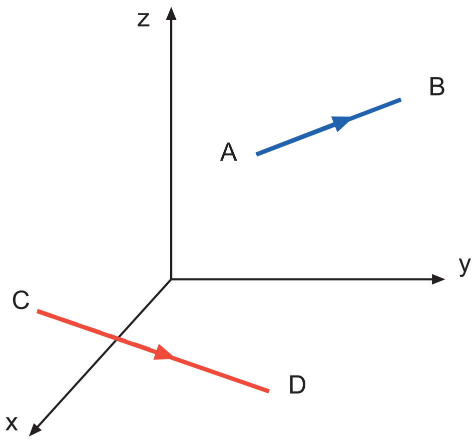
A vector quantity is a quantity which has both direction and magnitude.

Geometrically, a vector can be represented by a directed line segment, that is a line from one point to another which has a direction arrow on it.

The two lines MN and PQ represent two vectors in two dimensions.



The next diagram shows two lines AB and CD representing two vectors in three dimensions.

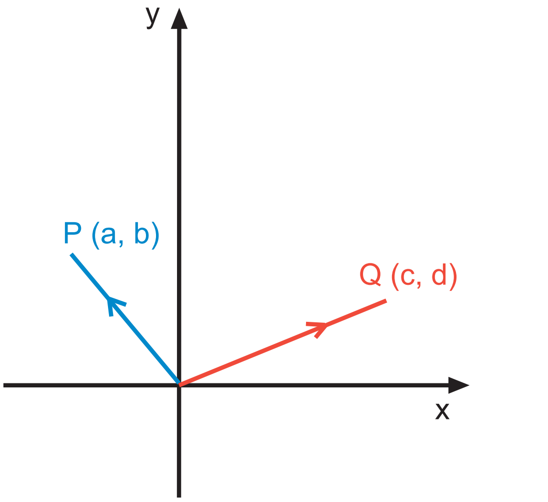
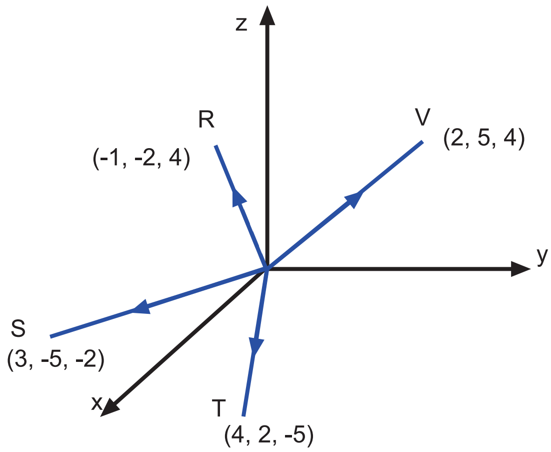


There is a particular type of vector called a position vector.

Position vector

A position vector is a vector which starts at the origin.

Here are position vectors in two and three dimensions.

2 dimensions	
3 dimensions	

A vector from the origin O, (that is a position vector), for example, to the point P, may be expressed by a small letter in the form **p** or by $\underline{p}$ or by the directed line segment $\overrightarrow{OP}$. The vector from the origin to the point R can be expressed as **r**, $\underline{r}$ or $\overrightarrow{OR}$

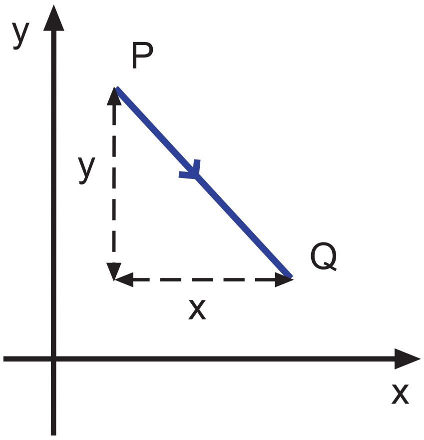
In text books, the bold form **p** may be common but is substituted by $\underline{p}$ in written work.

A vector from the point A to the point B (where neither A nor B is the origin) is expressed either as $\overrightarrow{AB}$, as a combination of position vectors (which will come later) or in component form.

Examples

1. Vector in component form

Examine the following diagram.



The displacement from the point P to the point Q is a movement of x units in the positive x-axis direction and -y units in the positive y-axis direction.

These are called the x-component and the y-component of a vector in two dimensions.

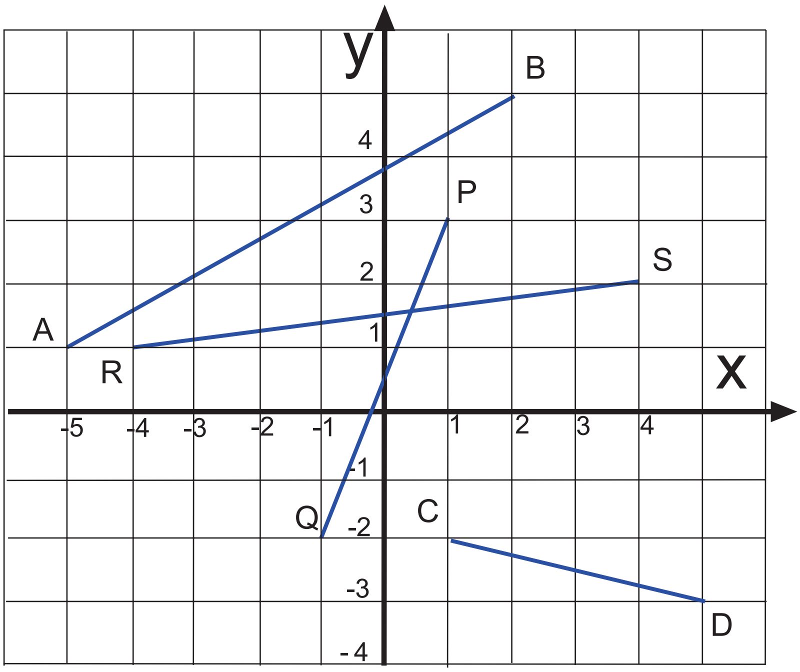
This vector can be written in component form as $\begin{pmatrix} x \\ -y \end{pmatrix}$

2. What is the component form of the position vector to the point B (2, 3)?

Answer: The component form is

$\begin{pmatrix} 2 \\ 3 \end{pmatrix}$

Here is an selection of lines in a two dimensional space.



The vector from A to B can be written as $\overrightarrow{AB}$

In component form it is written as $\begin{pmatrix} 7 \\ 4 \end{pmatrix}$ since it has a displacement from A to B of 7 units along the x-axis and 4 units up the y-axis.

The direction is important. The vector from B to A would be written as $\overrightarrow{BA}$ which is $\begin{pmatrix} -7 \\ -4 \end{pmatrix}$ in component form.

This gives a useful property of vectors.

For a vector between the two points A and B
$\overrightarrow{AB} = -\overrightarrow{BA}$

Q4: Give the component form of the following vectors which can be formed in the previous diagram.

1. $\overrightarrow{CD}$
2. $\overrightarrow{PQ}$
3. $\overrightarrow{RS}$
4. $\overrightarrow{SR}$
5. $\overrightarrow{QP}$
6. $\overrightarrow{DC}$

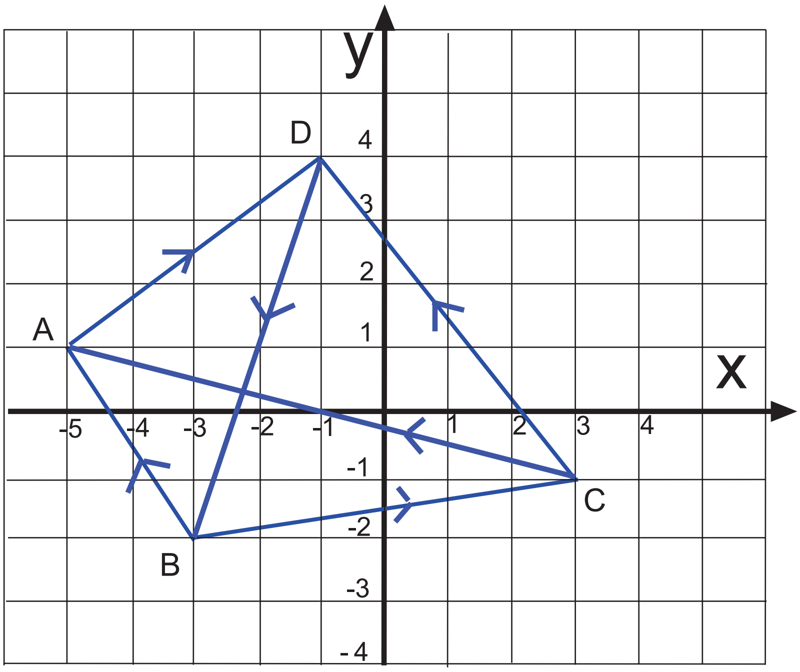
A vector in three dimensions can also be expressed in component form where the third component is called the z-component.

For example, a vector from the origin to the point R with coordinates (e, f, g) can be expressed as the ordered triple $\begin{pmatrix} e \\ f \\ g \end{pmatrix}$

The x-component is e, the y-component is f and the z-component is g

Q5: Express the vector from the point O to the point Q (c, d) in the four possible forms given earlier.

Q6: Name all the vectors shown in the diagram as directed line segments and for each, give its component form.

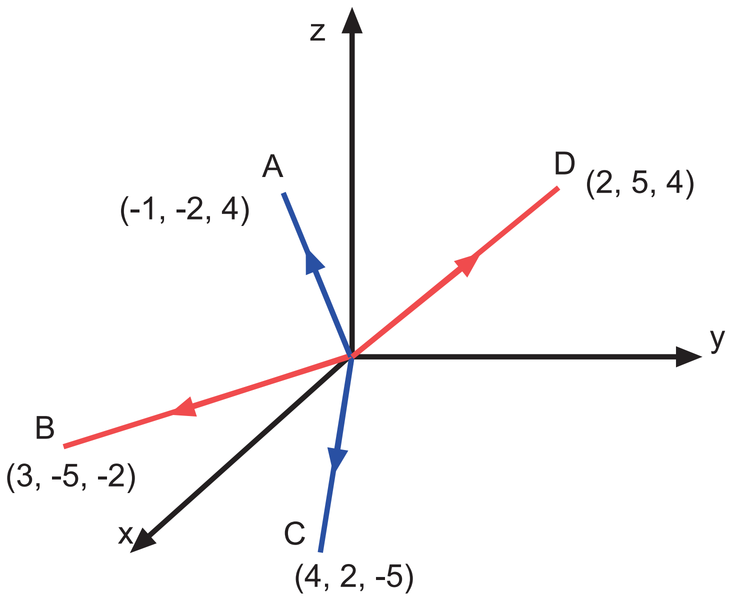


Q7: Express the vectors $\overrightarrow{OB}$, $\overrightarrow{OD}$ and $\overrightarrow{OR}$ in component form when:

a) B = (-1, 2, 0), D = (0, 1, -4) and R = (0, -1, 1)

b) B = (-1, 1, -1), D = (3, 2, 1) and R = (0, 1, 0)

Q8: Write the vectors in the diagram in component form

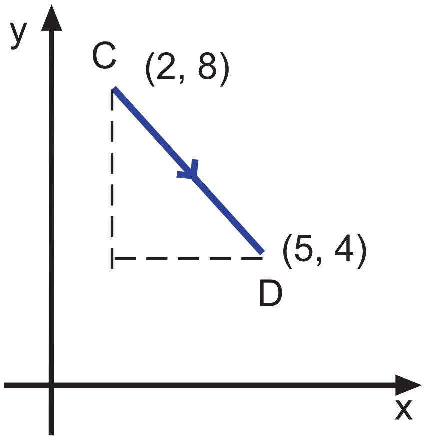


1.3 Length of a vector and standard basis

Learning Objective

Find a vector length and express a vector using standard basis

Remember that vectors have a magnitude and a direction. The direction is indicated by an arrow and the magnitude is the length.



In the previous diagram the length of the vector $\overrightarrow{CD}$ can be found using Pythagoras theorem.

Q9: What is the length of this vector?

Note that in the calculation, (5 - 2) is the x-component of the vector and (4 - 8) is the y-component. This is the basis of the definition.

Length of a vector in two dimensions

Let **p** be the vector $\begin{pmatrix} a \\ b \end{pmatrix}$ then the length of **p** written as $|\mathbf{p}|$ is defined as $|\mathbf{p}| = \sqrt{a^2 + b^2}$

A similar result holds for 3 dimensions

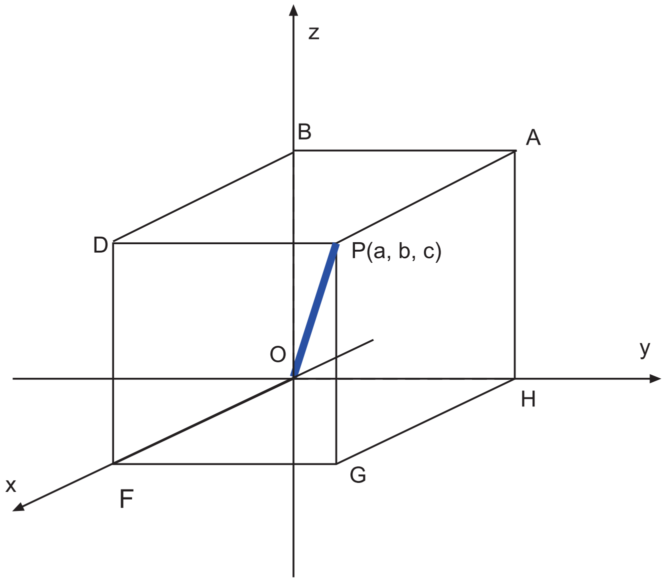
Length of a vector in three dimensions

Let **p** be the vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ then the length of **p** is defined as $|\mathbf{p}| = \sqrt{a^2 + b^2 + c^2}$

Challenge proof

20 min

If there is time try to prove this result by constructing a cuboid with one vertex at the origin and P at (a, b, c) being the lead diagonal through the cuboid. Recall the work on 3 dimensional objects in the trig. formula topic if necessary.



For position vectors, the length calculation comes directly from the definition. For vectors between two points other than the origin, the length is found by using the components of the vector as shown with the last example.

The length of a vector is the distance between the two points which form the vector.

Examples

1. Length of a position vector in two dimensions

Find the length of the position vector to the point B (-5, 12)

Answer:

The length is given by $|\mathbf{b}| = \sqrt{(-5)^2 + (12)^2} = 13$

2. Length of a position vector in three dimensions

Find the length of $\mathbf{a} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$ correct to 5 decimal places

Answer:

The length is given by $|\mathbf{a}| = \sqrt{2^2 + 3^2 + 4^2} = 5.38516$

3. Length of a vector between two points in two dimensions

Find the length of $\overrightarrow{AB}$ when A = (4, 3) and B = (-1, -2) correct to 2 decimal places.

Answer:

The length is given by $\left|\overrightarrow{AB}\right| = \sqrt{(-1-4)^2 + (-2-3)^2} = 7.07$

(-1 - 4) is the x-component and (-2 - 3) is the y-component of the vector.

4. Length of a vector between two points in three dimensions

Find the length of $\overrightarrow{AB}$ when A = (1, 2, 4) and B = (-2, 3, -1) correct to 2 decimal places.

Answer:

The length is given by $\left|\overrightarrow{AB}\right| = \sqrt{(-2-1)^2 + (3-2)^2 + (-1-4)^2} = 5.92$

Vector length exercise

15 min

There is an alternative exercise on the web.

Q10: Find the magnitude (length) of the position vectors to the following points correct to 2 decimal places:

a) A (1, -3)
b) B (-4, 5)
c) C (3, -3)

Q11: If X = (-3, 1), Y = (3, 2), Z = (-2, -2) and W = (5, -1), find the length of the following vectors, correct to 2 decimal places (draw the vectors on squared paper if it helps):

a) $\overrightarrow{XY}$
b) $\overrightarrow{XZ}$
c) $\overrightarrow{ZW}$
d) $\overrightarrow{WY}$
e) $\overrightarrow{XW}$
f) $\overrightarrow{YZ}$

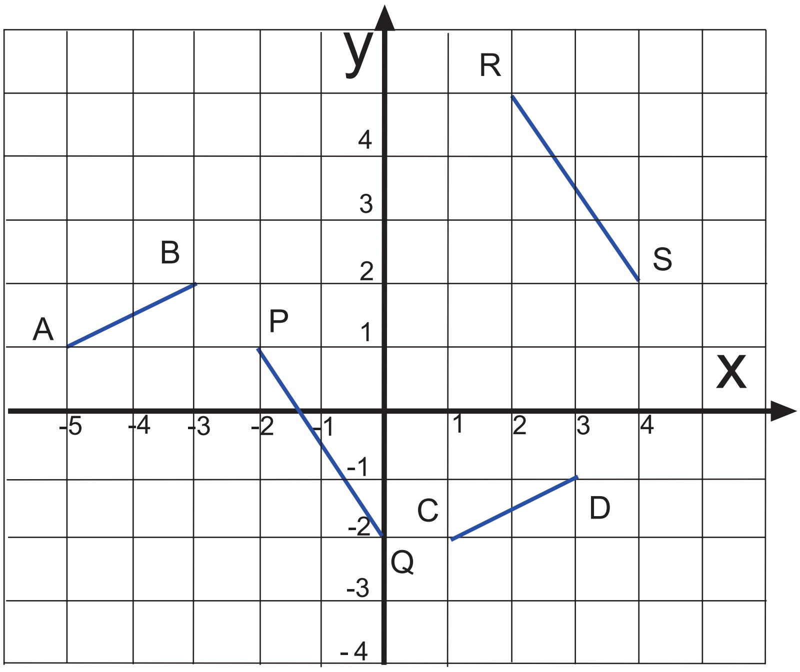
Q12: If P = (1, 2, 3), Q = (-2, -4, 1), R = (-3, 1, 2) and S = (-4, -3, -2), find the length of the following vectors correct to 2 decimal places. (Use the same technique as in the previous question but include the z-component):

a) $\overrightarrow{PS}$
b) $\overrightarrow{RS}$
c) $\overrightarrow{SQ}$
d) $\overrightarrow{RP}$
e) $\overrightarrow{QP}$
f) $\overrightarrow{QR}$

Q13: Determine the distance between the two points S (1, 2, -4) and T (-2, 5, 3) correct to 2 decimal places.

Equal vectors

Look at the following diagram in which AB is parallel to CD and RS is parallel to PQ.



$\overrightarrow{AB} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\overrightarrow{CD} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

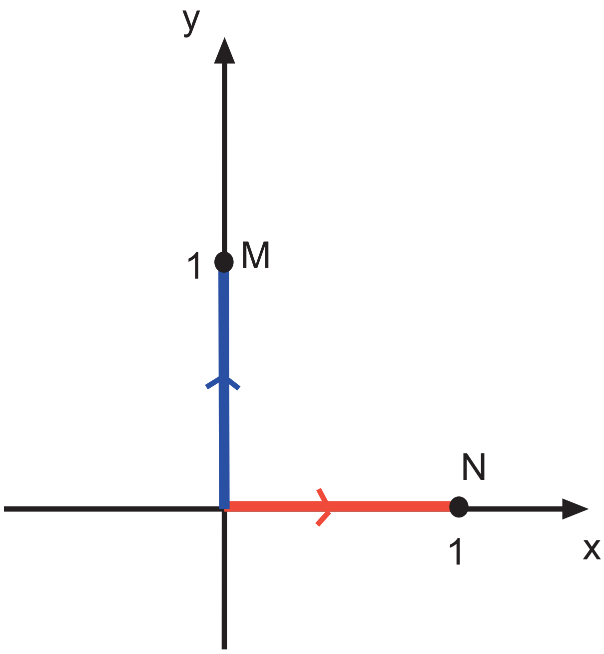
These two vectors have the same magnitude and direction and are therefore *equal*. The same is true for the vectors $\overrightarrow{PQ}$ and $\overrightarrow{RS}$.

More formally, if $\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} d \\ e \\ f \end{pmatrix}$ then a = d, b = e and c = f

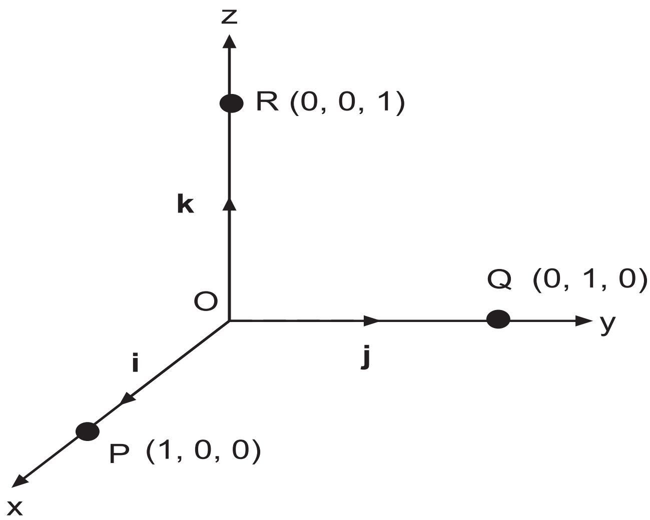
Unit vector

As the name would suggest a unit vector is a vector of magnitude 1. That is, the length (or magnitude) of the vector is 1

In two dimensions, there are two special vectors which lie along the x and y axes. The point N has coordinates (1, 0) and the point M has coordinates (0, 1)



In three dimensions, it follows that there are 3 special unit vectors which lie along the x, y and z axes. The point P has coordinates (1, 0, 0), Q has (0, 1, 0) and R has (0, 0, 1)



Each of the vectors $\overrightarrow{OM}$, $\overrightarrow{ON}$, $\overrightarrow{OP}$, $\overrightarrow{OQ}$, and $\overrightarrow{OR}$ is one unit long.

$\overrightarrow{ON}$ = $\overrightarrow{OP}$ and is denoted by **i**,

$\overrightarrow{OM}$ = $\overrightarrow{OQ}$ and is denoted by **j** and

$\overrightarrow{OR}$ and is denoted by **k**.

i, **j** and **k** are called the **standard basis vectors**.

Thus **i** is the vector $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

similarly **j** is the vector $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

and **k** is the vector $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Every vector $\begin{pmatrix} p \\ q \\ r \end{pmatrix}$ can be written using standard basis as p**i** + q**j** + r**k**

Conversely every vector written in the form p**i** + q**j** + r**k** can be written as $\begin{pmatrix} p \\ q \\ r \end{pmatrix}$

Examples

1. Write the vector **p** = $\begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix}$ using the standard basis.

Answer:

p = 3**i** - 5**j** + 2**k**

2. Write the vector 2**i** + 7**j** in component form.

Answer:

$\begin{pmatrix} 2 \\ 7 \\ 0 \end{pmatrix}$

There will be more work on standard basis later.

1.4 Arithmetic with vectors

The arithmetic operations on vectors are straightforward. In fact some of the techniques have already been used in the previous section.

1.4.1 Adding, subtracting and scalar multiplication of vectors

Learning Objective

Perform arithmetic operations on vectors

Addition

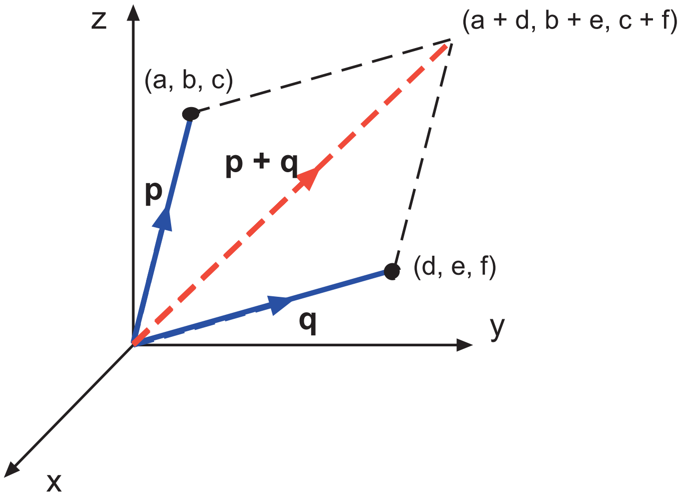
If **p** = $\begin{pmatrix} a \\ b \end{pmatrix}$ and **q** = $\begin{pmatrix} d \\ e \end{pmatrix}$ are vectors then **p + q** is the vector $\begin{pmatrix} a + d \\ b + e \end{pmatrix}$

Similarly in three dimensions:

If $\mathbf{p} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} d \\ e \\ f \end{pmatrix}$ are vectors then **p + q** is the vector $\begin{pmatrix} a + d \\ b + e \\ c + f \end{pmatrix}$

This new vector is called *the resultant*.

In graphical terms, the two vectors to be added (**p** and **q**) are used to form a parallelogram and the diagonal from the origin (**p + q**) will represent the resultant.



Examples

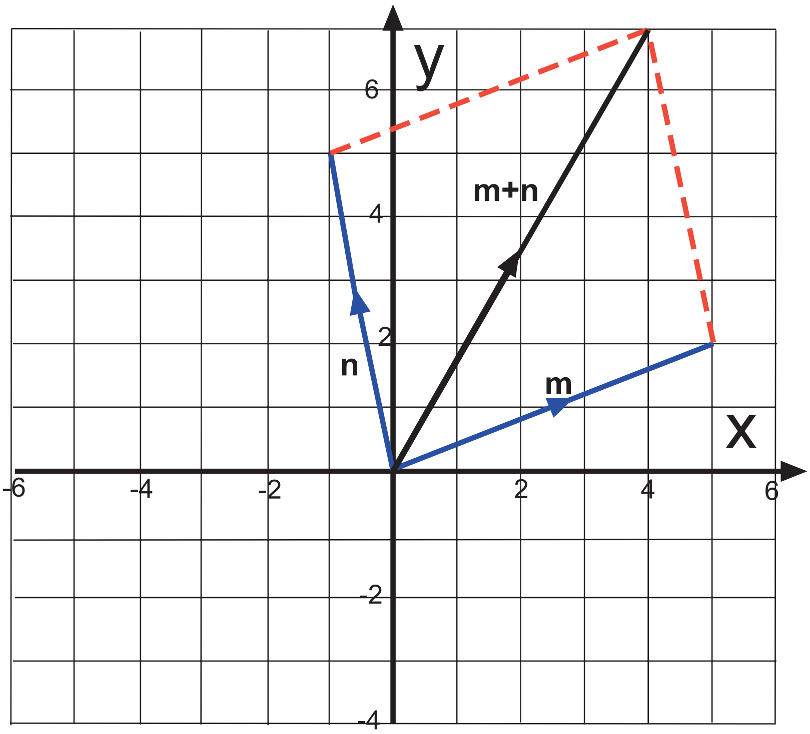
1. Adding two dimensional vectors

Add the vectors $\mathbf{m} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$ and $\mathbf{n} = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$

Answer:

The sum is found by adding the corresponding values so

$$\mathbf{m} + \mathbf{n} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} + \begin{pmatrix} -1 \\ 5 \end{pmatrix} = \begin{pmatrix} 5 + (-1) \\ 2 + 5 \end{pmatrix} = \begin{pmatrix} 4 \\ 7 \end{pmatrix}$$

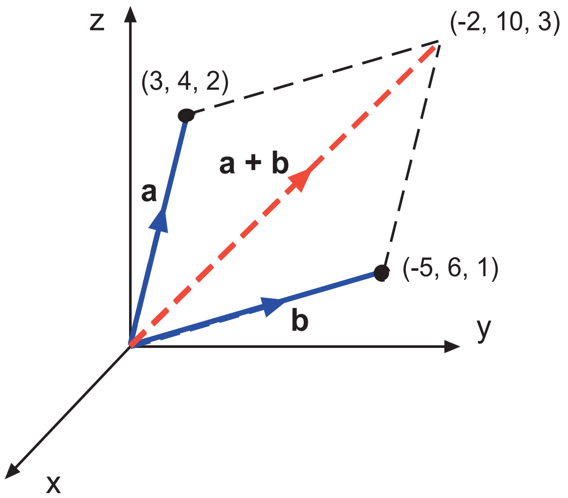


2. Adding three dimensional vectors

Add the vectors **a** = $\begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix}$ and **b** = $\begin{pmatrix} -5 \\ 6 \\ 1 \end{pmatrix}$

Answer:

$$\mathbf{a} + \mathbf{b} = \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix} + \begin{pmatrix} -5 \\ 6 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 + (-5) \\ 4 + 6 \\ 2 + 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 10 \\ 3 \end{pmatrix}$$



Adding vectors exercise

15 min

There is another exercise on the web if you prefer it.

Q14: If A = (-4, -1, 0), B = (2, -1, 1) and C = (4, -2, -3) add the following vectors:

a) **a** and **c**
b) **b** and **a**
c) **c** and **-c**

Q15: If A = (-1, 2, 3), B = (3, -4, 2) and C = (0, -1, -2) add the following vectors:

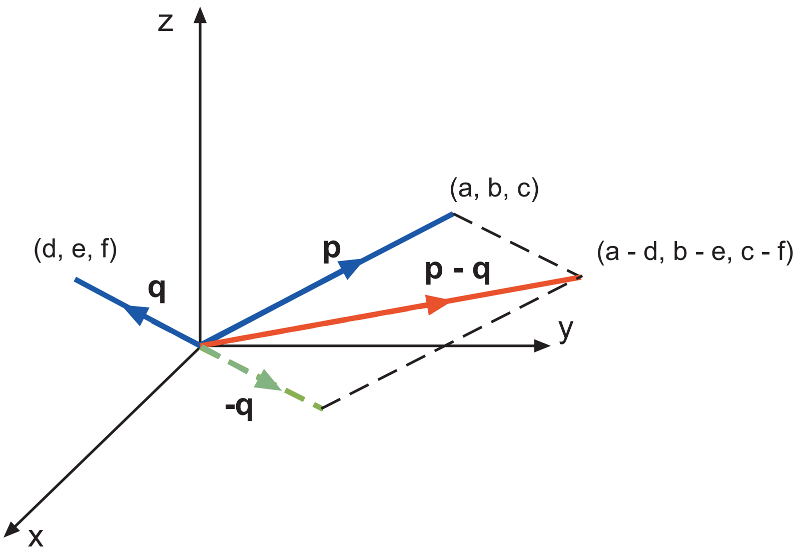
a) $\overrightarrow{AB}$ and $\overrightarrow{AC}$
b) $\overrightarrow{AB}$ and $\overrightarrow{BA}$
c) $\overrightarrow{BC}$ and $\overrightarrow{AC}$

Subtraction

If $\mathbf{p} = \begin{pmatrix} a \\ b \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} d \\ e \end{pmatrix}$ are vectors then $\mathbf{p - q}$ is the vector $\begin{pmatrix} a - d \\ b - e \end{pmatrix}$

Similarly if $\mathbf{p} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} d \\ e \\ f \end{pmatrix}$ are vectors then $\mathbf{p - q}$ is the vector $\begin{pmatrix} a - d \\ b - e \\ c - f \end{pmatrix}$

The following diagram shows how this relates to a parallelogram. Note that in subtraction, the vector **-q** instead of **q** forms one of the sides (**p** + (**-q**) = **p - q**)



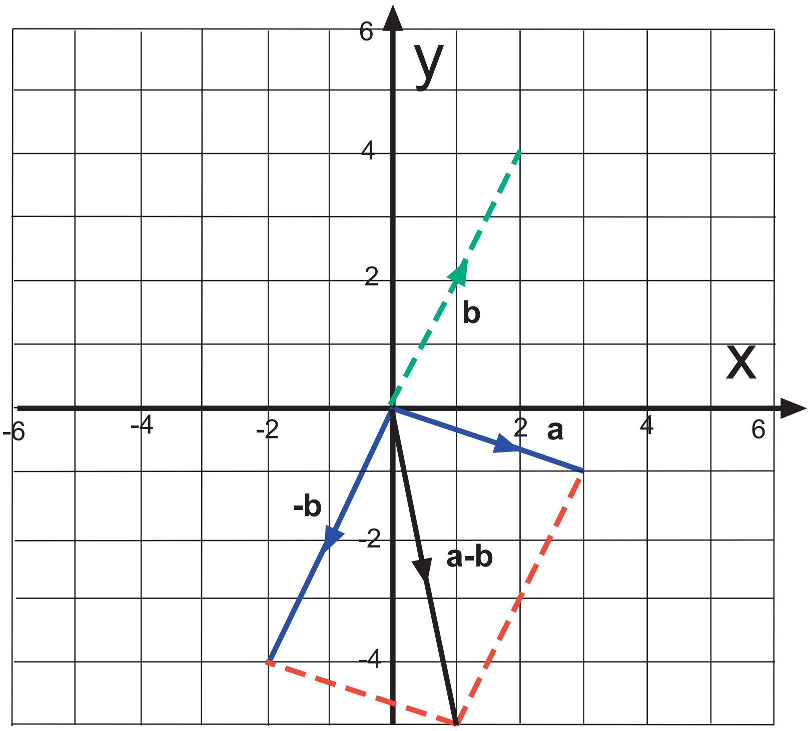
Examples

1. Subtracting two dimensional vectors

Evaluate **a** - **b** where **a** = $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$ and **b** = $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$

Answer:

$$\begin{pmatrix} 3 \\ -1 \end{pmatrix} - \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 3-2 \\ -1-4 \end{pmatrix} \begin{pmatrix} 1 \\ -5 \end{pmatrix}$$



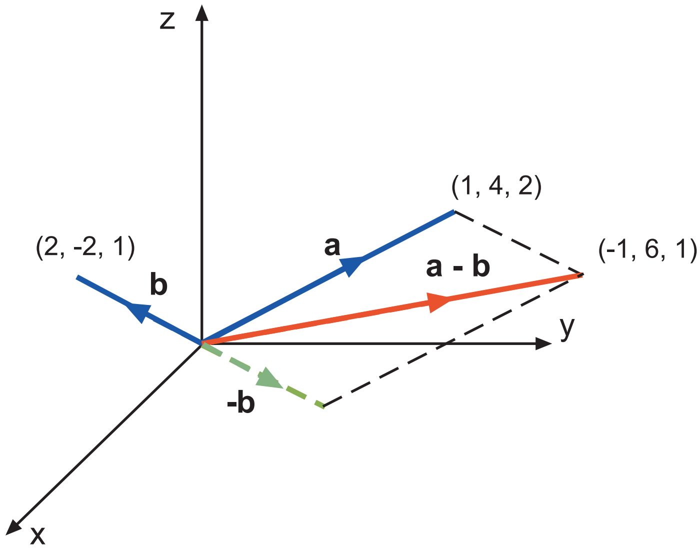
2. Subtracting three dimensional vectors

Evaluate

$$\begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$$

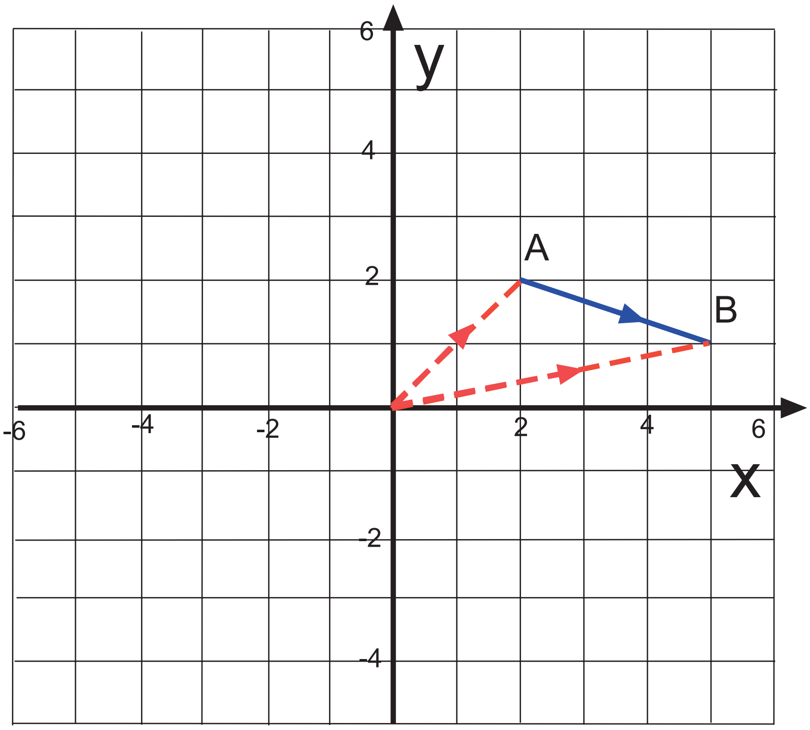
Answer:

$$\begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1-2 \\ 4-(-2) \\ 2-1 \end{pmatrix} = \begin{pmatrix} -1 \\ 6 \\ 1 \end{pmatrix}$$



Subtraction clearly shows the technique for determining any vector in component form.

Consider the following diagram.



The vector from A to B is the vector $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$ since it has a displacement of 3 units along the x-axis and -1 unit on the y-axis. The vector however, can be determined using the position vectors to the points A and B.

$\mathbf{a} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$

It is clear that $\mathbf{b} - \mathbf{a} = \overrightarrow{AB}$

The diagram also shows this. The direct route is from A to B following the arrow on the line AB. However, the journey from A to B can be made by travelling *against* the arrow on the line OA and then following the arrow on OB. This is **-a** + **b** or **b** - **a**.

Thus any vector can be expressed in terms of the relevant position vectors.

That is, $\overrightarrow{AB}$ = **b** - **a**

Subtracting vectors

15 min

There is an alternative exercise on the web.

Q16: If $\mathbf{a} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} 0 \\ 3 \\ -2 \end{pmatrix}$ find the following vectors:

a) **a** - **b**

b) **c** - **a**

c) **b** - **c**

Q17: If A = (0, 1, 0), B = (-1, 1, -1), C = (2, -3, -4) and D = (-2, 1, 3), find the following vectors:

a) $\overrightarrow{AD} - \overrightarrow{BD}$

b) $\overrightarrow{CD} - \overrightarrow{BA}$

c) $\overrightarrow{AC} - \overrightarrow{CD}$

d) $\overrightarrow{BD} - \overrightarrow{BA}$

Vectors in 2d addition and subtraction

10 min

There is a web animation to show addition and subtraction in 2 dimensions.

Multiplication by scalar

If $\mathbf{p} = \begin{pmatrix} a \\ b \end{pmatrix}$ and k is a real number, (a scalar) then $k\mathbf{p} = \begin{pmatrix} ka \\ kb \end{pmatrix}$

Similarly if $\mathbf{q} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and m is a real number, then $m\mathbf{q} = \begin{pmatrix} mx \\ my \\ mz \end{pmatrix}$

Example : Multiplication of a two dimensional vector by a scalar

If $\mathbf{s} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ find -4**s**

Answer:

$$-4\mathbf{s} = -4 \times \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} -4 \times 3 \\ -4 \times -2 \end{pmatrix} = \begin{pmatrix} -12 \\ 8 \end{pmatrix}$$

Example : Multiplication of a three dimensional vector by a scalar

If $\mathbf{a} = \begin{pmatrix} -2 \\ 3 \\ -1 \end{pmatrix}$ find 3**a**

Answer:

$$3\mathbf{a} = 3 \times \begin{pmatrix} -2 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \times -2 \\ 3 \times 3 \\ 3 \times -1 \end{pmatrix} = \begin{pmatrix} -6 \\ 9 \\ -3 \end{pmatrix}$$

Multiplication by a scalar exercise

10 min

There is an alternative exercise on the web if you prefer it.

Q18: If A = (2, 3, 0), B = (-1, -3, 1) and C = (3, -4, 2), multiply the vectors **a**, **b** and **c** by the scalar -3

Q19: **a** is multiplied by k, a positive integer, to give the vector k**a** where the components of **a** are all whole numbers. Find the value of k and the vector **a** if $k\mathbf{a} = \begin{pmatrix} 8 \\ -6 \\ 0 \end{pmatrix}$

This multiplication technique is particularly useful for determining whether two vectors are parallel or not.

Parallel vectors

If two vectors are parallel they have the same direction but their magnitudes are scalar multiples of each other.

Examples

1. Parallel vectors

Show that the two vectors $\mathbf{a} = \begin{pmatrix} -3 \\ 5 \\ 3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 6 \\ -10 \\ -6 \end{pmatrix}$ are parallel.

Answer:

Each component of **b** is -2 times the corresponding component of **b**

That is, **b** = -2**a** and the vectors are parallel.

2. Find a vector parallel to the position vector through the point (2, 4, 3) with z component equal to 6

Answer:

Let the vector be $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$

Then $\begin{pmatrix} a \\ b \\ c \end{pmatrix} = m \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix}$

Thus a = 2m, b = 4m and c = 3m

But c = 6 and so m = 2

The vector parallel to the position vector through the point (2, 4, 3) with z component equal to 6 is $\begin{pmatrix} 4 \\ 8 \\ 6 \end{pmatrix}$

This use of parallel vectors can be taken a stage further in order to determine *collinearity*.

Collinearity

All points which lie on a straight line are said to be collinear.

If $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are two vectors the following two vector conditions determine collinearity:

1. k is a scalar such that $\overrightarrow{AB} = k\overrightarrow{BC}$
2. the point B is common to both vectors

Example : Collinearity

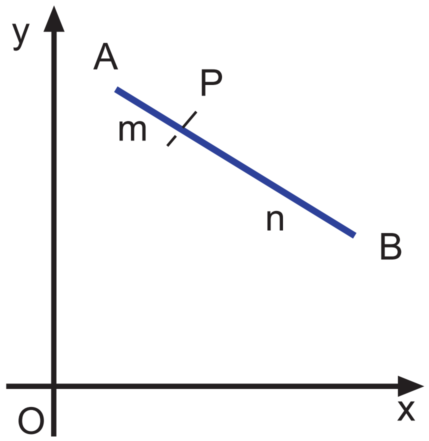
Determine whether the points A (1, 2, 3), B (3, 5, 4) and C (9, 14, 7) are collinear.

Answer:

$\overrightarrow{AB} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$ and $\overrightarrow{BC} = \begin{pmatrix} 6 \\ 9 \\ 3 \end{pmatrix}$

so $\overrightarrow{AB} = 3\overrightarrow{BC}$. Since the point B is common to both vectors, the three points are collinear.

This leads to another technique that can be used to determine the common point B or its postition vector. It is called the section formula.



Section formula

If **p** is the position vector of the point P which divides AB in the ratio m : n then

$$\mathbf{p} = \frac{n}{m+n}\mathbf{a} + \frac{m}{m+n}\mathbf{b}$$

Note that if a line AB is divided in the ratio of m : n by P then the ratio of AP : AB is actually m : m + n and this shows the derivation of the section formula.

The formula can be used as it stands but the alternative method can be useful for finding the coordinates of a point P which divides a line.

If there are three points A, B and C which are collinear and B divides the line AC in the ratio m : n then

$$\frac{\overrightarrow{AB}}{\overrightarrow{BC}} = \frac{m}{n} \text{ or } \overrightarrow{AB} = \frac{m}{n}\overrightarrow{BC}$$

Examples

1. Section formula

Find the position vector of the point P which divides the line AB in the ratio of 1 : 4 and where A = (5, 10) and B = (-5, 0)

Answer:

Using the section formula with m = 1, n = 4 gives

$$\mathbf{p} = \frac{4}{5}\mathbf{a} + \frac{1}{5}\mathbf{b} = \begin{pmatrix} 4 \\ 8 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 8 \end{pmatrix}$$

2. Alternative method

Find the position vector of the point B which divides the line AC in the ratio of 2 : 5 and where A = (-7, 14, 7) and C = (21, 0, 14)

Answer:

The ratio gives AB : AC as 2 : 7 but

$$\overrightarrow{AC} = \begin{pmatrix} 28 \\ -14 \\ 7 \end{pmatrix}$$

$$\overrightarrow{AB} = \frac{2}{7}\overrightarrow{AC} = \begin{pmatrix} 8 \\ -4 \\ 2 \end{pmatrix}$$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$\overrightarrow{OB} = \overrightarrow{AB} + \overrightarrow{OA}$$

$$\overrightarrow{OB} = \begin{pmatrix} 8 \\ -4 \\ 2 \end{pmatrix} + \begin{pmatrix} -7 \\ 14 \\ 7 \end{pmatrix} = \begin{pmatrix} 1 \\ 10 \\ 9 \end{pmatrix}$$

Thus B is the point (1, 10, 9)

Extra Help: Vector Pathways

An online exercise is provided to help you if you require additional assistance with this material, or would like to revise this subject.

Properties exercise

There is another exercise on the web if you prefer it.

15 min

Q20: Find a vector parallel to the position vector through the point (-5, 4, 1) with z component equal to -2

Q21: Determine if the following sets of points are collinear:

a) A (2, 0, 1), B (4, 2, -1) and C (6, 4, -3)
b) X (3, 2, -2), Y (1, 4, -1) and Z (-3, 8, 2)

Q22: Find the position vector of the point Q which divides the line AB in the ratio of 3 : 5 and where A = (-8, 16) and B = (24, 32)

Q23: Find the position vector of the point R which divides the line ST in the ratio of 1 : 3 and where S = (-4, 16) and T = (-8, 20)

Arithmetic using the standard basis vectors

Recall that **i**, **j** and **k** are called the **standard basis vectors**.

i is the vector $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

j is the vector $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

and **k** is the vector $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

If the vector $\begin{pmatrix} p \\ q \\ r \end{pmatrix}$ is written as p**i** + q**j** + r**k**, arithmetic can be performed on this form just as easily.

If **p** and **q** are written in terms of the standard bases then

p = a**i** + b**j** + c**k** and **q** = x**i** + y**j** + z**k**

thus **p + q** = (a + x)**i** + (b + y)**j** + (c + z)**k**

and **p - q** = (a - x)**i** + (b - y)**j** + (c - z)**k**

also m**p** = am**i** + bm**j** + cm**k**

Examples

1. Adding in standard basis form

Add the vectors -2**i** + 3**j** - 4**k** and -3**i** - 6**j** - 3**k**

Answer:

Adding the corresponding values gives

(-2**i** + 3**j** - 4**k**) + (-3**i** - 6**j** - 3**k**)

= (-2 - 3)**i** + (3 - 6)**j** + (-4 - 3)**k**

= -5**i** - 3**j** - 7**k**

2. Subtracting in standard basis form

Evaluate (**i** + 4**j** + 2**k**) - (2**i** -2**j** + **k**)

Answer:

(**i** + 4**j** + 2**k**) - (2**i** - 2**j** + **k**)

= **i** + 4**j** + 2**k** - 2**i** + 2**j** - **k**

= -**i** + 6**j** + **k**

3. Multiplying by a scalar in standard basis form

If **a** = 2**i** - 3**j** + **k** find -2**a**

Answer:

Each of the terms is multiplied by -2 to give

-2**a** = -4**i** + 6**j** - 2**k**

4. Finding parallel vectors in using standard basis form

Find a vector parallel to 3**i** - 2**j** + **k** with z component equal to 5

Answer:

Let the vector be a**i** + b**j** + c**k**

Then a**i** + b**j** + c**k** = m(3**i** - 2**j** + **k**)

Thus a = 3m, b = -2m and c = m

But c = 5 and so m = 5

The vector parallel to 3**i** - 2**j** + **k** with z = 5 is 5(3**i** - 2**j** + **k**) = 15**i** - 10**j** + 5**k**

Arithmetic on vectors using standard basis exercise

10 min

There is another exercise on the web for you to try if you prefer it.

Q24: Add the vectors 2**i** + 3**j** - 2**k** and **i** - 3**j** - **k**

Q25: Subtract 3**i** - 2**j** + 5**k** from -2**i**

Q26: Show that 2**i** - **j** + 3**k** and -4**i** + 2**j** - 6**k** are parallel vectors.

Q27: Find a vector parallel to 2**i** - 4**j** + **k** with z component equal to 3

1.5 Scalar product of vectors

Learning Objective

Find the scalar product of two vectors algebraically and geometrically

There are two ways of expressing the scalar product. The first is algebraically in component form.

Scalar product in component form (two dimnesions)

If $\mathbf{p} = \begin{pmatrix} a \\ b \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} d \\ e \end{pmatrix}$

then the scalar product is the number $\mathbf{p} \bullet \mathbf{q} = ad + be$

Scalar product in component form (three dimensions)

If $\mathbf{p} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} d \\ e \\ f \end{pmatrix}$

then the scalar product is the number $\mathbf{p} \bullet \mathbf{q} = ad + be + cf$

It is important to note that the scalar product of two vectors is *not a vector*. It is a *scalar*. The scalar product is also known as the dot product.

Examples

1. Scalar product (two dimensions)

Find the dot or scalar product of $\mathbf{a} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -2 \\ -1 \end{pmatrix}$

Answer:

$\mathbf{a} \bullet \mathbf{b} = (-2 \times -2) + (3 \times -1) = 1$

2. Scalar product (three dimensions)

Find the scalar product of $\mathbf{a} = \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}$

Answer:

a • **b** = -6 + 2 + 0 = -4

The scalar products of the standard basis vectors are useful to note:

i • **i** = **j** • **j** = **k** • **k** = 1 and

i • **j** = **j** • **k** = **i** • **k** = 0

also

i • **j** = **j** • **i**

j • **k** = **k** • **j**

i • **k** = **k** • **i**

Checking standard basis scalar products

10 min

Take these results and check them using the scalar product definition and the vectors **i**, **j**, **k** in component form.

These results can be used to give the scalar product in **i**, **j**, **k** notation of the two vectors $\mathbf{a} = \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}$

Example Find (-2**i** + **j** + 3**k**) • (3**i** + 2**j**) by multiplying out the brackets and using the properties of the vector product on standard bases.

Answer:

(-2**i** + **j** + 3**k**) • (3**i** + 2**j**)

= -6**i** • **i** - 4**i** •**j** + 3 **j** • **i** + 2**j** • **j** + 9**k** • **i** + 6**k** • **j**

= -6 -**i** • **j** + 2 + 9**i** • **k** + 6**j** •**k**

= -6 + 2 = -4

Algebraic scalar product exercise

10 min

There is an exercise similar to this on the web for you to try if you wish.

Q28: Find **a** • **b** when A = (2, 3) and B = (-3, 1)

Q29: Find **a** • **b** where $\mathbf{a} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -3 \\ 2 \\ -3 \end{pmatrix}$

Q30: Find the scalar product of the vectors 2**i** + 3**j** - 2**k** and **i** - 3**j** - **k**

Q31: Find the scalar product of the vectors $\begin{pmatrix} -2 \\ 5 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix}$

Algebraic rules of scalar products

There are several useful properties of scalar products.

Property 1
$\mathbf{a} \bullet (\mathbf{b} + \mathbf{c}) = \mathbf{a} \bullet \mathbf{b} + \mathbf{a} \bullet \mathbf{c}$

The proof of this result is given as proof 1 in the section headed proofs near the end of this topic.

Property 2
$\mathbf{a} \bullet \mathbf{b} = \mathbf{b} \bullet \mathbf{a}$

The proof of this result is given as proof 2 in the section headed proofs near the end of this topic.

Property 3
$\mathbf{a} \bullet \mathbf{a} = \lvert \mathbf{a} \rvert^2$ and is ≥ 0

The proof of this result is given as proof 3 in the section headed proofs near the end of this topic.

Property 4
$\mathbf{a} \bullet \mathbf{a} = 0$ if and only if $\mathbf{a} = 0$

The proof of this result is given as proof 4 in the section headed proofs near the end of this topic.

Q32: Show that the property $\mathbf{a} \bullet (\mathbf{b} + \mathbf{c}) = \mathbf{a} \bullet \mathbf{b} + \mathbf{a} \bullet \mathbf{c}$ holds for the three vectors $\mathbf{a} = 2\mathbf{i} - \mathbf{k}$, $\mathbf{b} = -3\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $\mathbf{c} = -\mathbf{i} + 2\mathbf{j} - \mathbf{k}$

Q33: Using $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ prove $\mathbf{a} \bullet \mathbf{b} = \mathbf{b} \bullet \mathbf{a}$

The second way of expressing the scalar product is geometrically.

Scalar product in geometric form

The scalar product of two vectors **a** and **b** is defined as

$\mathbf{a} \bullet \mathbf{b} = \lvert \mathbf{a} \rvert \lvert \mathbf{b} \rvert \cos\theta$ where θ is the angle between **a** and **b**, $0 \leq \theta \leq 180°$

The proof of this result can be found as proof 5 in the section headed proofs.

Example Find the scalar product of the vectors **a** and **b** where the length of **a** is 5, the length of **b** is 4 and the angle between them is 60°

Answer:

$\mathbf{a} \bullet \mathbf{b} = \lvert \mathbf{a} \rvert \lvert \mathbf{b} \rvert \cos\theta = 5 \times 4 \times \cos 60° = 10$

The geometric form of the scalar product is especially useful to find angles between vectors.

Example Find the angle between **i** + 4**j** and -8**i** + 2**j**

Answer:

a • **b** = -8 + 8 = 0

$|\ \mathbf{a}\ | = \sqrt{17}$

$|\ \mathbf{b}\ | = \sqrt{68}$

$\cos\theta = \dfrac{\mathbf{a} \bullet \mathbf{b}}{|\ \mathbf{a}\ |\,|\ \mathbf{b}\ |} = 0$ so $\theta = \dfrac{\pi}{2}$

The last example demonstrates an important geometric property of the scalar product.

Property 5
For non-zero vectors, **a** and **b** are perpendicular if and only if **a** • **b** = 0

The proof of this result is given as proof 6 in the section headed proofs near the end of this topic.

To show that two vectors are perpendicular (at right angles) however, the algebraic form is all that is required.

Examples

1. Perpendicular vectors in component form

Show that the following two vectors are perpendicular

$\begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$

a • **b** = -6 + 4 + 2 = 0

The vectors are therefore perpendicular.

2. Show that the two vectors **a** = -2**i** + 2**j** + **k** and **b** = 3**i** + 2**j** + 2**k** are perpendicular.

Answer:

a • **b** = $(-2 \times 3) + (2 \times 2) + (1 \times 2) = 0$

The vectors are perpendicular.

Geometric scalar product exercise

10 min

There is another exercise on the web for you to try if you prefer it.

Q34: Find the scalar product of two vectors that have lengths 4 and 8 units respectively and an angle of 45° between them. Give the answer to 3 decimal places.

Q35: Find the scalar product of two vectors whose lengths are 2.5 and 5 and have an angle of 150° between them. Give the answer to 3 decimal places.

Q36: Determine the angle between two vectors whose scalar product is -6 and whose lengths are 4 and 3

Q37: Find the angle between the two vectors 3**i** - 4**j** and 12**i** + 5**j**. Give the angle in degrees correct to 2 decimal places.

Q38: Find the angle ACB if A is (0, 1, 6), B is (2, 3, 0) and C is (-1, 3, 4)

Perpendicular vectors exercise

There is an exercise on the web for you to try if you like.

10 min

Q39: Find c so that c**i** + 2**j** - **k** is perpendicular to **i** - 3**k**

Q40: Find c so that 3**i** + c**j** + 4**k** is perpendicular to 2**i** -2**j** + 3**k**

Q41: Show that the vectors -4**i** - 3**j** + 2**k** and 2**i** - 2**j** + **k** are perpendicular.

1.6 Summary

The following points and techniques should be familiar after studying this topic:

- Stating a vector in component form.
- Calculating the length of a vector.
- Adding and subtracting vectors in 2 or 3 dimensions.
- Multiplying a vector by a scalar.
- Showing collinearity, parallel and perpendicular vectors.
- Finding the scalar product of two vectors.
- Using the scalar product to find the angle between two vectors.

1.7 Proofs

Proof 1: **a •(b + c)** = **a • b** + **a • c**

Let $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$

$$\mathbf{b} + \mathbf{c} = \begin{pmatrix} b_1 + c_1 \\ b_2 + c_2 \\ b_3 + c_3 \end{pmatrix}$$

$$\mathbf{a} \bullet (\mathbf{b} + \mathbf{c}) = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \bullet \begin{pmatrix} b_1 + c_1 \\ b_2 + c_2 \\ b_3 + c_3 \end{pmatrix} = a_1b_1 + a_1c_1 + a_2b_2 + a_2c_2 + a_3b_3 + a_3c_3$$

$$\mathbf{a} \bullet \mathbf{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \bullet \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = a_1b_1 + a_2b_2 + a_3b_3$$

$$\mathbf{a} \bullet \mathbf{c} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \bullet \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = a_1c_1 + a_2c_2 + a_3c_3$$

Thus

$$\begin{aligned} \mathbf{a} \bullet \mathbf{b} + \mathbf{a} \bullet \mathbf{c} &= a_1b_1 + a_2b_2 + a_3b_3 + a_1c_1 + a_2c_2 + a_3c_3 \\ &= a_1b_1 + a_1c_1 + a_2b_2 + a_2c_2 + a_3b_3 + a_3c_3 \end{aligned}$$

Proof 2: **a** • **b** = **b** • **a**

Let $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$

a • **b** =

$a_1b_1 + a_2b_2 + a_3b_3$ =

$b_1a_1 + b_2a_1 + b_3a_3$ (since ab = ba for any real numbers) =

b • **a**

Proof 3: **a** • **a** = $|\mathbf{a}|^2$ and is ≥ 0

Recall the definition of length.

If $\mathbf{p} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ then the length of $\mathbf{p} = \sqrt{a^2 + b^2 + c^2} = |\mathbf{p}|$

But using the scalar product

$\mathbf{p} \bullet \mathbf{p} = (a \times a) + (b \times b) + (c \times c) = a^2 + b^2 + c^2 = |\mathbf{p}|^2$

Since any square of a real number is positive then **p** • **p** is always greater than or equal to zero.

Proof 4: **a** • **a** = 0 if and only if **a** = 0

Let $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$

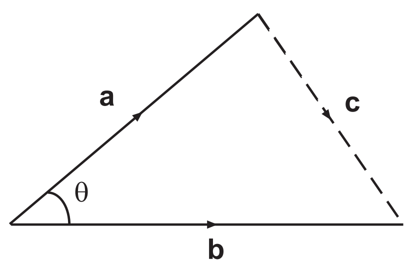
then $\mathbf{a} \bullet \mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = a_1^2 + a_2^2 + a_3^2$

The square of a real number is positive.

So **a** • **a** = 0 only if $a_1^2 = a_2^2 = a_3^2 = 0$

Thus $a_1 = a_2 = a_3 = 0 \Rightarrow \mathbf{a} = 0$

Proof 5: $\mathbf{a} \bullet \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos \theta$



Let **a** = $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and **b** = $\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

c = **b - a**

$$
\begin{aligned}
|\,\mathbf{c}\,|^2 &= |\,\mathbf{b} - \mathbf{a}\,|^2 \\
&= (b_1 - a_1)^2 + (b_2 - a_2)^2 + (b_3 - a_3)^2 \\
&= b_1^2 - 2a_1b_1 + a_1^2 + b_2^2 - 2a_2b_2 + a_2^2 + b_3^2 - 2a_3b_3 + a_3^2 \\
\text{Thus} \qquad &= (a_1^2 + a_2^2 + a_3^2) + (b_1^2 + b_2^2 + b_3^2) - 2(a_1b_1 + a_2b_2 + a_3b_3) \\
&= |\,\mathbf{a}\,|^2 + |\,\mathbf{b}\,|^2 - 2(\mathbf{a} \bullet \mathbf{b}) \text{ but} \\
|\,\mathbf{c}\,|^2 &= |\,\mathbf{a}\,|^2 + |\mathbf{b}\,|^2 - 2|\,\mathbf{a}\,|\,|\,\mathbf{b}\,| \cos\theta \text{ by the cosine rule so} \\
(\mathbf{a} \bullet \mathbf{b}) &= |\,\mathbf{a}\,|\,|\,\mathbf{b}\,| \cos\theta
\end{aligned}
$$

Proof 6: For non-zero vectors, **a** and **b** are perpendicular if and only if **a** • **b** = 0

a •**b** = | **a** | | **b** | cos θ

If **a** • **b** = 0 then | **a** | | **b** | cos θ = 0

$\Rightarrow$ cos θ = 0 since **a** and **b** are non-zero vectors.

Thus θ = 90° and the vectors **a** and **b** are perpendicular.

Conversely if **a** and **b** are perpendicular then | **a** | | **b** | cos θ = 0

and so **a** • **b** = 0

1.8 Extended Information

The web sites for this topic are well worth a visit as they comprehensively cover the same ground and provide good backup material. There are also sites with a different style of assessment which may prove useful for additional revision work.

Stevin

Simon Stevin was a Flemish mathematician of the 16/17th century. He was an outstanding engineer and used the concept of vector addition on forces. This was however a long time before vectors were generally accepted.

Hamilton

This Irish mathematician was the first, in 1853, to use the term 'vector'.

Gibbs

In the late 1800s, Josiah Gibbs used vectors in his lectures. He made major contributions to the work in vector analysis.

Weatherburn

An Australian, Charles Weatherburn published books in 1921 on vector analysis.

There are many more mathematicians who played a part in the development of vectors. Try to find some more if there is time.

1.9 Review exercise

Review exercise

30 min

There is another exercise on the web if you prefer it.

Q42: A, B and C have coordinates (2, 3, -1), (-1, 2, 2) and (-7, 0, 8).

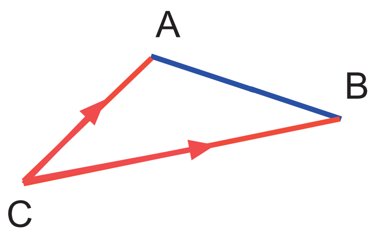
a) Write down the component form of $\overrightarrow{AC}$

b) Hence show that the points are collinear.

Q43: The point H divides the line KJ in the ratio 2: 1

P and Q have coordinates of (-9, 3, 6) and (6, -3, 0) respectively. Find the coordinates of H.

Q44: The diagram shows triangle ABC



where $\overrightarrow{CA} = \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}$ and $\overrightarrow{CB} = \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix}$

a) Find the value of $\overrightarrow{CA} \bullet \overrightarrow{CB}$

b) Use the result of (a) to find the size of angle ACB

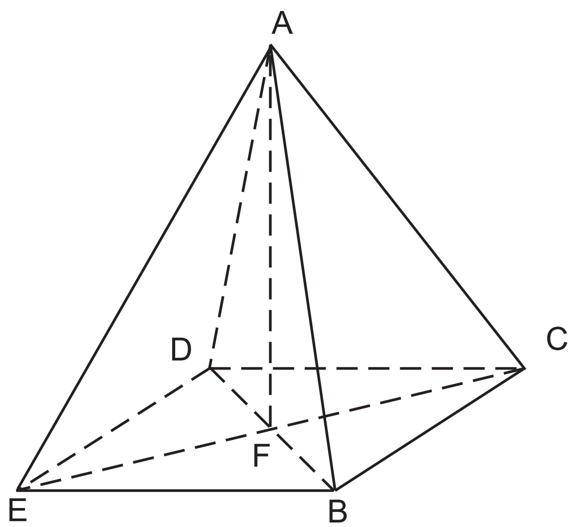
1.10 Advanced review exercise

Advanced review exercise

There is another exercise on the web if you prefer it.

30 min

Q45:



ABCDE is a square pyramid with a base of side length 6 cm. The sloping sides are all equilateral triangles. $\overrightarrow{BA} = \mathbf{p}$, $\overrightarrow{BE} = \mathbf{q}$ and $\overrightarrow{BC} = \mathbf{r}$

a) Evaluate $\mathbf{p} \bullet \mathbf{r}$

b) Hence find $\mathbf{p} \bullet (\mathbf{r} + \mathbf{q})$

c) Express $\overrightarrow{AD}$ in terms of **p, q** and **r**

d) Hence find the angle DAB

Q46: The angle between two vectors **a** and **b** is 60 °. Vector **a** has length 3 units and vector **b** has length 4 units.

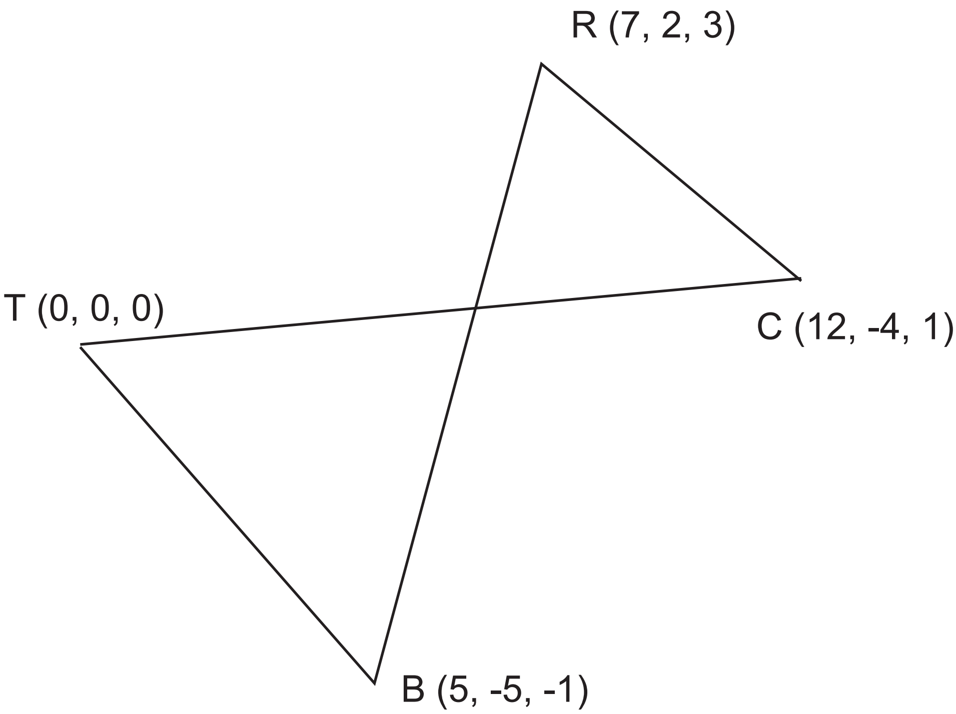
1. Evaluate:
   a) $\mathbf{a} \bullet \mathbf{a}$
   b) $\mathbf{b} \bullet \mathbf{a}$
   c) $\mathbf{b} \bullet \mathbf{b}$
2. Another vector is defined by $\mathbf{v} = 3\mathbf{a} - 2\mathbf{b}$. Evaluate $\mathbf{v} \bullet \mathbf{v}$ and write down the length of **v**

Q47: ABCD is a quadrilateral with vertices A (4, -1, 3), B (8, 3, -1), C (0, 4, 4) and D (-4, 0, 8)

a) Find the coordinates of M, the mid point of AB
b) Find the coordinates of the point T, which divides CM in the ratio 2 : 1
c) Show that B, T and D are collinear and find the ratio in which T divides BD

1989 SCE Higher paper II

Q48:



This diagram is drawn to the scale of 1 unit : 2 kilometres. The point T represents a transmitter, C represents a signal reflector dish, B is the bridge of a ship and R is a reflector on an aircraft.

a) Find the distance in kilometres, from the bridge of the ship to the reflector on the aircraft.
b) Prove that the direction of the beam TC is perpendicular to the direction of the beam BR
c) Calculate the size of angle TCR

adapted from 1992 SCE Higher paper II

1.11 Set review exercise

Set review exercise

20 min

Work out your answers and then access the web to input your answers in the online test called set review exercise.

Q49: A = (-2, 3, -1), B = (4, 3, -5) and C = (-8, 3, 3)

a) What are the vector conditions which determine the collinearity of A,B and C?

b) Find the components of the vector $\overrightarrow{AB}$

c) Find the components of the vector $\overrightarrow{BC}$

d) Are the points collinear?

Q50: Find the position vector of the point P which divides the line AB in the ratio 2 : 3 and where A = (-15, 10, 20) and B = (-5, -10, 15)

Q51: Find the scalar product of the position vectors to the points A = (-2, 2, 3) and to B = (1, -1, -4)

Q52: A = (5, -3, -2), B = (4, -1, 3) and C = (-1, -2, 2)

a) Find the vector from A to B using standard form.

b) Find the vector from A to C using standard form.

c) What is the scalar product of the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$?

d) What is the length of $\overrightarrow{AB}$?

e) What is the length of $\overrightarrow{AC}$?

f) Find the angle BAC

Topic 2

Further differentiation and integration

Contents

Learning Objectives

Use further differentiation and integration.

Minimum performance criteria:

- *Differentiate k sin x, k cos x*
- *Differentiate using the function of a function rule.*
- *Integrate functions of the form $f(x) = (x + q)^n$, n rational except for -1, $f(x) = p \cos x$ and $f(x) = p \sin x$*

Prerequisites

You should already have a basic knowledge of differentiation and integration.

2.1 Revision exercise

Learning Objective

Identify areas that need revision

20 min

Revision exercise

There is an on-line exercise at this point which you might find helpful.

Q1: Differentiate the following functions

a) $f(x) = 3x^2 + \frac{1}{2x}$

b) $f(x) = (2x + 1)(x - 3)$

c) $f(x) = \sqrt{x}\,(x^2 - 6)$

d) $f(x) = \frac{4x^3 + 2x - 3}{x}$

Q2: Find the following integrals

a) $\int 6x^2 - x^{1/3}\,dx$

b) $\int \frac{x^2 - 4x^5}{x}\,dx$

c) $\int \sqrt{z}\left(z - \frac{1}{3z}\right)dz$

Q3: Evaluate the following

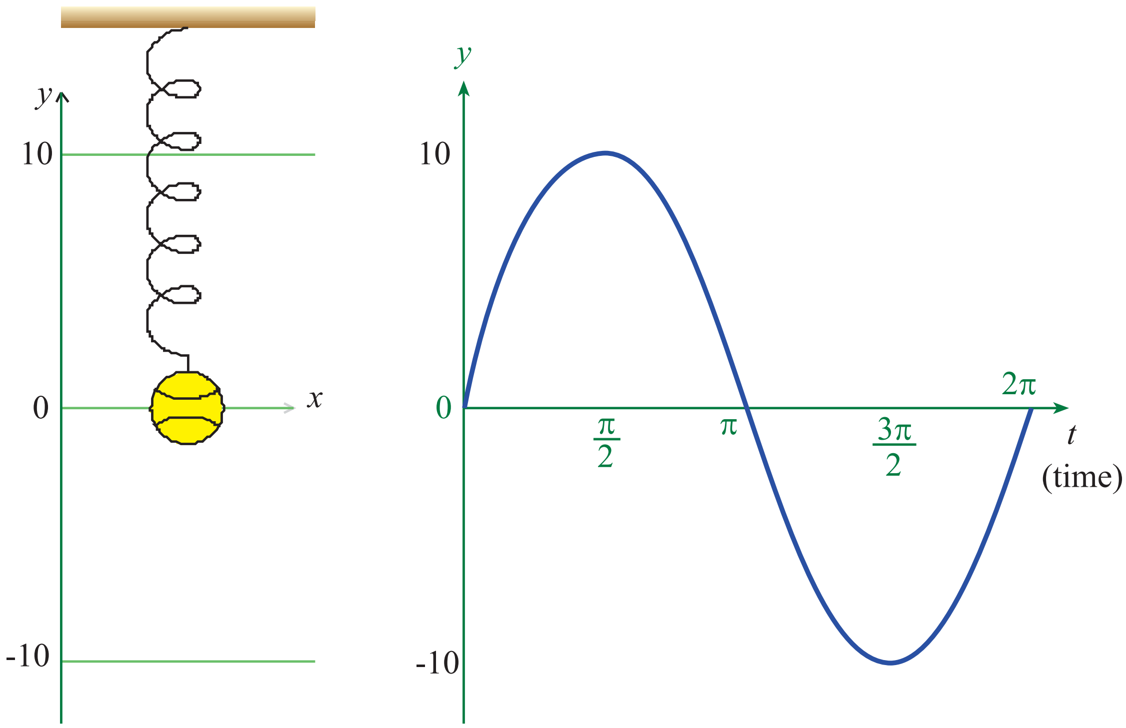
a) $\int_0^3 (1 + 4x)\,dx$

b) $\int_{-1}^2 (x + 1)^2\,dx$

c) $\int_1^4 \frac{3 - t}{\sqrt{t}}\,dt$

2.2 Introduction

A ball, attached to the end of a stretched spring, is released at time t = 0. The displacement y (cm) of the ball from the x -axis at time t (seconds) is given by the formula y (t) = 10 sin t



We might wish to know the answers to the following questions:

- What is the speed of the ball after 2 seconds?
- When is the ball first stationary?

Since speed is the rate of change of distance with respect to time, the speed of the ball is given by the differential equation

$$\frac{dy}{dt} = \frac{d}{dt}(10 \sin t)$$

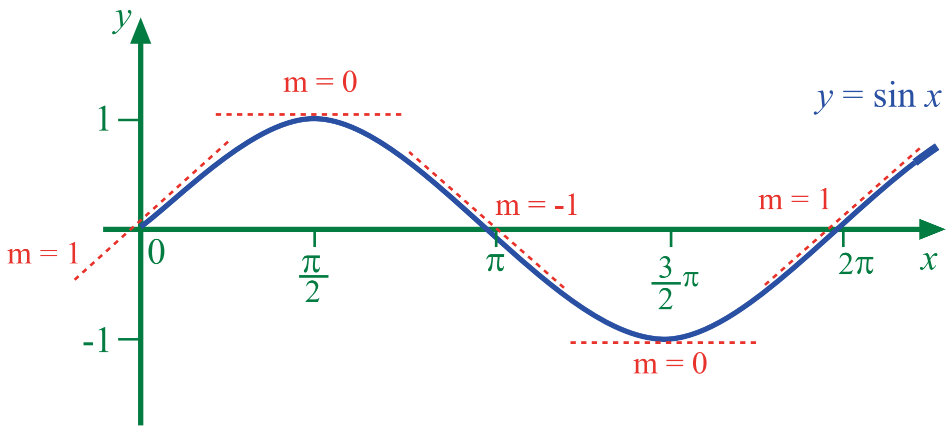
Thus in order to answer these questions we need to find the derivative of the sine function. You will learn how to do this and more in the following section of work.

2.3 Differentiation of sin x and cos x

Learning Objective

Differentiate sine and cosine functions

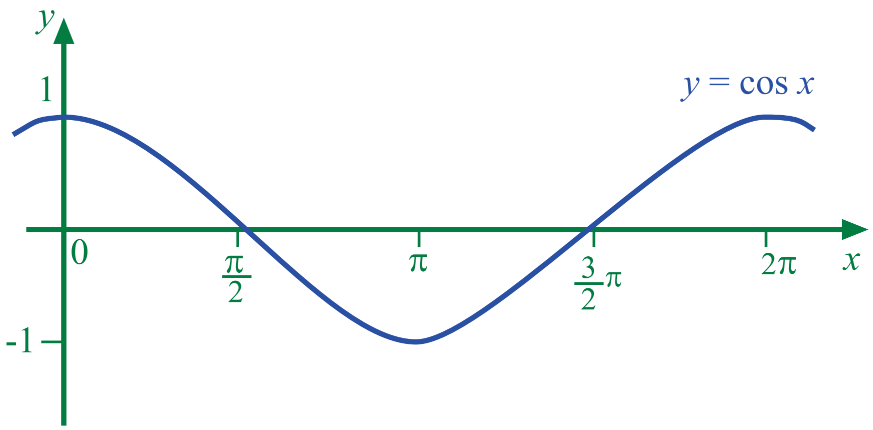
The graph for y = sin x is shown here.



Notice that the tangent to the curve is drawn at various points. The value for the gradient of the tangent at these points is recorded in the following table.

x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
m_T	1	0	- 1	0	1

When these points are plotted and joined with a smooth curve the result is as follows.



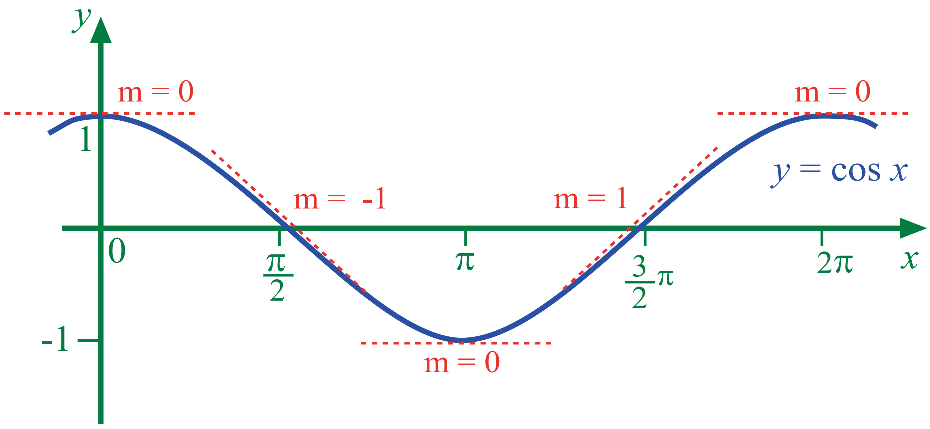
Since the gradient of the tangent at the point (a, f (a)) on the curve y = f (x) is f ' (a) then the above graph represents the graph of the derivative of sin x

Then for y = sin x it appears that $\frac{dy}{dx}$ = cos x

(You could check this further by calculating gradients at intermediate points).

Q4:

Study the graph for y = cos x as shown here.



Complete the following table for the gradient of the tangents.

x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
m_T					

Plot the points and join with a smooth curve.

Q5: When y = cos x it appears that $\frac{dy}{dx}$ = ?

a) sin x
b) cos x
c) - sin x
d) - cos x

When f (x) = sin x then f ' (x) = cos x alternatively $\frac{d}{dx}(\sin x) = \cos x$
When f (x) = cos x then f ' (x) = - sin x alternatively $\frac{d}{dx}(\cos x) = -\sin x$
Note: x must be measured in ***radians***

Examples

1. Find f ' (x) when f (x) = 3 cos x

Solution

When f (x) = 3 cos x

then f ' (x) = - 3 sin x

2. Calculate $\frac{d}{dx}\left(2 \sin x - \cos x + 4 x^2\right)$

Solution

$$\begin{aligned}\frac{d}{dx}\left(2 \sin x - \cos x + 4 x^2\right) &= 2 \cos x - (-\sin x) + 8 x\\ &= 2 \cos x + \sin x + 8 x\end{aligned}$$

Now try the questions in the following exercise.

60 min

Exercise 2

There is an on-line exercise at this point which you might find helpful.

Q6: Find f ' (x) when

a) $f(x) = 2 \sin x$

b) $f(x) = 3 \cos x$

c) $f(x) = -5 \sin x$

d) $f(x) = -4 \cos x$

e) $f(x) = 5 \sin x - \cos x$

f) $f(x) = 3 \cos x + 2 \sin x$

g) $f(x) = 3 x^2 - 5 \cos x$

h) $f(x) = 7 \sin x + \cos x - 6$

i) $f(x) = \frac{2 x \sin x - 5}{x}$

j) $f(x) = \frac{4 - 3\sqrt{x}\cos x}{\sqrt{x}}$

Q7: Calculate the following

a) $\frac{d}{dx}(3 \cos x)$

b) $\frac{d}{dt}(5 \sin t - 2 \cos t)$

c) $\frac{d}{du}\left(\frac{5}{u^2} - 3 \sin u\right)$

d) $\frac{dy}{d\theta}\left(7 \cos \theta + \sqrt{\theta^3}\right)$

Q8: Find the gradient of the curve with equation y = 2 sin x at the following points.

a) $x = 0$

b) $x = \frac{\pi}{4}$

c) $x = \frac{\pi}{3}$

d) $x = \frac{\pi}{2}$

Q9: Find the gradient of the curve with equation y = 3 sin x - $\sqrt{2}$ cos x at the following points

a) x = 0
b) x = $\frac{\pi}{4}$
c) x = $\frac{\pi}{2}$
d) x = $\frac{3\pi}{2}$

Q10: Calculate f ' (x) for f (x) = 3 sin x and hence show that f (x) has a maximum turning point at $\left(\frac{\pi}{2}, 3\right)$ and a minimum turning point at $\left(\frac{3\pi}{2}, -3\right)$

Q11: Find the equation of the tangent to the curve y = 2 sin x at x = $\frac{\pi}{3}$

Q12: Find the equation of the tangent to the curve y = 3 cos x + sin x at x = $\frac{\pi}{4}$

Q13: Show that the graph of y = x + sin x is never decreasing.

Q14:

a) Find the gradient of the tangents to the curve y = x^2 + 2 sin x at the points where x = 0 and x = $\frac{\pi}{2}$
b) Calculate the acute angle, in degrees, between these tangents. Round your answer to 1 decimal place.

Q15: Find the equation of the tangent to the curve y = $\frac{4x^2}{\pi}$ - 2 cos x at x = $\frac{\pi}{2}$

2.4 Differentiation of $(x + a)^n$

Learning Objective

Differentiate functions of the type $(x + a)^n$

We can differentiate expressions such as $(x + 5)^2$ and $(x - 4)^3$ by expanding and differentiating term by term.

Examples

1. Find f ' (x) when f (x) = $(x + 5)^2$

Solution

We need to expand the expression first.

$$\begin{aligned} f(x) &= (x + 5)^2 \\ &= (x + 5)(x + 5) \\ &= x^2 + 10x + 25 \end{aligned}$$

Now we can differentiate term by term

$$\begin{aligned} f'(x) &= 2x + 10 \\ &= 2(x + 5) \end{aligned}$$

2. Find f ’ (x) when f (x) = $(x - 4)^3$

Solution

Again we need to expand the expression first.

$$\begin{aligned} f(x) &= (x - 4)^3 \\ &= (x - 4)(x - 4)(x - 4) \\ &= (x - 4)\left(x^2 - 8x + 16\right) \\ &= x^3 - 8x^2 + 16x - 4x^2 + 32x - 64 \\ &= x^3 - 12x^2 + 48x - 64 \end{aligned}$$

Now we can differentiate term by term

$$\begin{aligned} f'(x) &= 3x^2 - 24x + 48 \\ &= 3\left(x^2 - 8x + 16\right) \\ &= 3(x - 4)^2 \end{aligned}$$

Use the method above to differentiate the following functions. Factorise your answers.

Q16:

a) $f(x) = (x + 3)^2$
b) $f(x) = (x + 3)^3$
c) $f(x) = (x - 2)^2$
d) $f(x) = (x - 2)^3$

Are you beginning to see a pattern to your answers?

Make a prediction for the derivatives of the following functions ***without*** expanding the expressions.

Q17:

a) $f(x) = (x + 3)^4$
b) $f(x) = (x + 3)^5$
c) $f(x) = (x + 3)^6$
d) $y = (x - 2)^4$
e) $y = (x - 2)^5$
f) $y = (x - 2)^6$

In general we can write

$$\boxed{\text{When } f(x) = (x + a)^n \text{ then } f'(x) = n(x + a)^{n-1}, \ (n \in \mathbb{Q}, a \in \mathbb{R})}$$

We can use this rule for more complex functions as in the following examples.

Examples

1. Differentiate $f(x) = \dfrac{1}{(x - 3)^5}$

Solution

First we must rewrite the expression for f (x).

$$\begin{aligned} f(x) &= \frac{1}{(x-3)^5} \\ &= (x-3)^{-5} \end{aligned}$$

Now we can differentiate using the above rule.

$$\begin{aligned} f'(x) &= -5\,(x-3)^{-6} \\ &= \frac{-5}{(x-3)^6} \end{aligned}$$

2. Find $\dfrac{d}{dx}\left(\sqrt{x+1}\right)^5$

Solution

$$\begin{aligned} \frac{d}{dx}\left(\sqrt{x+1}\right)^5 &= \frac{d}{dx}(x+1)^{5/2} \\ &= \frac{5}{2}(x+1)^{3/2} \\ &= \frac{5}{2}\left(\sqrt{x+1}\right)^3 \end{aligned}$$

Now try the questions in the following exercise.

Exercise 3

50 min

There is an on-line exercise at this point which you might find helpful.

Differentiate the following.

Q18:

a) $(x + 6)^8$

b) $(x - 2)^5$

c) $(x + 8)^{-3}$

d) $(x - 4)^{3/2}$

e) $\dfrac{1}{(x + 3)^2}$

f) $\sqrt{(x - 6)^5}$

g) $\dfrac{1}{x - 1}$

h) $\sqrt[3]{(x + 5)^2}$

Q19: Find the value of f ' (0) and f ' (1) when $f(x) = (x - 1)^4$

Q20: Find the equation of the tangent to the curve $y = \dfrac{1}{(x+1)^2}$ at the point where x = 2

Q21: Find

a) $\dfrac{d}{dx}\left(\sin x + (x+9)^8\right)$

b) $\dfrac{d}{dx}\left(\dfrac{1}{x-5} - 3\cos x\right)$

c) $\dfrac{d}{dx}\left(4\sin x + \sqrt[4]{x-2}\right)$

d) $\dfrac{d}{dx}\left(3x^8 - (x+4)^5\right)$

2.5 Differentiation of $(ax + b)^n$

Learning Objective

Differentiate functions of the type $(ax + b)^n$

We now know that, for example, when $y = (x+1)^3$ then $\dfrac{dy}{dx} = 3(x+1)^2$

Consider $y = (2x+1)^3$

How will the coefficient 2 affect the derivative?

Example Find $\dfrac{dy}{dx}$ when $y = (2x+1)^3$

Solution

We need to expand the expression first.

$$\begin{aligned} y &= (2x+1)^3 \\ &= (2x+1)(2x+1)(2x+1) \\ &= (2x+1)\left(4x^2 + 4x + 1\right) \\ &= 8x^3 + 8x^2 + 2x + 4x^2 + 4x + 1 \\ &= 8x^3 + 12x^2 + 6x + 1 \end{aligned}$$

Now we can differentiate term by term.

$$\begin{aligned} \frac{dy}{dx} &= 24x^2 + 24x + 6 \\ &= 6(4x^2 + 4x + 1) \\ &= 6(2x+1)^2 \\ &= 3(2x+1)^2 \times 2 \end{aligned}$$

Notice that the coefficient of 2 provides a factor of 2 in the derivative.

Use this method to differentiate the following functions. Write your answers as in the example above.

Q22:

a) $y = (3x + 1)^2$
b) $y = (5x - 2)^2$
c) $y = (2x + 5)^3$
d) $y = (4x - 1)^3$

Have you spotted a pattern?

Make a prediction for the derivatives of the following functions ***without*** expanding the expressions.

Q23:

a) $y = (2x + 1)^4$
b) $y = (3x + 1)^4$
c) $y = (5x + 1)^4$
d) $y = (5x - 2)^8$
e) $y = (7x + 5)^4$

You may have noticed that

$$\text{When } f(x) = (ax + b)^n \text{ then } f'(x) = n\,(ax + b)^{n-1} \times a$$
$$= an\,(ax + b)^{n-1}, \quad (n \in \mathbb{Q}, a \in \mathbb{R})$$

You can see how this rule is used in the following examples.

Examples

1. Differentiate $f(x) = (9x - 4)^5$

Solution

$f'(x) = 5\,(9x - 4)^4 \times 9 \times 9 \times 9 \times 9 = 45\,(9x - 4)^4$

2. Differentiate $y = \dfrac{1}{(6x - 5)^{1/3}}$

Solution

First we must rewrite the expression for y

$$y = \frac{1}{(6x - 5)^{1/3}}$$
$$= (6x - 5)^{-1/3}$$

Now we can differentiate using the rule.

$$\frac{dy}{dx} = -\frac{1}{3}(6x - 5)^{-4/3} \times 6$$
$$= -2(6x - 5)^{-4/3}$$
$$= -\frac{2}{(6x - 5)^{4/3}}$$

Now try the questions in the following exercise.

20 min

Exercise 4

There is an on-line exercise at this point which you might find helpful.

Differentiate the following

Q24:

a) $(3x + 5)^7$

b) $(1 - 4x)^3$

c) $(6x + 1)^{1/2}$

d) $(2 - 3x)^4$

e) $(2x + 1)^{-5}$

f) $\dfrac{1}{(7x + 2)^4}$

g) $\dfrac{1}{\sqrt{5 - 4x}}$

h) $\dfrac{1}{5x + 6}$

2.6 The Chain Rule

Learning Objective

Differentiate composite functions using the chain rule.

Function notation

$h(x) = (ax + b)^n$ is an example of a composite function.

Let $f(x) = ax + b$ and $g(x) = x^n$ then we can write

$$h(x) = (ax + b)^n$$
$$= (f(x))^n$$
$$= g(f(x))$$

When $f(x) = ax + b$ then $f'(x) = a$

When $g(x) = x^n$ then $g'(x) = nx^{n-1}$

We could also write that when $g(f) = (f)^n$ then $g'(f) = n(f)^{n-1} = n(ax + b)^{n-1}$

However, we already know that when $y = (ax + b)^n$ then $\frac{dy}{dx} = an\left(ax + b\right)^{n-1}$

We can now write this in function notation in the following way.

When $h(x) = g(f(x))$ then

$$\begin{aligned} h'(x) &= an\left(ax + b\right)^{n-1} \\ &= n\left(ax + b\right)^{n-1} \times a \\ &= g'\left(f(x)\right) \times f'(x) \quad \text{(from (10.1) and (10.2))} \end{aligned}$$

This result is known as **the chain rule**.

Leibniz notation

We can also write the chain rule in Leibniz notation.

Again let $y = (ax + b)^n$ but this time let $u = ax + b$

We can now write $y = u^n$

Since $u = ax + b$ then $\frac{du}{dx} = a$

Since $y = u^n$ then $\frac{dy}{du} = nu^{n-1} = n\left(ax + b\right)^{n-1}$

As before, we already know that when $y = (ax + b)^n$ then $\frac{dy}{dx} = an\left(ax + b\right)^{n-1}$

We can rewrite this in Leibniz notation in the following way.

When $y = u^n$ then

$$\begin{aligned} \frac{dy}{dx} &= an\left(x + a\right)^{n-1} \\ &= n\left(x + a\right)^{n-1} \times a \\ &= \frac{dy}{du} \times \frac{du}{dx} \quad \text{(from (10.3) and (10.4))} \end{aligned}$$

It will be useful to remember both forms of the chain rule. (However, you are not required to be able to prove either of them)

The chain rule

Function notation

$$h'(x) = g'\left(f(x)\right) \times f'(x)$$

Leibniz notation

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

The chain rule allows us to differentiate many types of composite functions as you can see in the following examples. Note that either function or Leibniz notation can be used.

Example : Bracketed function raised to a power.

Find h ' (x) when h (x) = $\sqrt{(x^2 + 6x)}$

Solution (using function notation)

$$\begin{aligned} h(x) &= \sqrt{(x^2 + 6x)} \\ &= \left(x^2 + 6x\right)^{1/2} \end{aligned}$$

Let f (x) = $(x^2 + 6x)$ then f ' (x) = 2x + 6

Let g (x) = $x^{1/2}$ then $g'(x) = \frac{1}{2} x^{-1/2}$

We can also write that when g (f) = $(f)^{1/2}$ then $g'(f) = \frac{1}{2}(f)^{-1/2}$

Now we can use the chain rule to find h' (x)

$$\begin{aligned} h'(x) &= g'(f) \times f'(x) \\ &= \frac{1}{2}(f)^{-1/2} \times f'(x) \\ &= \frac{1}{2}\left(x^2 + 6x\right)^{-1/2} \times (2x + 6) \\ &= (x + 3)\left(x^2 + 6x\right)^{-1/2} \\ &= \frac{x + 3}{(x^2 + 6x)^{1/2}} \\ &= \frac{x + 3}{\sqrt{(x^2 + 6x)}} \end{aligned}$$

Solution (using Leibniz notation)

Let $u = x^2 + 6x$ then $\frac{du}{dx} = 2x + 6$

Let $y = \sqrt{u} = u^{1/2}$ then $\frac{dy}{du} = \frac{1}{2} u^{-1/2}$

Thus, by the chain rule

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= \frac{1}{2} u^{-1/2} \times (2x + 6) \\ &= \frac{1}{2}\left(x^2 + 6x\right)^{-1/2} \times (2x + 6) \\ &= \frac{x + 3}{\sqrt{(x^2 + 6x)}} \end{aligned}$$

This is the same result as that obtained using function notation.

The chain rule also gives us a method for differentiating composite functions involving trig functions.

Examples

1. Trig. function with a multiple angle.

Find h ’ (x) when h (x) = sin (5x)

Solution (using function notation)

Let f (x) = 5x then f ’ (x) = 5

Let g (f) = sin (f0 then g ’ (f) = cos (f)

Thus, by the chain rule

$$\begin{aligned} h'(x) &= g'(f) \times f'(x) \\ &= \cos(f) \times 5 \\ &= \cos(5x) \times 5 \\ &= 5\cos(5x) \end{aligned}$$

Solution (using Leibniz notation)

Let y = sin u with u = 5x

When y = sin u then $\frac{dy}{du} = \cos u$

When u = 5x then $\frac{du}{dx} = 5$

Thus by the chain rule

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ = 5 cos (5x)

= cos u $\times 5$ = cos u $\times 5$ = cos u $\times 5$ = cos u $\times 5$

This is the same result as that obtained using function notation.

2. Trig. function raised to a power.

Find the derivative of $\cos^3 x$

Solution (using function notation)

Remember that $\cos^3 x$ can be rewritten as $(\cos x)^3$

Let f (x) = cos x then f’ (x) = - sin x

Let g (f) = $(f)^3$ then g ’ (f) = $3\,(f)^2$

Now we can use the chain rule to find the derivative of $\cos^3 x$

$$\begin{aligned} \frac{d}{dx}\left(\cos^3 x\right) &= \frac{d}{dx}(\cos x)^3 \\ &= g'(f) \times f'(x) \\ &= 3\,(f)^2 \times f'(x) \\ &= 3\,(\cos x)^2 \times (-\sin x) \\ &= -3 \sin x \cos^2 x \end{aligned}$$

Solution (using Leibniz notation)

Let u = cos x then $\frac{du}{dx} = -\sin x$

Let $y = u^3$ then $\frac{dy}{du} = 3u^2$

Thus, by the chain rule

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ $\quad = 3\ (\cos x)^2\ (-\sin x)$

$= 3\ u^2 \times \times \times \times (-\sin x)$

$= -3 \sin x \cos^2 x$

This is the same result as that obtained using function notation.

Now try the questions in the following exercise.

60 min

Exercise 5

There is an on-line exercise at this point which you might find helpful.

Differentiate the following. You can choose to use either function or Leibniz notation, whichever you find easier.

Q25:

a) $h(x) = (2x - 5)^3$
b) $h(x) = (x^3 + 2x)^7$
c) $h(x) = (4 - 3x^2)^5$
d) $h(x) = \sqrt{16x^2 + 9}$
e) $h(x) = \frac{1}{3x^5 + 2}$
f) $h(x) = \left(\frac{1}{x} + x\right)^5$

Q26:

a) $y = \sin(2x)$
b) $y = \cos\left(\frac{1}{2}x\right)$
c) $y = \cos(2x - 3)$
d) $y = \sin\left(5 - \frac{x}{2}\right)$
e) $y = \sin(x^2 + 4)$
f) $y = \cos\left(6x + \frac{\pi}{4}\right)$

Q27:

a) $f(x) = \sin^2 x$
b) $f(x) = \cos^5 x$
c) $f(x) = \frac{1}{\sin^3 x}$
d) $f(x) = \sqrt{\sin x}$

Q28: When f (x) = $(1 - 2 \cos x)^2$ calculate f' $\left(\frac{\pi}{6}\right)$ and f' $\left(\frac{\pi}{3}\right)$

Q29: Find the equation of the tangent to the graph y = $\frac{3}{(2x + 1)^2}$ at the point where x = 1

Q30:

a) Find the coordinates of the stationary points on the curve f (x) = $\sin^2 x + \sin x$ for $0 \leqslant x \leqslant 2\pi$

b) Make a sketch of the curve for $0 \leq x \leq 2\pi$.

Q31: Show that the graph of y = 5x + sin 2x is never decreasing.

2.7 Integration of sin *x* and cos *x*

Learning Objective

Integrate sine and cosine functions

We have already seen that

$$\frac{d}{dx}(\sin x) = \cos x$$
$$\frac{d}{dx}(\cos x) = -\sin x$$

Since integration is the reverse process to differentiation it therefore follows that

$$\int \cos x dx = \sin x + C$$
$$\int \sin x dx = -\cos x + C$$

Again, note that x must be measured in ***radians***.

Examples

1. Find $\int (8 + \sin x)\, dx$

Solution

$\int (8 + 3\sin x)\, dx = 8x - 3\cos x + C$

2. Evaluate $\int_0^{\pi/4} (\sqrt{2}\cos x + 3\sin x)\, dx$

Solution

$$\int_0^{\pi/4} \left(\sqrt{2}\cos x + 3\sin x\right) dx = \left[\sqrt{2}\sin x - 3\cos x\right]_0^{\pi/4}$$
$$= \left(\frac{\sqrt{2}}{\sqrt{2}} - \frac{3}{\sqrt{2}}\right) - (0 - 3)$$
$$= 1 - \frac{3\sqrt{2}}{2} + 3$$
$$= 4 - \frac{3\sqrt{2}}{2}$$

Now try the questions in the following exercise.

60 min

Exercise 6

There is an on-line exercise at this point which you might find helpful.

Q32: Find the following

a) $\int 3\sin x\, dx$

b) $\int \sqrt{5}\cos x\, dx$

c) $\int -\pi \sin\theta\, d\theta$

d) $\int \left(8 - \sqrt{2}\cos u\right) du$

e) $\int \left(\sqrt{x} + 3\sin x\right) dx$

f) $\int (6\sin t - \cos t)\, dt$

g) $\int \left(\frac{3}{\sqrt{\omega}} + 7\sin\omega\right) d\omega$

h) $\int \left(\frac{\cos x}{3} - \sqrt{5}\sin x + 2\pi\right) dx$

Q33: Evaluate the following integrals

a) $\int_0^{\pi/6} 4\cos x\, dx$

b) $\int_0^{\pi/4} \sqrt{2}\sin x\, dx$

c) $\int_0^{\pi/3} (\cos t - 5\sin t)\, dt$

d) $\int_{\pi/3}^{\pi/2} (6 - 5\sin u)\, du$

e) $\int_0^{\pi/3} (2x + 3\cos x)\, dx$

f) $\int_{\pi/2}^{3\pi/2} (2\sin\alpha - 5\cos\alpha + \pi)\, d\alpha$

Q34: Evaluate the following integrals

a) $\int_{0}^{\pi} \sin x \, dx$

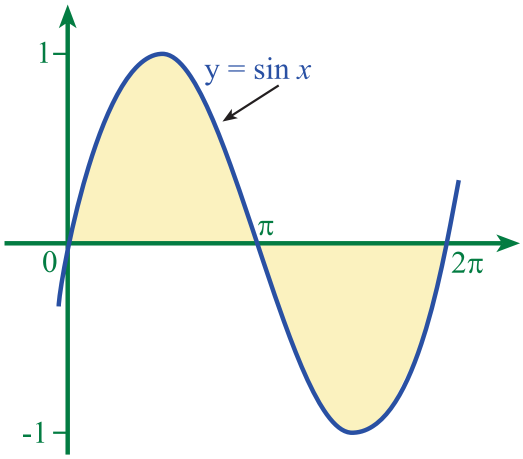
b) $\int_{\pi}^{2\pi} \sin x \, dx$

c) $\int_{0}^{2\pi} \sin x \, dx$

d)

The graph shown here is for y = sin x

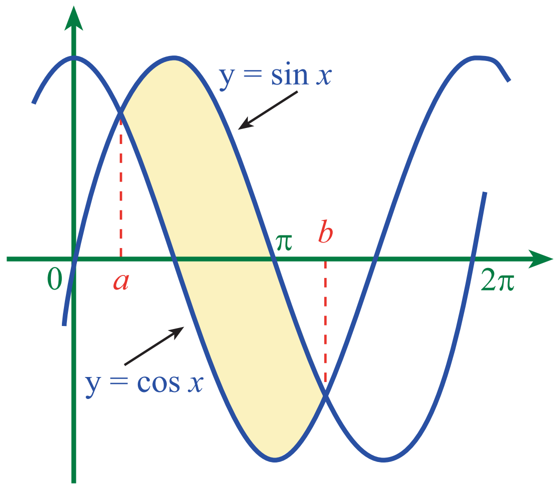
Calculate the shaded area between the curve and the x-axis.



Q35:

a) Find the coordinates of the points of intersection at a and b

b) Calculate the shaded area in the diagram.



Q36: A particle starts from the origin at time, t = 0 and moves in a straight line along the x-axis so that its speed at time t is given by the formula v (t) = 5 + 2 sin t

a) Calculate the formula for s (t), the distance of the particle from the origin, at time t. (Remember that v = $\frac{ds}{dt}$)

b) How far is the particle from the origin at t = $\frac{\pi}{3}$?

2.8 Integration of $(ax + b)^n$

Learning Objective

Integrate functions of the type (ax + b)

We have already seen that

$$\frac{d}{dx}\left((ax+b)^n\right) = an\,(ax+b)^n$$

and therefore similarily

$$\frac{d}{dx}\left((ax+b)^{n+1}\right) = a(n+1)\;(ax+b)^n$$

Dividing both sides by a (n + 1) then gives us

$$\frac{d}{dx}\left(\frac{(ax+b)^{n+1}}{a\,(n+1)}\right) = (ax+b)^n$$

Since integration is the reverse process to differentiation we can now write:

$$\boxed{\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a\;(n+1)} + C, \quad n \neq -1}$$

See the following examples.

Examples

1. Find $\int (4x + 3)^4 dx$

Solution

$$\int (4x+3)^4 dx = \frac{(4x+3)^5}{} \quad = \frac{1}{20}(4x+3)^5 + C$$

2. Find $\int (2x + 5)^{3/4} dx$

Solution

$$\begin{aligned}\int (2x+5)^{3/4} dx &= \frac{(2x+5)^{7/4}}{2 \times \frac{7}{4}} + C\\ &= \frac{4}{7} \times \frac{(2x+5)^{7/4}}{2} + C\\ &= \frac{2}{7}(2x+5)^{7/4} + C\end{aligned}$$

3. Evaluate $\int_1^2 \frac{dx}{(3x-1)^2}$

Solution

$$\int_1^2 \frac{dx}{(3x-1)^2} = \int_1^2 (3x-1)^{-2}\,dx$$
$$= \left[\frac{(3x-1)^{-1}}{3 \times (-1)}\right]_1^2$$
$$= -\frac{1}{3}\left[\frac{1}{3x-1}\right]_1^2$$
$$= -\frac{1}{3}\left[\left(\frac{1}{5}\right) - \left(\frac{1}{2}\right)\right]$$
$$= -\frac{1}{3} \times -\frac{3}{10}$$
$$= \frac{1}{10}$$

Now try the questions in the following exercise.

Exercise 7

There is an on-line exercise at this point which you might find helpful.

60 min

Q37: Find the following

a) $\int (2x+5)^7 dx$

b) $\int (3t-1)^4 dt$

c) $\int (4x-3)^{-2} dx$

d) $\int (2-3x)^{-6} dx$

e) $\int (4r+1)^{1/2} dr$

f) $\int (x+6)^{-4/5} dx$

Q38: Integrate the following

a) $\sqrt{x+9}$

b) $\dfrac{1}{(2x-1)^3}$

c) $\dfrac{1}{\sqrt{3x+4}}$

d) $\sqrt{(1-4x)^3}$

e) $\dfrac{1}{\sqrt[3]{5x-1}}$

f) $\dfrac{1}{\sqrt{(4x+7)^5}}$

Q39: Find the general solution of

a) $\dfrac{dy}{dx} = (1-2x)^4$

b) $\frac{ds}{dt} = (4t + 3)^{-2}$

Q40: Evaluate the following integrals

a) $\int_0^2 (3x - 1)^2 \, dx$

b) $\int_0^1 (1 - 3x)^3 \, dx$

c) $\int_0^4 \frac{dx}{\sqrt{1 + 2x}}$

d) $\int_0^1 \frac{dx}{(4x - 1)^3}$

e) $\int_0^9 \sqrt{9 - x} \, dx$

f) $\int_1^2 \frac{dx}{(2x - 3)^2}$

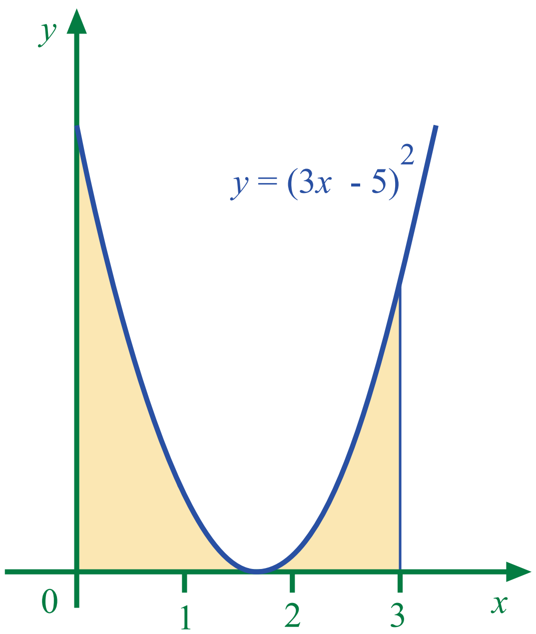
Q41: Find the particular solution of the following differential equations

a) $\frac{du}{dt} = \sqrt{5 - t}$ given that u = 0 when t = 5

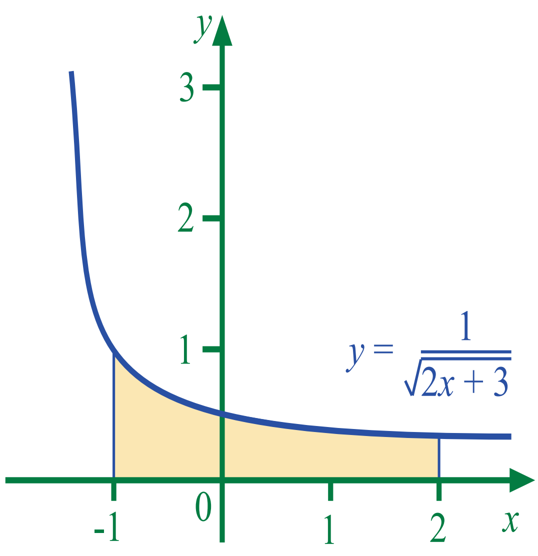
b) $\frac{dx}{dt} = \frac{1}{(3t + 1)^{2/3}}$ given that x = 5 when t = 0

Q42: Calculate the shaded area in the following diagrams.

a)



b)



2.9 Integration of sin(*ax* + *b*) and cos(*ax* + *b*)

Learning Objective

Integrate functions of the type sin (ax + b) and cos (ax + b)

We have already seen that

$$\frac{d}{dx}\left(\sin\left(ax + b\right)\right) = a\cos\left(ax + b\right)$$
$$\frac{d}{dx}\left(\cos\left(ax + b\right)\right) = -a\sin\left(ax + b\right)$$

Again, since integration is the reverse process to differentiation we can now write:

$$\int \cos\left(ax + b\right)dx = \frac{1}{a}\sin(ax + b) + C$$
$$\int \sin\left(ax + b\right)dx = -\frac{1}{a}\cos(ax + b) + C$$

Examples

1. Find $\int \cos\left(3x + 5\right)dx$

Solution

$$\int \cos\left(3x + 5\right)dx = \frac{1}{3}\sin\left(3x + 5\right) + C$$

2. Evaluate $\int_0^{\pi/4} \sin\left(2x + \frac{\pi}{2}\right)dx$

Solution

$$\begin{aligned}\int_0^{\pi/4} \sin\left(2x + \frac{\pi}{2}\right)dx &= \left[-\frac{1}{2}\cos\left(2x + \frac{\pi}{2}\right)\right]_0^{\pi/4}\\ &= -\frac{1}{2}\left[\cos\left(2x + \frac{\pi}{2}\right)\right]_0^{\pi/4}\\ &= -\frac{1}{2}\left[\cos\pi - \cos\frac{\pi}{2}\right]\\ &= -\frac{1}{2}\left(-1 - 0\right)\\ &= \frac{1}{2}\end{aligned}$$

Now try the questions in the following exercise.

Exercise 8

There is an on-line exercise at this point which you might find helpful.

60 min

Q43: Find the following

a) $\int \cos(2x - 1)dx$

b) $\int \sin(3x + 5)\, dx$

c) $\int 6 \cos(2x - 3)\, dx$

d) $\int \sin\left(4x + \frac{\pi}{3}\right) dx$

e) $\int \cos\left(\frac{1}{3}x\right) dx$

f) $\int 5 \sin\left(\frac{1}{2}x + \frac{\pi}{3}\right) dx$

Q44: Evaluate the following

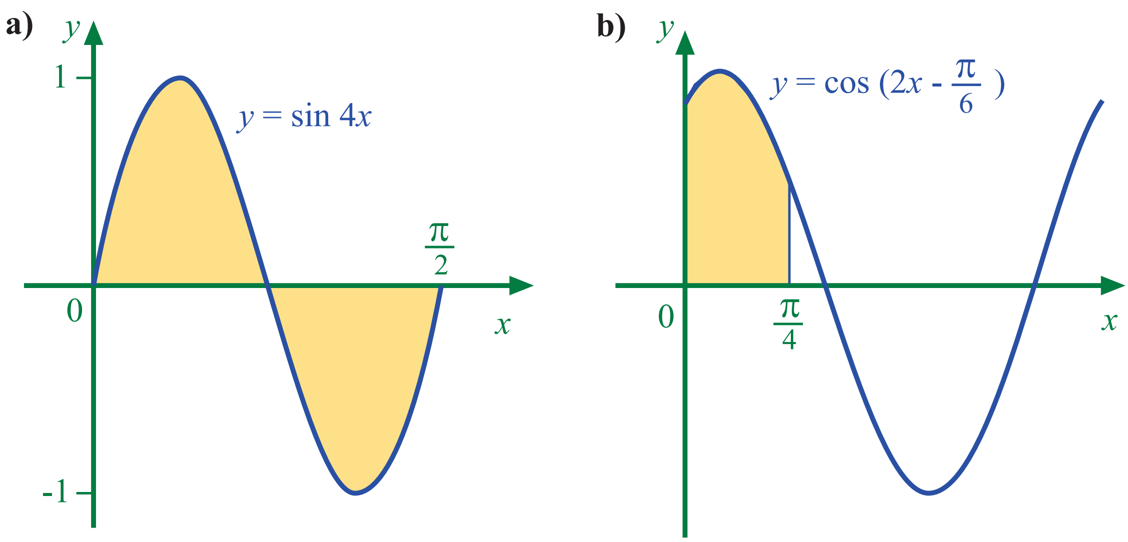
a) $\int_0^{\pi/4} \cos 2x dx$

b) $\int_0^{\pi/3} \sin 2x dx$

c) $\int_0^{\pi/2} 3 \cos\left(\frac{t}{2}\right) dt$

d) $\int_{\pi/6}^{\pi/4} \cos\left(3x + \frac{\pi}{4}\right) dx$

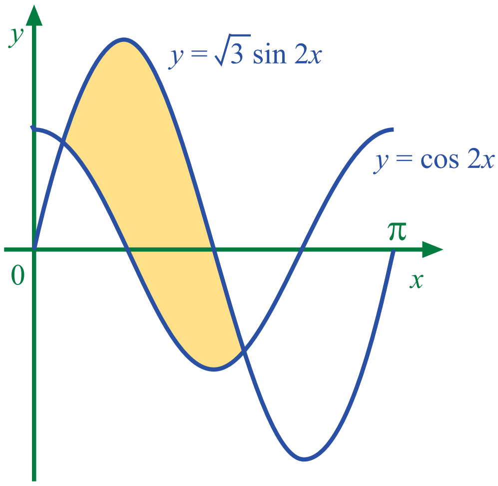
Q45: Calculate the area of each shaded region.



Q46:

a) Show that the curves $y = \sqrt{3} \sin 2x$ and $y = \cos 2x$ intersect at $x = \frac{\pi}{12}$ and $x = \frac{7\pi}{12}$

b) Hence calculate the shaded area in the following diagram.



Q47:

a) Use the double angle formula $\cos 2x = 2\cos^2 x - 1$ to show that $\cos^2 x = \frac{1}{2}\cos 2x + \frac{1}{2}$

b) Hence find $\int \cos^2 x dx$

Q48:

a) Use the double angle formula $\cos 2x = 1 - 2\sin^2 x$ to express $\sin^2 x$ in terms of $\cos 2x$

b) Hence find $\int \sin^2 x dx$

Q49: Use the double angle formula $\sin 2x = 2\sin x \cos x$ to find $\int \sin x \cos x dx$

Q50: Use double angle formulae to evaluate the following

a) $\int \left(\sin^2 x - \cos^2 x\right) dx$

b) $\int \cos^2 3x dx$

c) $\int 4\sin 5x \cos 5x dx$

d) $\int \left(x - 3\sin^2 2x\right) dx$

2.10 Summary

Learning Objective

Recall the main learning points from this topic

1. $$\frac{d}{dx}(\sin x) = \cos x$$
$$\frac{d}{dx}(\cos x) = -\sin x$$

2. $$\frac{d}{dx}(x + a)^n = n(x + a)^{n-1}, \quad (a \in \mathbb{R}, n \in \mathbb{Q})$$

3. $\frac{d}{dx}(ax + b)^n = an(ax + b)^{n-1}, \quad (a \in \mathbb{R}, n \in \mathbb{Q})$

The chain rule

Function notation

4. $h'(x) = g'(f(x)) \times f'(x)$

Leibniz notation

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

5. $\int \cos x dx = \sin x + C$

$\int \sin x dx = -\cos x + C$

6. $\int (ax + b)^n dx = \frac{(ax + b)^{n+1}}{a(n + 1)} + C, \quad (a \neq 0, n \neq -1)$

7. $\int \cos(ax + b) dx = \frac{1}{a}\sin(ax + b) + C, \quad (a \neq 0)$

$\int \sin(ax + b) dx = -\frac{1}{a}\cos(ax + b) + C, \quad (a \neq 0)$

2.11 Extended information

There are links on the web which give a variety of web sites related to this topic.

2.12 Review exercise

10 min

Review exercise in further diff and int

There is an on-line exercise at this point which you might find helpful.

Q51:

a) Differentiate - 3 cos x with respect to x

b) Given y = 4 sin x, find $\frac{dy}{dx}$

Q52: Find $f'(x)$ when $f(x) = (x - 5)^{-4}$

Q53:

a) Find $\int \frac{2}{3} \sin x \, dx$

b) Integrate -4 cos x with respect to x

c) Evaluate $\int_3^4 (x - 2)^3 dx$

2.13 Advanced review exercise

Advanced review exercise in further diff and int

There is an on-line exercise at this point which you might find helpful.

60 min

Q54:

a) If $f(x) = \cos^2 x + \dfrac{3}{5x^2}$, find $f'(x)$

b) Given that $f(x) = 2(5 - 3x)^4$, find the value of $f'(2)$

c) Differentiate $3x^{2/3} + 5 \sin 3x$ with respect to x

d) Find the derivative of $6\sqrt{x} + \sin^2 x$

e) Find $\dfrac{dy}{dx}$ given that $y = \sqrt{1 - \sin x}$

Q55:

a) Find $\int (4x^3 - \sqrt{x} - \sin x)$

b) Evaluate $\int_0^{\pi/2} \sin 2x \, dx$

c) Evaluate $\int_{-1}^{0} (3x + 1)^3 \, dx$

d) Find $\int \sqrt{1 + 2x \, dx}$ and hence find the ***exact*** value of $\int_0^4 \sqrt{1 + 2x \, dx}$

Q56:

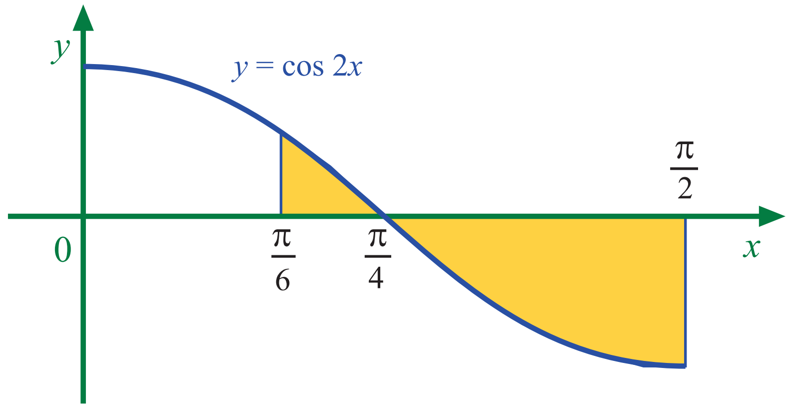
a) Show that $(\cos x + \sin x)^2 = 1 + \sin 2x$

b) Hence evaluate $\int_0^{\pi/4} (\cos x + \sin x)^2 \, dx$

Q57: Differentiate $\sin^5 x$ with respect to x.
Hence find $\int \sin^4 x \cos x \, dx$

Q58: By writing cos 3x as cos (2x + x) show that $\cos 3x = 4\cos^3 x - 3\cos x$
Hence find $\int \cos^3 x \, dx$

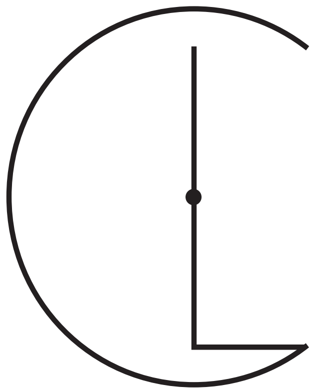
Q59: An artist has designed a "bow" shape which he finds can be modelled by the shaded area below. Calculate the area of this shape.

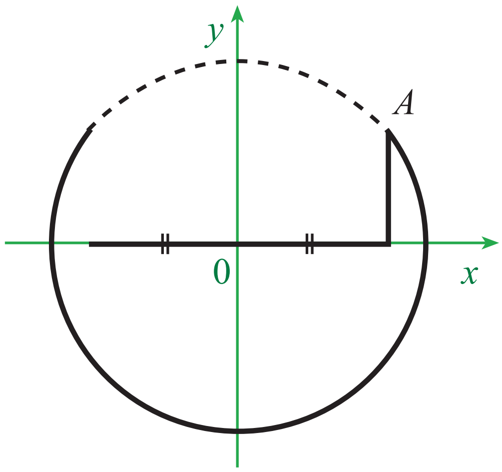


(Higher Mathematics)

Q60:

Linktown Church is considering designs for a logo for their parish magazine. The "C" is part of a circle and the centre of the circle is the mid-point of the vertical arm of the "L". Since the "L" is clearly smaller than the "C", the designer wishes to ensure that the total length of the arms of the "L" is as long as possible.





The designer decides to call the point where the "L" and "C" meet A and chooses to draw coordinate axes so that the A is in the first quadrant. With axes as shown, the equation of the circle is $x^2 + y^2 = 20$

a) If A has coordinates (x, y), show that the total length T of the arms of the "L" is given by

$$T = 2x + \sqrt{20 - x^2}$$

b) Show that for a stationary value of T, x satisfies the equation

$$x = 2\sqrt{20 - x^2}$$

c) By squaring both sides, solve this equation.
Hence find the greatest length of the arms of the "L".

(Higher Mathematics)

2.14 Set review exercise

Set review exercise in further diff and int

There is an on-line exercise at this point which you might find helpful.

10 min

Q61:

a) Differentiate 4 cos x with respect to x

b) Given y = 3 sin x, find $\frac{dy}{dx}$

Q62: Find f ' (x) when f (x) = $(x + 3)^{2/3}$

Q63:

a) Find $\int 5 \sin x \, dx$

b) Integrate $-\frac{3}{4}\cos x$ with respect to x

c) Evaluate $\int_1^3 (x - 1)^4 \, dx$

Topic 3

Logarithmic and exponential functions

Contents

Learning Objectives

- *Use properties of logarithmic and exponential functions*

Minimum performance criteria:

- *Simplify a numerical expression using the laws of logarithms*
- *Solve simple logarithmic and exponential equations*

Prerequisites

A sound knowledge of the following subjects is required for this unit:

- Indices
- 2D coordinates and plotting graphs
- Logarithmic and exponential graphs

3.1 Revision exercise

This topic concentrates on using relationships and solving equations involving logarithmic and exponential functions. It is advisable that the work on features of exponential and logarithmic graphs is well known before starting this topic.

Revision exercise

30 min

There is a web exercise if you prefer it.

Q1: Simplify the following:

a) $a^5 \times a^2$

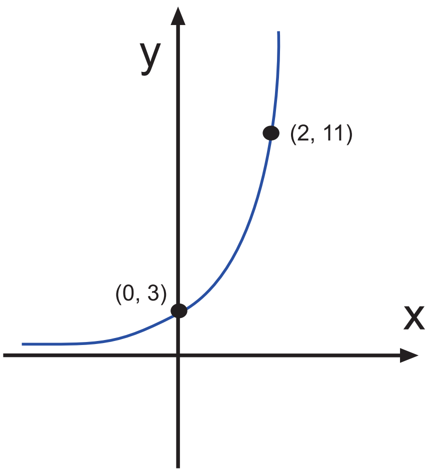
b) $b^9 \div b^3$

c) $(a^2)^3$

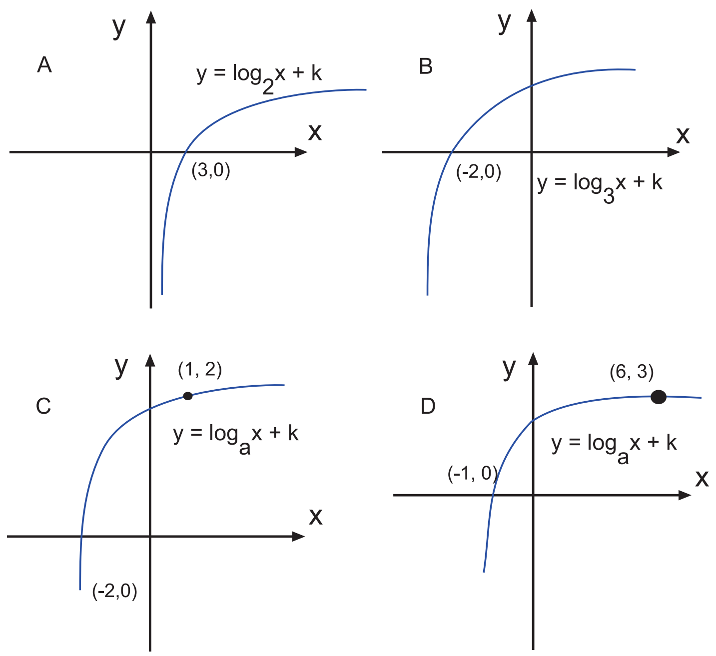
d) $\dfrac{a^5b^4c}{abc} \times \dfrac{b^2c}{a^2} \div \dfrac{ab}{c^2}$

Q2: State the points at which the function $y = x^2 - 3x + 2$ crosses the axes.

Q3: Find the equation of the function which has the form $f(x) = a^x + k$ as represented on the graph:



Q4: Identify the functions in the following sketches in the form shown on each by referring to the related exponential graphs:

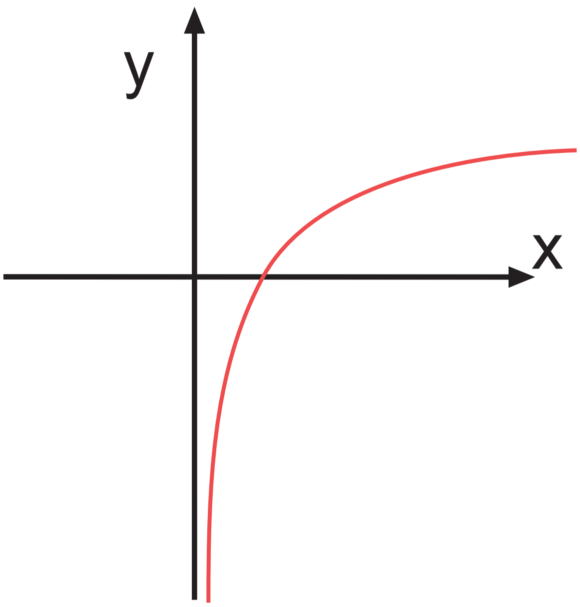


Q5: Sketch the graph of the function $f(x) = 2^x - 3$

3.2 Logarithmic and exponential functions.

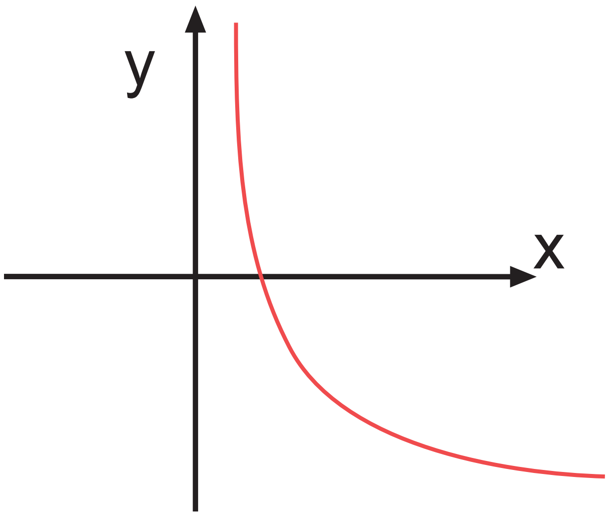
It is worth recalling some of the facts about logarithmic and exponential graphs at this stage.

The graphs of logarithmic functions take one of two forms: The following form is by far the better known.



This occurs when the base of the log is greater than one. For example, $\log_{10} x$

The second form is related to the exponential function of the form $y = a^x$ where $a < 1$. In other words, this log graph is the inverse of the exponential graph where a $<$ 1 and will have a base number less than one. These are not commonly used.



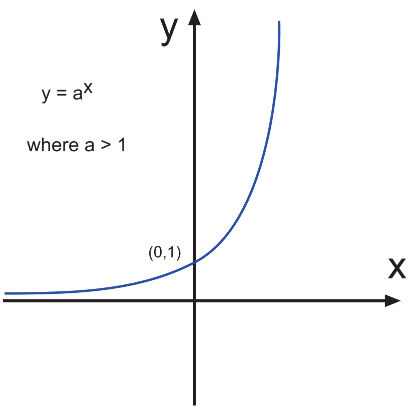
10 min

Logarithmic examples

There is a selection of log graphs on the web to view. These illustrate the similarity of log graphs but show the difference in the y scale which will identify each one.

For exponential graphs, there are also two forms.

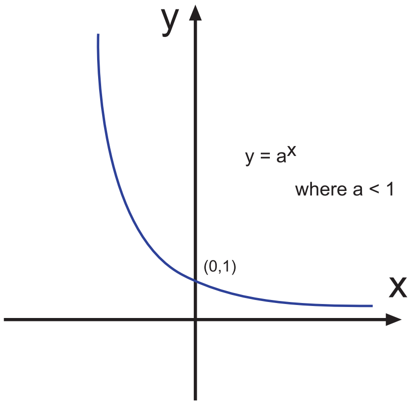
If a $>$ 1 the graph is increasing with the shape



This type of exponential function is called a *growth function.*

Examples of exponential growth could include: a sum of money on deposit earning compound interest and simple models of population increase.

If a $<$ 1 the graph is decreasing with the shape



Note that $y = a^x = (^1/_a)^{-x}$.

For example: $0.25^x = 4^{-x}$

This type of exponential function is called a *decay function.*

Examples of exponential decay could include: radioactive material life, leakage of batteries/ tanks/containers, evaporation of liquids and machinery value over time.

Exponential examples

10 min

There is a selection of exponential graphs on the web to view.

Also recall the statement: the easiest way to plot a log function is to relate it to the exponential function and reflect the graph in the line y = x. This reflection be seen clearly from the diagrams.

The graphs of $y = a^x$ for $a > 0$ will pass through the point (0, 1).

In a similar way, the graphs of $y = \log_a x$ will pass through the point (1, 0).

Before continuing, here is the definition of an exponential function.

Exponential function
A function of the form $f(x) = a^x$ where $a > 0$ and $a \neq 1$ is called an exponential function

The relationship between logarithmic and exponential functions can be stated as:

If $f(x) = a^x$ then the inverse function $f^{-1}(x) = \log_a x$

Simply, this says that $y = \log_a x$ is the inverse function of $y = a^x$.

This gives an important mathematical relationship which can be useful for solving equations in log and exp functions.

Log and exp relationship

$$y = a^x \Leftrightarrow \log_a y = x$$

where $a > 1$ and $y > 0$

Note that at all times the argument of a log function must be a positive value. That is, if $y = \log_a x$ then x is positive.

When dealing with logarithmic and exponential functions, certain symbolism is used.

From the relationship just established, it is clear that the log function has a *base number* (for example a). This number can be any positive number but a quick look at a calculator will show that there are two common bases.
These are base 10 (common log), denoted by 'log' and base e (natural log), denoted by 'ln'. Thus $\log_e x = \ln x$.

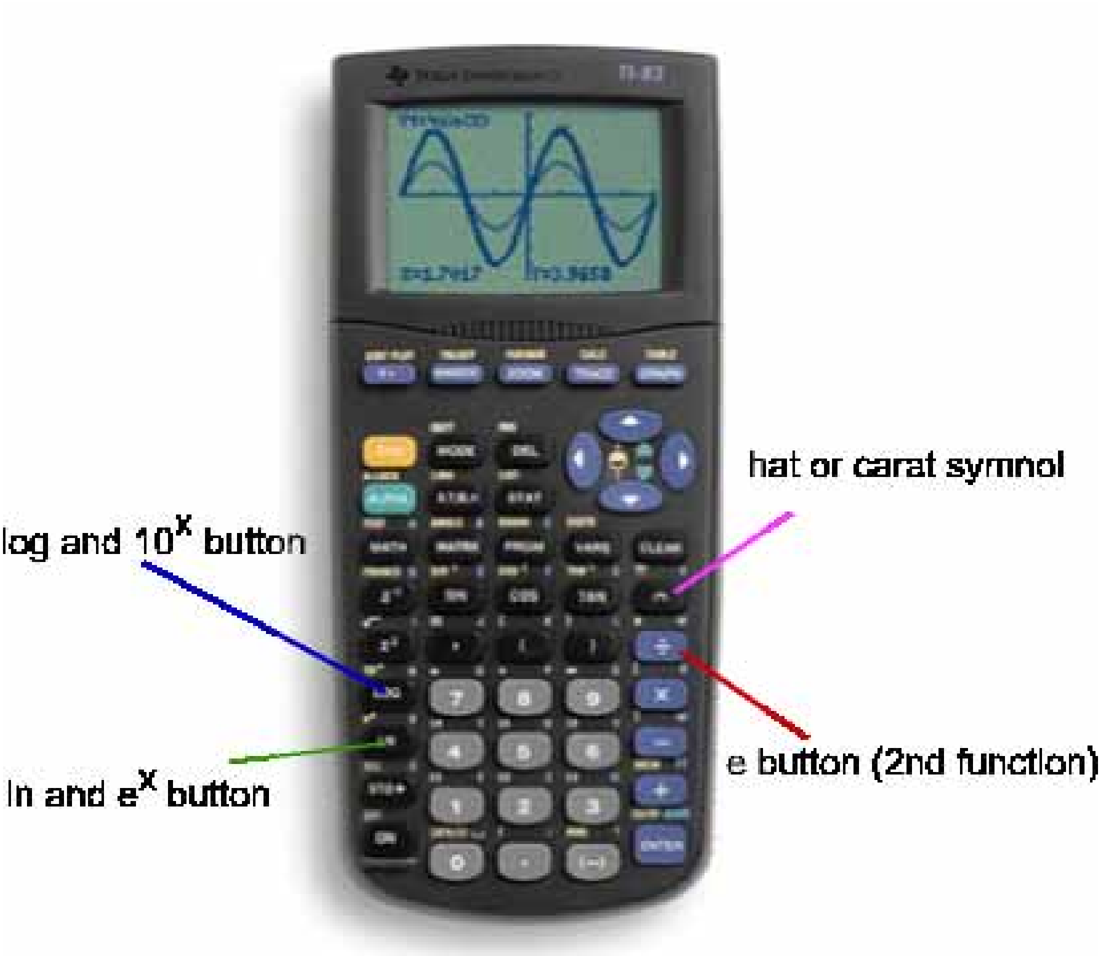
Although either is correct to use, this topic will mainly use $\log_e x$ for the function 'log to the base e of x'.

The number 'e' is a special number and is given on any calculator. It is rather like the number π as it too is a never ending decimal.

On a calculator use e^x for powers of e and the button 'e' for the value of e^1.

The 'exp' button when available is confusing but actually means ' $\times$ 10 to a power' and is used for powers of 10. In more modern calculators this has been replaced by the symbol 10^x. These two buttons 10^x and e^x save time and effort, but if there is any doubt, use the power key which may be denoted by the hat or carat symbol or by y^x or x^y.

Here is a picture of a TI83 calculator with the relevant buttons marked.



Calculator button exercise

10 min

Q6: Use a calculator to find, correct where appropriate to 2 d.p. the following log functions:

a) $\log_{10} 1$
b) $\log_e 1$
c) $\log_{10} 10$
d) $\log_e e$
e) $\log_{10} 100$
f) $\log_{10} 0.001$
g) $\log_e 0.001$
h) $\ln 3$
i) $\ln (e^2)$

Q7: Use a calculator to find, correct to 2 d.p. where appropriate, the following exponential functions:

a) e
b) e^0
c) e^1
d) $10^{1.5}$
e) $4^{-0.4}$
f) $e^{-0.01}$
g) $4e^3$
h) e^5
i) 2^e

The derivation of the number e is interesting and the following investigation gives a flavour of the work involved to find an accurate value for e.

Investigation to find the value of e

20 min

If time permits, try this investigation to reveal the value of e

Take the sequence $\left\{\left(1 + \frac{1}{n}\right)^n\right\}$ and investigate the values of the terms in this sequence with a graphics calculator. On a TI83 calculator the following sequence can be used. Press 'y =' and enter the expression. Use 'tblset' to set the second entry of the intervals to 10. Press '2nd' 'graph' to access the table of values. Other calculators will work in a similar way.

Check some of the values. Change 'tblset' to increments of 100 and note the value of y for x = 3000 upwards.

Clear the calculator and check the value of e directly from the keypad. Note the similarity.

Further work on this limit is too advanced for this course but this investigation gives some insight into the derivation of e

It will be the calculator value of e which will be used in any calculations to come.

Examples

1. If $\log_5 x = 2$, find x

Answer:

The relationship given earlier, viz, $\log_a x = y \Leftrightarrow a^y = x$ gives

$5^2 = x \Rightarrow x = 25$

2. Write $3^4 = x$ in logarithmic form.

Answer:

$a = 3$, $y = 4$ thus $\log_a x = y$ becomes $\log_3 x = 4$

3. State the value of y in logarithmic form when $5^y = 2.3$

Answer:

$a = 5$ and $x = 2.3$ gives

$\log_5 2.3 = y$

Relationship exercise

20 min

There is an alternative exercise on the web if you prefer it.

Q8: Write the following in log form.

a) $a^y = x$
b) $a^x = 3$
c) $2^x = 9$
d) $b^5 = 3$
e) $6^2 = p$
f) $2^{-3} = y$

Q9: Write the following in exponential form.

a) $\log_a x = y$
b) $\log_a x = 9$
c) $\log_4 x = 2$
d) $\log_x 2 = k$
e) $\log_x y = 2$
f) $\log_2 s = 4$
g) $\log_3 b = 5$

Q10: Evaluate the following.

a) $10^y = 5$

b) $e^2 = x$

c) $\log_e 3 = k$

d) $\log_{10} x = 0.5$

e) $\log_a 8 = 3$

f) $\log_b 81 = 4$

g) $\log_3 x = 2$

h) $\log_2 y = 5$

3.3 Laws of logarithms

Learning Objective

Apply the log laws to solving equations

With indices there are certain rules called the laws of indices which make calculations easier.

For example: $a^2 \times a^5 = a^7$ uses the law of adding the indices.

Similarly there are several relationships involving logarithms which are useful in solving equations. These relationships are collectively known as *The Laws of Logarithms* or 'the log laws'.

The proofs and derivations of these laws are given near the back of this topic in the section headed Proofs.

The laws have been given numbers purely for ease of referencing.

Log Laws

Law 1:

$\log_a 1 = 0$

Law 2:

$\log_a a = 1$

Law 3:

$\log_a (bc) = \log_a b + \log_a c$

Law 4:

$\log_a (^b/_c) = \log_a b - \log_a c$

Law 5:

$\log_a b^n = n \log_a b$

These laws are required for manipulation of the expressions such as those which follow. Although it is important to learn these laws, familiarity with them will increase with practice.

Examples

1. Find x when $\log_x 36 = 2$

Answer:

$\log_x 36 = 2 \Rightarrow x^2 = 36 \Rightarrow$

$x = 6$ (note that x is positive)

(This example uses the previously stated relationship $y = a^x \Leftrightarrow \log_a y = x$ which is very important and is frequently used in equation solving).

2. Simplify $\log_4 28 - 2\log_4 2$

Answer:

$\log_4 28 - 2\log_4 2$

$= \log_4 (7 \times 4) - \log_4 2^2$

$= \log_4 7 + \log_4 4 - \log_4 4$ (law 3)

$= \log_4 7$

3. Evaluate $\log_5 40 - \log_5 8$

Answer:

$\log_5 40 - \log_5 8$

$= \log_5 ({}^{40}/_{8})$ (law 4)

$= \log_5 5$

$= 1$ (law 2)

4. Evaluate $(\log_4 45 - \log_4 5) - 2\log_4 3$

Answer:

$(\log_4 45 - \log_4 5) - 2\log_4 3$

$= \log_4 ({}^{45}/_{5}) - 2\log_4 3$ (law 4)

$= \log_4 9 - 2\log_4 3$

$= \log_4 9 - \log_4 9$ (law 5)

$= 0$ (by arithmetic or by law 4 and law 1!)

5. Evaluate $\log_3 81$

Answer:

$\log_3 81$

$= \log_3 3^4 = 4\log_3 3$ (law 5)

$= 4$ (law 2)

6. Evaluate $2\log_4 8$

Answer:

$2\log_4 8$

$= \log_4 8^2 = \log_4 64$ (law 5)

$= \log_4 4^3 = 3\log_4 4$ (law 5)

$= 3$ (law 2)

7. Simplify ${}^{1}/_{3}\log_2 27 - {}^{1}/_{2}\log_2 9$

Answer:

${}^{1}/_{3}\log_2 27 - {}^{1}/_{2}\log_2 9$

$= \log_2 3 - \log_2 3$ (law 5 twice)

$= 0$

8. Simplify $6 \log_5 25 + {}^1/_4 \log_5 5$

Answer:

$6 \log_5 25 + {}^1/_4 \log_5 5$

$= 12 \log_5 5 + {}^1/_4 \log_5 5$ (law 5)

$= 12 + {}^1/_4$ (law 2)

$= 12{}^1/_4$

The examples demonstrate the importance of checking the base of the logarithm before carrying out any manipulation. The base can normally provide some clue of the simplification required.

In many cases, however, it is possible to reach the answer using different approaches and combinations of the laws. Do bear this in mind when checking the workings in any given answers.

A word of warning:

Pay particular attention to the format of this law: $\log_a ({}^b/_c) = \log_a b - \log_a c$

Note that the left hand side, $\log_a ({}^b/_c)$ should *not* be confused with $\frac{\log_a b}{\log_a c}$ which, on this course, *can only be evaluated as it stands.*

Log laws exercise

20 min

There is an alternative exercise on the web if you prefer it.

Q11: Evaluate the following:

a) $3 \log_2 6 - 3 \log_2 3 + 3 \log_2 2$
b) $e \log_e 1$
c) $\log_5 625 - \log_4 64$
d) $2 \log_x x^2 + \log_x 1$
e) $\log_e 32 - 5 \log_e 2$

Q12: Simplify the following:

a) $2 \log_x 5 + \log_x 4 - \log_x 50$
b) $\log_3 12 - \log_3 16$
c) $2 \log_{10} x + \log_{10} (x - 2)$
d) $\log_a 4 + \log_b 25 - \log_b 5 - \log_a 2$
e) $\log_{10} 30 - \log_2 20$

Q13: Find the value of x in the following:

a) $2 \log_2 6 - \log_2 9 = x$
b) $-2 + \log_2 x = 0$
c) $\log_e x^2 - \log_e 36 = \log_e 1$
d) $3 \log_3 x - \log_x 27 = 0$

3.4 Solving log and exp equations

Learning Objective

Solve expressions with log and exp functions

The last section touched on aspects of equation solving using the log and exp relationships established. However, it is possible to take this further and use the laws of indices and logs to solve more complex equations.

Examples

1. Solve $\log_{30} (x - 2) + \log_{30} (x - 1) = 1$

Answer:

$\log_{30} (x - 2) + \log_{30} (x - 1) = 1$

$\log_{30} (x - 2)(x - 1) = 1 \Rightarrow$

$(x - 2)(x - 1) = 30^1 = 30 \Rightarrow$

$x^2 - 3x + 2 - 30 = 0$

$(x - 7)(x + 4) = 0$

$x = 7$ or $x = -4$

-4 is a solution to the quadratic but is not a solution to the original log equation

since for $x = -4$, $\log_{30} (x - 1)$ is undefined and that solution must be rejected.

$x = 7$ is the only solution

2. Solve $3^x = 4$

Answer:

It may seem obvious to start with the relationship $a^y = x \Leftrightarrow \log_a x = y$

This would give x in log terms but there is no easy way of evaluating log to bases other than e and 10

The solution lies in taking the log of each side (to base 10 or e)

$\log_{10} 3^x = \log_{10} 4$

$x \log_{10} 3 = \log_{10} 4$

$x = \log_{10} 4 \div \log_{10} 3 = 1.2619$

3. Solve $\log_{10} (x + 2) + \log_{10} (x - 1) = 1$

Answer:

$\log_{10} (x + 2) + \log_{10} (x - 1) = 1$

$\log_{10} (x + 2)(x - 1) = 1 \Rightarrow$

$(x + 2)(x - 1) = 10^1 = 10 \Rightarrow$

$x^2 + x - 2 -10 = 0$

$(x - 3)(x + 4) = 0$

$x = 3$ or $x = -4$

-4 is a solution to the quadratic but is not a solution to the original log equation since for x = -4, $\log_{10}(x - 1)$ is undefined and that solution must be rejected.

x = 3 is the only solution

4. Solve $\log_6 (x^2 - 9) - \log_6 (x - 3) - 2 = 0$

Answer:

$\log_6 (x^2 - 9) - \log_6 (x - 3) - 2 = 0$

$\log_6 (x^2 - 9) - \log_6 (x - 3) = 2 \log_6 6 \Rightarrow$

$\log_6 (x + 3) = \log_6 36$

$x + 3 = 36 \Rightarrow$

$x = 33$

Equation solving exercise

20 min

There is an alternative exercise on the web if you prefer it.

Q14: Solve $\log_{15} (x - 3) + \log_{15} (x - 1) = 1$

Q15: Solve $\log_3 (x^2 - 4) - \log_3 (x - 2) - 2 = 0$

Q16: Solve $2^x = 7$ correct to 3 d.p.

Q17: Solve $3^x = e$ correct to 3 d.p.

Q18: Solve $\log_x 125 = 3$

Q19: Solve $\log_5 (x - 3)^2 = 2$

Q20: Solve $1 + \log_3 2 = x$ correct to 2 d.p.

Q21: Solve $\log_2 (x^2 - 4) - \log_2 (x - 2) = 5$

The previous examples and exercises dealt with solutions of an equation in an algebraic unknown (x). However, there are many real life situations which can be modelled and solved using log and exp functions. The easiest way to explain the types of problems which can be solved is through example. Several of these follow. The techniques can be used in any situation which is similar to those shown.

There are three main types of problem: exponential decay, exponential growth and experimental data graphs suggesting an exponential relationship.

The experimental data problem is worthy of a closer look before embarking on the examples.

Scientists undertaking experiments are normally interested in finding a relationship between the variables which they are investigating. In some cases, where two variables are involved, the data may suggest an exponential formula.

Not only can the graphs of the variables be plotted against each other, but in particular instances, if an exponential relationship is suspected, then it is easier to plot the logs of the variables against each other. In this way, if the resulting graph is a straight line, then the variables are connected by an exponential formula. The reasoning behind this follows.

Suppose that results suggest that these two variables x and y are connected according to the formula $y = ax^n$

$y = ax^n \Rightarrow$

$\log y = \log ax^n \Rightarrow$

$\log y = \log a + \log x^n \Rightarrow$

$\log y = n \log x + \log a$

This can be written as $Y = mX + c$ where $Y = \log y$, $X = \log x$, $m = n$ and $c = \log a$

$Y = mX + c$ is indeed the formula for a straight line.

By solving for m and c in the normal way to give the equation of a straight line, the resulting information is adequate to find the constant a and hence the exponential equation required.

It is useful to remember the relationship.

$y = ax^n \Rightarrow \log y = n \log x + \log a$
(which is in the form $Y = mX + c$)

Examples

1. Exponential decay problem

The mass M_1 of a radioactive substance at time t years is given by the equation $M_1 = M_0\ e^{-kt}$ where M_0 is the initial mass and k is a constant.

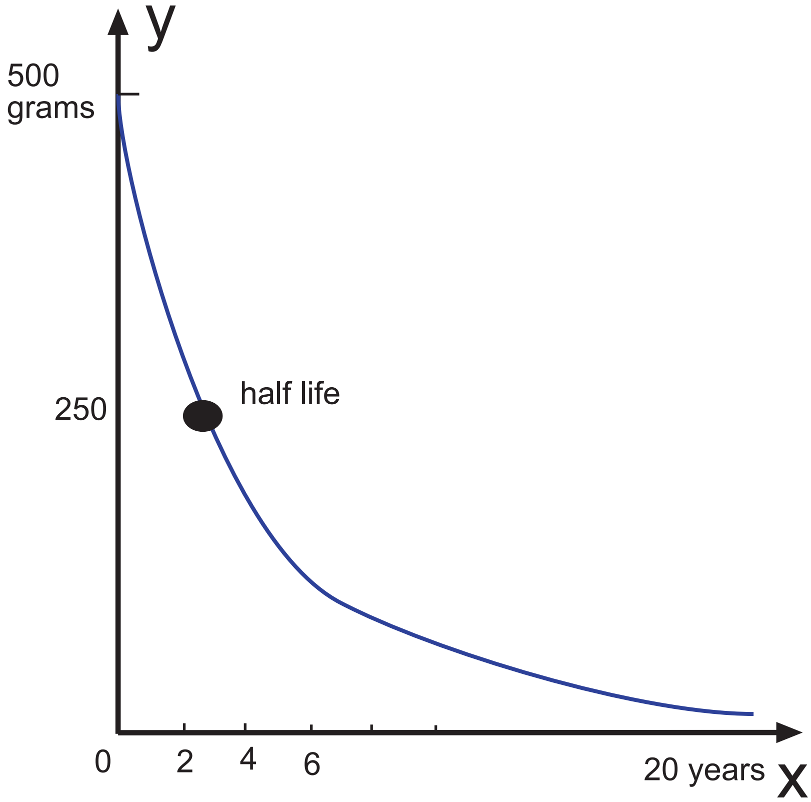
a) 90 grams of this substance decays to leave 30 grams over 4 years. Find k to 3 d.p.
b) If another sample is found weighing 500 grams, how much, to the nearest gram, will remain in 20 years?
c) What is the half life of this substance?
d) Illustrate this information on a graph.

Answer:

a) $M_1 = M_0\ e^{-kt}$
$^{30}/_{90} = e^{-4k}$
$\log_e (^{30}/_{90}) = -4k$
$k = 0.275$
b) $M_1 = 500\ e^{-20(0.275)} = 2$ grams.
c) $M_1 = 0.5\ M_0 \Rightarrow$
$0.5 = e^{-0.275t} \Rightarrow$
$\log_e 0.5 = -0.275t \Rightarrow$
$t = 2.52$ years

d)

The graph should contain at least the half life point and the axes must go as far as the values given in the question. More information may be included if the graph is clear.



2. Exponential growth

A sample of bacteria multiplies at a rate according to the formula $B = 25\ e^{kt}$ where k is a constant, B is the number of bacteria in thousands present after t minutes.

a) How many bacteria are assumed to be taken initially for the sample?

b) If after 18 minutes there are 35000 bacteria, what is the value of k (to 5 d.p.)?

c) How long to the nearest second will it take for the bacteria to double?

d) How many whole bacteria are present after one hour.

Answer:

a) 25000 since $e^{kt} = 1$ when $t = 0$

b) $^{35}/_{25} = e^{18k} \Rightarrow$

$k = \log_e (^{35}/_{25}) \div 18 \Rightarrow k = 0.01869$

c) $2 = e^{0.01869t} \Rightarrow$

$t = \log_e 2 \div 0.01869 = 37.09$ minutes

The bacteria double after 37 minutes and 5 seconds.

d) $B = 25\ e^{60(0.01869)} = 76.7287$

There will be 76728 whole bacteria present.

3. Experimental data

In an experiment the following data were recorded:

x	1.2	2.0	2.9	3.8
y	3.2	4.0	4.6	5.0

It is thought that x and y follow the relationship $y = ax^n$, show that this is correct and find the formula.

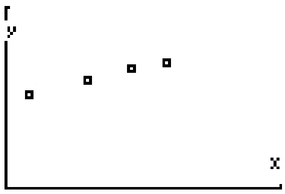
Answer:

Use a table of values for the log of each variable and plot the graph to verify that it is close to a straight line.

In this case $\log_{10}$ has been used.

x	0.08	0.3	0.46	0.58
y	0.51	0.6	0.66	0.70

The graph can be done by calculator unless a question specifically asks for the graphs. Here is the graph from the calculator.



Because the data is experimental, the relationship may not be exact (as in this case) but should be close enough to confirm a straight line fit.

Choose two points on the line and form two equations of the form Y = mX + c. (It is best to choose two points which will make the maths simple.)

Choose, for example (0.3, 0.6) and (0.58, 0.70)

0.6 = 0.3m + c: call equation1

0.7 = 0.58m + c: call this equation 2

Subtract 1 from 2 to give

$0.1 = 0.28m \Rightarrow m = 0.36$ and so $n = 0.36$

Substituting in equation 1 gives

$0.6 = 0.3 \times 0.36 + c \Rightarrow c = 0.49$

Since $c = \log_{10} a$ then $0.49 = \log_{10} a \Rightarrow$

$a = 10^{0.49} = 3$

The equation is $y = 3x^{0.36}$

Log and exp problems

There is a web exercise if you prefer it.

30 min

Q22: In an experiment, a gas cloud volume expands at a rate according to the formula $C = C_0\, e^{kt}$ where k is a constant, t is the time in seconds and C_0 is the initial volume of the cloud in cubic centimetres. In one experiment, the initial volume of the gas cloud is 50 ccs. This measures 500 ccs after 40 seconds.

a) What is the value of k?

b) How long to the nearest second, does it take a gas cloud to double in size?

c) The maximum size of a cloud in the experiment is 1000000 ccs. Assuming that the smallest initial volume is 75 ccs, what is the maximum time that the experiment can run ?

Q23: A radioactive substance decays at a rate given by the formula $M_1 = M_0\ e^{-kt}$ where M_0 is the initial mass and M_1 is the mass after t years. Find the half life of the element when k = 0.019 and how long it takes for the mass to reduce to 5g when M_0 = 35 grams.

Q24: Bacteria increase according to the formula $N = 150\ e^{1.02t}$ where N is the number of bacteria and t is the time in hours.

a) How many bacteria were present at the start of the experiment?

b) How long does it take for the bacteria to double?

c) A sample of 3000 bacteria is placed in a dish. After 4 hours, the thermostat control breaks down and the bacteria begin to die. If the numbers decrease according to the formula $D = D_0\ e^{-0.03t}$, how much longer, to the nearest minute, will it take for the number of bacteria return to the original number of 3000?

Q25: A chemical evaporates under certain conditions according to the formula $E_1 = E_0\ e^{-kt}$ where E_0 is the initial volume and E_1 is the volume after t hours.

a) If 1 litre of the chemical has evaporated after 20 hours to leave 1 ml, find k correct to 4 d.p.

b) What volume of chemical to the nearest ml was present before evaporation when, after half an hour, there is 45 ml left?

Q26: In an experiment the following data were recorded:

x	1.1	1.5	1.9	2.4
y	3.0	5.2	7.9	12.1

It is thought that x and y follow the relationship $y = ax^n$, show that this is correct and find the formula.

Q27: A container is pressurised to 80 psi. After seven hours the pressure has dropped to 30 psi.

a) The pressure P is given by the formula $P = P_0\ e^{-kt}$ where t is in hours and P_0 is the initial pressure. Find the value of k.

b) For safety, the gas in the container must be kept at a pressure greater than 95 psi. If, at the start of an experiment, the container is pressurised to 150 psi, what is the maximum time to the nearest minute that the experiment can last before the gas is no longer safe?

Q28: In an experiment the logs of the two variables were plotted and showed a straight line relationship. This led the scientists to expect that the two variables T and Q were connected according to a formula of the form $T= k\ Q^s$

The table of log values is shown. Find the values of k and s which satisfy the relationship $T = k\ Q^s$

$\log_{10}T$	0.04	0.08	0.11	0.15	0.18	0.20	0.23
$\log_{10} Q$	0.31	0.33	0.34	0.35	0.36	0.37	0.38

3.5 Summary

The following points and techniques should be familiar after studying this topic:

- The relationship between log functions and exponential functions.
- The laws of logarithms.
- Simplifying expressions using the log laws.
- Solving logarithmic and exponential equations.

3.6 Proofs

Law 1: $\log_a 1 = 0$

Using indices, $1^0 = 1$, $2^0 = 1$ and so on.

Thus $a^0 = 1$ for any integer a

The relationship $a^y = x \Leftrightarrow \log_a x = y$ then gives

$a^0 = 1 \Leftrightarrow \log_a 1 = 0$ as required.

Law 2: $\log_a a = 1$

Using indices, $1^1 = 1$, $2^1 = 2$ and so on

Thus $a^1 = a$ for any integer a

The relationship $a^y = x \Leftrightarrow \log_a x = y$ then gives

$\log_a a = 1$ as required.

Law 3: $\log_a (bc) = \log_a b + \log_a c$

Let $\log_a b = p$ and $\log_a c = q$

The relationship $a^y = x \Leftrightarrow \log_a x = y$ then gives

$b = a^p$ and $c = a^q$

So $bc = a^p \times a^q$ which by the laws of indices gives

$bc = a^{p+q}$

The relationship $a^y = x \Leftrightarrow \log_a x = y \Rightarrow$

$\log_a (bc) = p + q$

That is, $\log_a (bc) = \log_a b + \log_a c$ as required.

Law 4: $\log_a ({}^b/_c) = \log_a b - \log_a c$

Let $\log_a b = p$ and $\log_a c = q$

The relationship $a^y = x \Leftrightarrow \log_a x = y$ then gives

$b = a^p$ and $c = a^q$

So $^b/_c = a^p \div a^q$ which by the laws of indices gives

$bc = a^{p - q}$

The relationship $a^y = x \Leftrightarrow \log_a x = y \Rightarrow$

$\log_a (^b/_c) = p - q$

That is, $\log_a (^b/_c) = \log_a b - \log_a c$ as required.

Law 5: $\log_a b^n = n \log_a b$

Let $\log_a b = p$

The relationship $a^y = x \Leftrightarrow \log_a x = y$ then gives

$b = a^p$

So $b^n = (a^p)^n$ which by the laws of indices gives

$b^n = a^{np}$

The relationship $a^y = x \Leftrightarrow \log_a x = y \Rightarrow$

$\log_a b^n = np$

That is, $\log_a b^n = n \log_a b$ as required.

3.7 Extended Information

The web sites for this topic are well worth a visit as they comprehensively cover the same ground and provide good backup material. There are also sites with a different style of assessment which may prove useful for additional revision work.

Napier

John Napier in the early 1600's chose the name logarithm. In his work he wanted to avoid the use of fractions when performing the calculations of trig. functions. It is worth pointing out that Naperian logs are not natural logs (that is they are not logs to the base e).

Briggs

Briggs and Napier agreed that a decimal based logarithm would be an advantage and this was achieved in 1617. These logarithms were known as Brigg's logarithms or common logarithms and are used today with the symbol 'log'.(see calculator buttons)

Kepler

He was a German mathematician who also produced extensive log tables. Since his work focused on planetary motion, this was of great use in these calculations.

Burgi

This Swiss watchmaker undertook similar studies on logs. He worked with Kepler and although his findings mirrored those of Napier, it was Napier who achieved the recognition due to earlier publication of his work than Burgi.

3.8 Review exercise

Review exercise

25 min

There is another exercise on the web if you prefer it.

Q29: Simplify the following:

a) $\log_2 16 - \log_2 4$
b) $\log_e e^2$
c) $\log_7 5 - \log_7 1$

Q30: Find x when $\log_x 36 = 2$

Q31: Solve $\log_5 (2x + 1) + \log_5 (x + 2) = 1$

Q32: Evaluate $\log_e x = 1$ and give a value for x correct to two decimal places.

Q33: If $3 = 10^t$, find a value for t correct to two decimal places.

Q34: Bacteria multiply at a rate given by the formula $N = e^{1.4t}$ where t is the time in hours and N is the number of bacteria in thousands.

a) After 5 hours, how many whole bacteria are there?
b) How long to the nearest minute does it take until there are 500,000,000 bacteria?

Q35: Simplify $2 \log_3 5 + \log_3 4 - (\log_3 2 + \log_3 10)$

3.9 Advanced review exercise

Advanced review exercise

25 min

There is another exercise on the web if you prefer it.

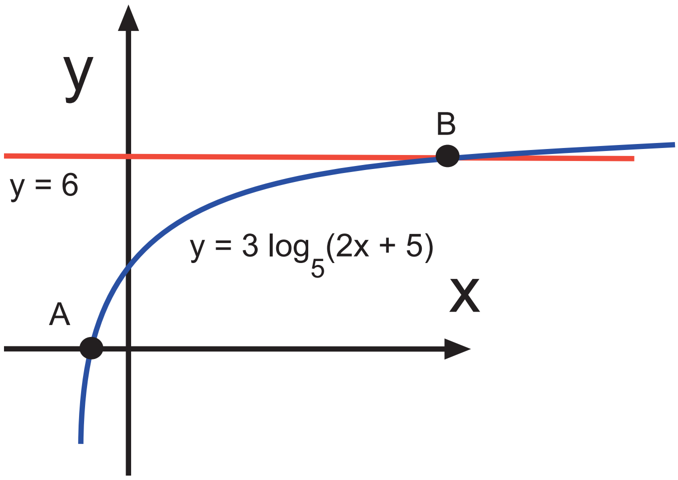
Q36: Bacteria increase according to the formula $N = 3\ e^{1.28t}$ where N is the number of bacteria in thousands and t is the time in hours.

a) How many bacteria were present at the start of the experiment?
b) How many whole bacteria are present after 80 minutes?
c) If after 2 hours the bacteria are transferred to another dish and the temperature is increased and the bacteria multiply faster according to the formula $N = e^{2.4t}$, how many bacteria will there be after 4 hours?

Q37: In an experiment on growth, a straight line graph was plotted of $\log_{10}$ w against $\log_{10}$ t where t is the time in years and w is the weight in kilograms. Two points matching the graph closely were (1.5, 3.7) and (2, 5.7). The relationship between w and t takes the form $w = a\ t^b$. Find a and b to satisfy this relationship.

Q38: A radioactive element decays at a rate given by the formula $M_1 = M_0\ e^{-kt}$ where M_0 is the initial mass and M_1 is the mass after t years. Find the half life of the element when k = 0.003

Q39: Part of the graph of $y = 3 \log_5 (2x + 5)$ is shown in the diagram. This graph crosses the x-axis at point A and the straight line y = 6 at the point B. Find the coordinates of point A and the x coordinate of point B.



Q40: A faulty high pressure container is filled to a pressure of 150 units. The pressure in the faulty container is given by the formula $P_1 = P_0\ e^{-kt}$ where t is the time in hours after being pressurised and P_0 is the initial pressure.

a) After 8 hours the pressure drops to 100 units. Find a value for k correct to 4 d.p.
b) The container contents, to avoid damage, must be kept at a minimum pressure of 75 units. If the container is initially pressurised to 200 units, how long, to the nearest minute, will the contents last without damage.

3.10 Set review exercise

Set review exercise

20 min

Work out your answers and then access the web to input your answers in the online test called set review exercise.

Q41: Simplify the following:

a) $\log_a 4 + \log_a 3$
b) $3 \log_5 4 - \log_5 8$
c) $2 \log_7 7$

Q42: If $x = \dfrac{\log_e 7}{\log_e 2}$, find an approximation for x

Q43: If $\log_2 y = 3.4$, write down an expression for the exact value of y

Q44: If $y = e^{3.4}$ find an approximation for y to 2 d.p.

Q45: Simplify the following:

a) $4 \log_3 3$
b) $2 \log_2 4 + \log_2 3$
c) $2 \log_3 6 - \log_3 3$

Q46: If $a = \log_e 5 \times 2 \log_e 3$, find an approximate value for a

Q47: If $\log_e p = 2$ find an approximate value for p

Q48: $m = e^e$ find an approximate value for m

Topic 4

Further trigonometric relationships

Contents

Learning Objectives

- *Apply further trigonometric relationships*

Minimum performance criteria:

- *Express a cos θ + b sin θ in the form r cos($\theta \pm \alpha$) or r sin($\theta \pm \alpha$)*

Prerequisites

A sound knowledge of the following subjects is required for this unit:

- Addition angle formulae expansions
- The associated angles for trig. ratios using the quadrant diagrams
- Techniques for solving trig. equations

4.1 Revision exercise

Revision exercise

30 min

There is a web exercise if you prefer it.

Q1: Expand the following:

a) cos (2x - y)
b) sin (a + b)
c) cos (m + n)
d) sin (p - q)

Q2: If sin x is positive in the solution to an equation in x and one solution is 25° in the first quadrant, what is the other solution in the range $0 \leq x < 360°$?

Q3: If tan x is negative, in which quadrants can the angle x lie?

Q4: If cos y is negative and one solution of y for the equation is 165°, what is the other in the range $0 \leq x < 360°$?

Q5: Solve 2 cos 2x = 1 for $0 \leq x < 180°$

4.2 Graphs of combined trigonometric functions

Learning Objective

Sketch and recognise features of the graphs of combined trigonometric functions

Examples of graphs for different trigonometric functions that demonstrate the relationship of each with the standard trig graphs of sin x, cos x and tan x should be a familiar concept.

The following example revises some of the relevant points.

Example Determine the relationship of the graph of y = 3 cos (2x - 30)° - 4 to the graph of
y = cos x

Answer:

The term - 4 will only move the graph down the axis by 4 units but will not affect the shape.

At this stage rearrange the equation f (x) = 3 cos (2x - 30) - 4 to give
f (x) = 3 cos 2(x - 15) - 4

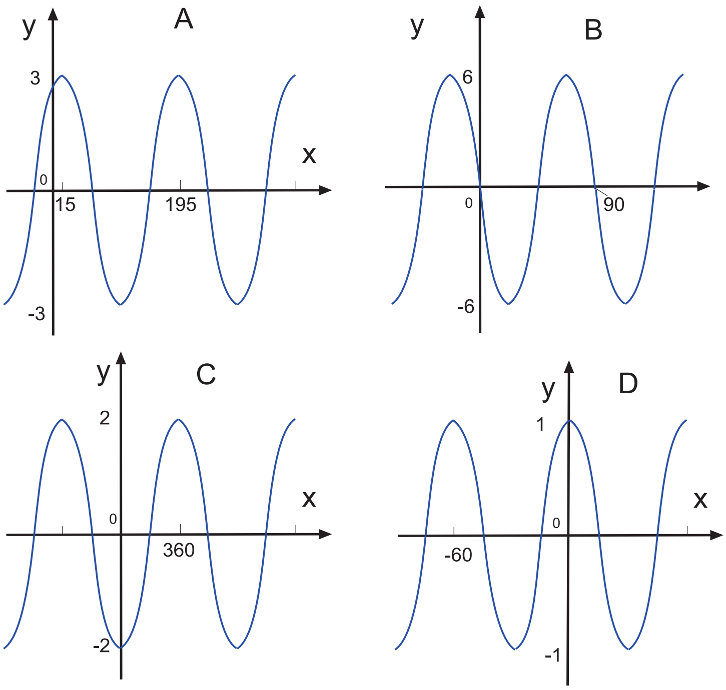
Now, in a similar way to the effect which 4 has, the term -15 will move the graph to the right by 15 units but will not affect the shape.

It is only the values of 3 and 2 which will affect the shape and so affect the amplitude and period of a function.

The amplitude of this graph is 3 and the period is 180° (since the 2 indicates that the graph will repeat twice within the period of a cos graph. That is, twice within 360°)

The amplitude of such a function can be found from the maximum and minimum values of the function. Conversely, if an accurate graph of the function is available, it can be used to determine the maximum and minimum values of the function.

Q6: Give the equation of the following graphs using sine functions for A and B and cosine functions for C and D:



Now consider the function sin x + cos x

The result is another *wave function*. It has an amplitude of 1.414 (or $\sqrt{2}$) and the graph has moved to the left of a normal sin function by 45°. This is termed a phase shift of -45° and the angle is known as a phase angle of -45°

However, the graph with amplitude $\sqrt{2}$ and which has moved to the left of the normal sine graph can be written as $\sqrt{2}$ sin (x + 45)°

This is the basis of a formula which can be used both to aid with the sketching of multiple trig term functions and to facilitate the manipulation of the algebra to reach the solutions of equations involving such terms.

5 min

y = sin x + cos x

The formation of this function is shown on the web.

20 min

Calculator investigation

Using a graphics calculator plot the graph of 2 sin x + 2 cos x

Q7: What is the amplitude of the graph?

Q8: What is the phase angle in relation to the normal cos expression?

Q9: What happens if there is no access to a graphics calculator ? :-
Look at the answers to see screen dumps from a TI83 for these graphs or try the following web site which has an excellent on-line graphics calculator.

http://www.univie.ac.at/future.media/moe/fplotter/fplotter.html

Another alternative is to construct a table of values and plot the graphs on paper.

Continue with this investigation by plotting the graphs of the following expressions by graphic calculator, on-line calculator or by hand. Examine the maximum values and the phase angles in relation to the sine graph.

a) 5 sin x + 11 cos x

b) sin x + 3 cos x

c) 2 sin x + 3 cos x

d) $\sqrt{3}$ sin x + 2 cos x

e) sin x + $\sqrt{3}$ cos x

f) 3 sin x + 4 cos x

Q10: Determine the connection between the coefficients of the sin and cos terms and the maximum values of the functions.

Q11: Consider the phase angle in each case and determine a relationship using the tangent function, to the coefficients of the terms in the expression.

4.3 Combined trig. function formulae

Learning Objective

Construct and apply the formulae to solve equations

Examination of a selection of graphs shows that a combination of sin and cos functions can be expressed and graphed as a single sine function.

In fact, not only can these combinations be graphed as a sine function, but they can also be graphed as a cosine function.

Equation of the form a sin x + b cos x

An **equation of the form a sin x + b cos x** can be expressed in one of the following four forms:

1. k sin (x + α)
2. k sin (x - α)
3. k cos (x + α)
4. k cos (x - α)

where k is the amplitude of the function and α is the phase angle.

The strategy for finding this combined function is the same regardless of which formula is used.

Strategy for expressing a sin x + b cos x as a single trig. function

1. Equate the expression a sin x + b cos x with the required form.
2. Use the addition formulae to expand the single combined trig. function using say, k for amplitude and α for the phase angle.
3. Equate the coefficients of sin x:- equation 1
4. Equate the coefficients of cos x:- equation 2
5. Square equations 1 and 2 and add them:- this gives the solution for k
6. Divide one equation by the other in such a way as to form tan α
7. Solve for α in the correct quadrant by referring to the sin and cos values in equations 1 and 2

Some examples will help to make this clearer.

Examples

1. 3 sin x + 4 cos x as k sin (x + α)

Express 3 sin x + 4 cos x in the form k sin (x + α) where k $>$ 0 and 0 $\leq \alpha <$ 360°

Answer:

Let 3 sin x + 4 cos x = k sin (x + α) = k sin x cos α + k cos x sin α

Equate the coefficients of sin x: 3 = k cos α:- equation 1

Equate the coefficients of cos x: 4 = k sin α:- equation 2

Squaring and adding these two equations gives:

9 + 16 = $k^2 \cos^2\alpha + k^2 \sin^2 \alpha = k^2$

k = $\sqrt{25}$ = 5

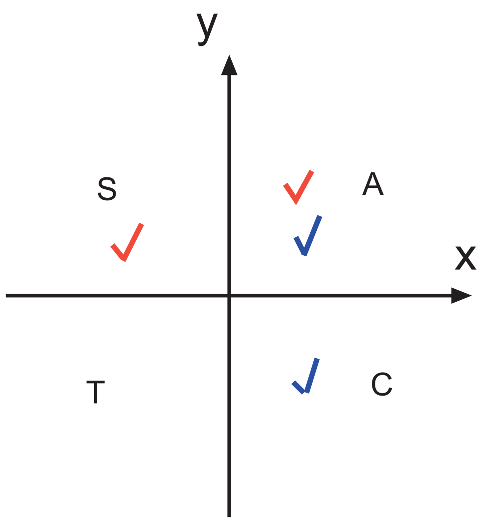
Dividing the equations ($^2/_1$) to form tan α gives:

$^4/_3$ = tan α (note the k cancels and $^{\sin}/_{\cos}$ = tan).

In the first quadrant $\alpha = \tan^{-1}(^4/_3) = 53.13°$

From equation 1: cos is positive

From equation 2: sin is positive



Then the angle required lies in the first quadrant and so α = 53° to the nearest degree.

3 sin x + 4 cos x = 5 sin (x + 53)°

2. a sin x + b cos x as k sin (x - α)

Express 2 sin x + 3 cos x in the form k sin (x - α) where k $>$ 0 and 0 $\leq \alpha <$ 360°

Answer:

Let 2 sin x + 3 cos x = k sin (x - α) = k sin x cos α - k cos x sin α

Equate the coefficients of sin x: 2 = k cos α:- equation 1

Equate the coefficients of cos x: 3 = - k sin $\alpha \Rightarrow$ -3 = k sin α:- equation 2

Squaring and adding these two equations gives:

4 + 9 = $k^2 \cos^2\alpha + k^2 \sin^2 \alpha = k^2$

k = $\sqrt{13}$ (leave as a surd).

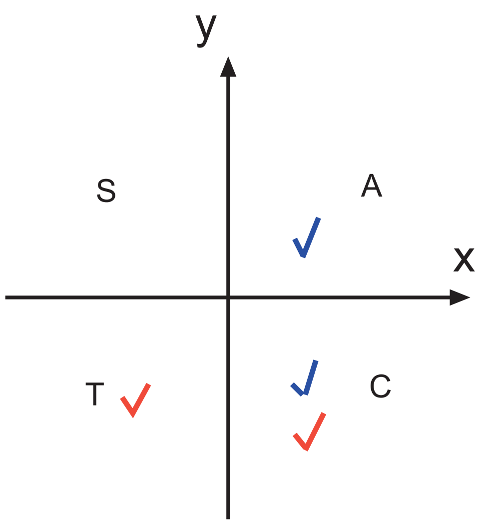
Dividing the equations ($^2/_1$) to form tan α gives:

$-{}^3/_2 = \tan \alpha$ (note the k cancels and ${}^{\sin}/_{\cos} = \tan$)

In the first quadrant $\alpha = \tan^{-1}({}^3/_2) = 56.31°$

From equation 1: cos is positive

From equation 2: sin is negative



Then the angle required lies in the fourth quadrant and so α = -56° = 304° to the nearest degree.

2 sin x + 3 cos x = $\sqrt{13}$ sin (x - 304)°

3. a sin x + b cos x as k cos (x + α)

Express 4 sin x + 7 cos x in the form k cos (x + α) where $k > 0$ and $0 \leq \alpha < 360°$

Answer:

Let 4 sin x + 7 cos x = k cos (x + α) = k cos x cos α - k sin x sin α

Equate the coefficients of sin x: 4 = -k sin $\alpha \Rightarrow$ -4 = k sin α:- equation 1

Equate the coefficients of cos x: 7 = k cos α:- equation 2

Squaring and adding these two equations gives:

$16 + 49 = k^2 \sin^2\alpha + k^2 \cos^2 \alpha = k^2$

$k = \sqrt{65}$

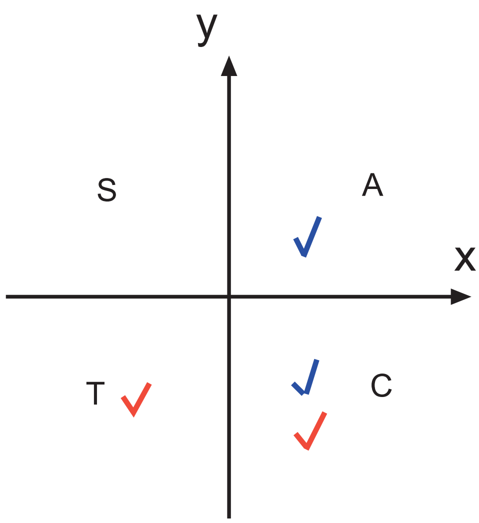
Dividing the equations (${}^1/_2$) to form tan α gives:

$-{}^4/_7 = \tan \alpha$ (note the k cancels and ${}^{\sin}/_{\cos} = \tan$)

In the first quadrant $\alpha = \tan^{-1}({}^4/_7) = 29.74°$

From equation 1: sin is negative

From equation 2: cos is positive



Then the angle required lies in the fourth quadrant and so α = 360 - 30° = 330° to the nearest degree.

4 sin x + 7 cos x = $\sqrt{65}$ cos (x + 330)°

4. a sin x + b cos x as k cos (x - α)

Express 5 sin x + 2 cos x in the form k cos (x - α) where k $>$ 0 and 0 $\leq \alpha <$ 360°

Answer:

Let 5 sin x + 2 cos x = k cos (x - α) = k cos x cos α + k sin x sin α

Equate the coefficients of sin x: 5 = k sin α :- equation 1

Equate the coefficients of cos x: 2 = k cos α:- equation 2

Squaring and adding these two equations gives:

25 + 4 = k^2 $\sin^2\alpha$ + k^2 $\cos^2 \alpha$ = k^2

k = $\sqrt{29}$

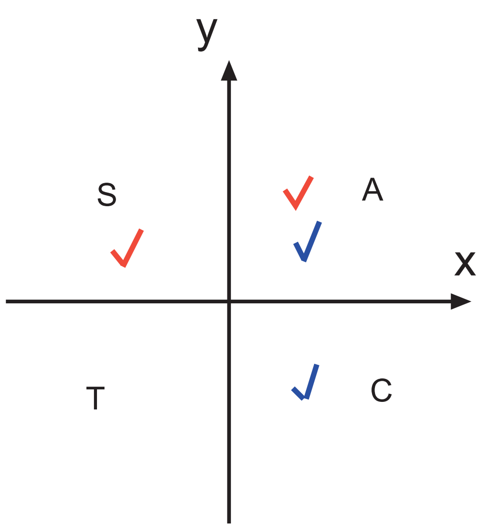
Dividing the equations ($^1/_2$) to form tan α gives:

$^5/_2$ = tan α (note the k cancels and $^{\sin}/_{\cos}$ = tan)

In the first quadrant α = $\tan^{-1}(^5/_2)$ = 68.20°

From equation 1: sin is positive

From equation 2: cos is positive



Then the angle required lies in the first quadrant and so α = 68° to the nearest degree.

5 sin x + 2 cos x = $\sqrt{29}$ cos (x - 68)°

The same technique works with negative values of a and b. The difference lies only in determining the quadrant in which the answer lies.

Wave function exercise

There is a web exercise if you prefer it.

30 min

Q12: Express 4 sin x + cos x in the form k cos (x - α) where k $>$ 0 and 0 $\leq \alpha <$ 360°

Q13: Express 6 sin x + 3 cos x in the form k cos (x + α) where
k $>$ 0 and 0 $\leq \alpha <$ 360°

Q14: Express 3 sin x - 2 cos x in the form k sin (x - α) where k $>$ 0 and 0 $\leq \alpha <$ 360°

Q15: Express -sin x - 6 cos x in the form k sin (x + α) where k $>$ 0 and 0 $\leq \alpha <$ 360°

Q16: Express 2 cos x - 4 sin x in the form k cos (x - α) where k $>$ 0 and 0 $\leq \alpha <$ 360°

Q17: Express -$\sqrt{3}$ sin x + 2 cos x in the form k sin (x - α)
where k $>$ 0 and 0 $\leq \alpha <$ 360°

4.4 Maxima/minima values and solving equations

4.4.1 Maxima/minima values of the expression a sin x + b cos x

Learning Objective

Find maximum and minimum values of expressions of the form asin x + bcos x = c

From the work covered in this topic, it is now straightforward to express
a sin x + b cos x in terms of a single cosine or sine function. The form of this new function has two variables say k and α, although it is important to realise that other symbols can be used. (r instead of k and θ instead of α are very common.)

The amplitude of this single combined function is the value k

By definition, the amplitude of a wave function is half of the distance between the maximum and the minimum of the wave.
This is in fact how k is determined but the graphs of the functions in this topic are all symmetrical about the x-axis making this calculation unnecessary.

It follows that for a straightforward combined wave function,
the maximum value is k and the minimum value is - k

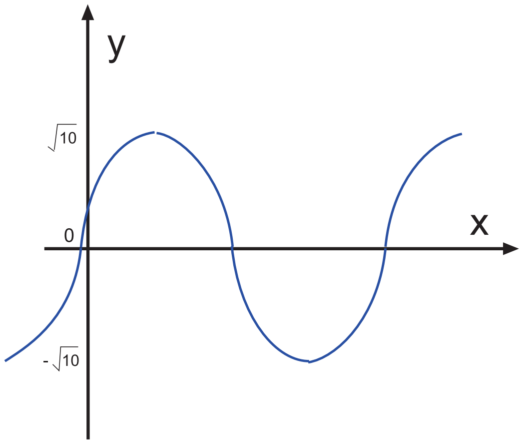
Example If 3 sin x + cos x can be expressed as $\sqrt{10}$ sin (x + 18)°, find the maximum and minimum values of the function.

Answer:

It has been stated that the maximum is k; that is the maximum is $\sqrt{10}$

Similarly the minimum is -k; that is the minimum is $-\sqrt{10}$

Examine the basic graph:



The maximum and minimum values are clearly the values $\pm$k and the amplitude is confirmed as half the distance between the maximum and the minimum.

The maximum value occurs when sin (x + 18)° is at its maximum. That is at the value 1: then the maximum of $\sqrt{10}$ sin (x + 18)° is $\sqrt{10}$

The minimum occurs when sin (x + 18)° has the value -1: that is the minimum of $\sqrt{10}$ sin (x + 18)° is $-\sqrt{10}$

From this it is possible to determine the maximum and minimum values for a variety of expressions.

Example What is the maximum and minimum values of the expression 4 + 5 cos (x - 37)°?

Answer:

The maximum is 4 + 5 = 9 and occurs when cos (x - 37)° = 1

The minimum is 4 - 5 = -1 and occurs when cos (x - 37)° = -1

Some care is needed though when the expression is slightly different.

Example What is the maximum and minimum values of the expression 3 - 7 sin (x - 20)°?

Answer:

In this case the maximum occurs when *sin (x - 20)* ° = *-1*as this gives a maximum of 3 - (-7) = 10

It follows that the minimum occurs when sin (x - 20)$^\circ$ = 1 as this gives 3 - 7 = -4

Using the techniques of this topic it is now possible to find the maximum and minimum of expressions such as -2 + 3 sin x - 5 cos x and the value of x at which these maxima and minima occur.

The general technique is as follows.

First of all express 3 sin x - 5 cos x as a combined trig function.

Secondly use this new function to determine the maximum and minimum values using the knowledge of max/min values of sin and cos functions.

Solve the single expression in sin or cos = $\pm$ 1 to find the appropriate values of x

The first exercise will give practice in finding maximum and minimum values.

If only the maximum or minimum values are required, there is a shortcut. Since any of the four combined trig functions has the same maximum of 1 and minimum of -1 it follows that it is only the amplitude (the value of k) which needs to be calculated.

Example What is the maximum and minimum values of the expression - 3 sin x + 4 + 5 cos x?

Answer:

Isolate the combination of trig functions: -3 sin x + 5 cos x

Here in general terms a = -3 and b = 5

But previously it was shown that k = $\sqrt{}$(a^2 + b^2)

In this case therefore, k = $\sqrt{}$(9 + 25) = $\sqrt{}$ 34

Returning to the original expression, - 3 sin x + 4 + 5 cos x = 4 + (-3 sin x + 5 cos x)

If the combined expression for -3 sin x + 5 cos x is X then 4 + $\sqrt{}$34X will give

a maximum at 4 + $\sqrt{}$34 when X = 1

a minimum at 4 - $\sqrt{}$34 when X = -1

Maximum and minimum exercise

30 min

There is another exercise on the web if you prefer it.

Q18: Find the maximum and minimum values of the following expressions:

a) 2 - 4 sin (x - 300)$^\circ$
b) -5 - cos (x + 34)$^\circ$
c) 5 - 5 sin (x - 30)$^\circ$
d) 3 + 2 sin x - 4 cos x
e) -1 - cos x + sin x
f) 2 sin x - 2 + cos x

4.4.2 Solving equations

Learning Objective

Find the values of x for which maximum and minimum values of expressions of the form asin x + bcos x = c occur

The technique is best shown by example.

Example Solve the equation 2 sin x - 5 cos x = 4, $0 \leq x < 360°$

Answer:

2 sin x - 5 cos x can be expressed as say, k sin (x + α)

2 sin x - 5 cos x = k sin x cos α + k cos x sin α

Equate the coefficients of sin x: 2 = k cos α:- equation 1

Equate the coefficients of cos x: -5 = k sin α:- equation 2

Squaring and adding gives 4 + 25 = $k^2(\cos^2\alpha + \sin^2\alpha) = k^2$

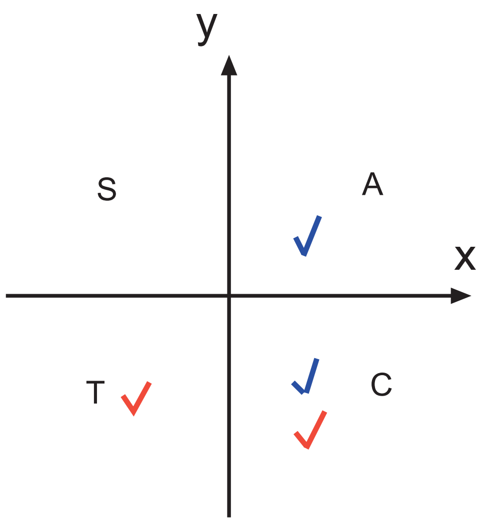
k = $\sqrt{29}$

Dividing gives tan α = $^{-5}/_{2}$

In the first quadrant this solves to give α = 68° to the nearest degree.

from equation 1: cos is positive

from equation 2: sin is negative



The angle lies in quadrant four.

α = 360 - 68° = 292°

2 sin x - 5 cos x can be expressed as $\sqrt{29}$ sin (x + 292)°

So 2 sin x - 5 cos x = $\sqrt{29}$ sin (x + 292)° = 4 $\Rightarrow$

sin (x + 292)° = 4 $\div \sqrt{29}$

Thus the first quadrant angle α = 48° to the nearest degree.

Since sin is positive the solutions lie in quadrants one and two.

x + 292 = 48° or x + 292 = 132°

$0 \leq x < 360° \Rightarrow 292 \leq x + 292 < 652$

Check that the solutions lie within the range and that there are no others.

Thus x + 292 = 48 + 360 = 408° and x + 292 = 132 + 360 = 492°

x = 116° and x = 200°

In every case this method will provide the solutions. Any of the four combined trig functions can be used if one of the other forms is preferred.

Solving equations exercise

30 min

There is an alternative exercise on the web if you prefer it.

Q19: Find the maximum value of the expression 7 sin x + 3 cos x
and the corresponding value for x, $0 \leq x < 360°$

Q20: Find the minimum value of the expression
5 sin x + 2 cos x and the corresponding value for x, $0 \leq x < 360°$

Q21: Solve the equation - sin x - 4 cos x = 3, $0 \leq x < 360°$

Q22: Solve the equation 2 sin x - 4 cos x = -3, $0 \leq x < 360°$

Q23: Solve the equation -3 sin x + 6 cos x = 2, $0 \leq x < 360°$

Q24: Find the maximum and minimum values of the expression $\sqrt{3}$ sin x + cos x and the corresponding value for x, $0 \leq x < 360°$

4.5 Solving problems with trig. formulae

Learning Objective

Apply the correct techniques and trig. formulae to solve problems

This topic has extended the work using trigonometric formulae to cover the remaining formulae required in this course. In reality the formulae and techniques will be used to solve real life problems. Some of the possible situations are explained in the examples and exercises which follow. In all cases, adherence to the techniques shown in the topic will ensure that the solutions are obtained in a logical and concise manner.

Examples

1. At a harbour, the waves at high tide at a certain time of year can be modelled by the formula h = 2 sin t + 4 cos t where h is the height of the wave in metres and t is the time in minutes. Find the maximum height of the waves. If the harbour wall is 3m higher than the still water level at high tide, will the sea come over the wall?

Answer:

Let 2 sin t + 4 cos t = k sin (t + α) [Recall that any of the four formulae can be used and that the variables need not be k and α]

2 sin t + 4 cos t = k sin t cos α + k cos t sin α

Equate the coefficients of sin t: 2 = k cos α:- equation 1

Equate the coefficients of cos t: 4 = k sin α :- equation 2

Squaring and adding these two equations gives:

4 + 16 = k^2 $\cos^2\alpha$ + k^2 $\sin^2$ α = k^2

k = $\sqrt{20}$ (leave as a surd).

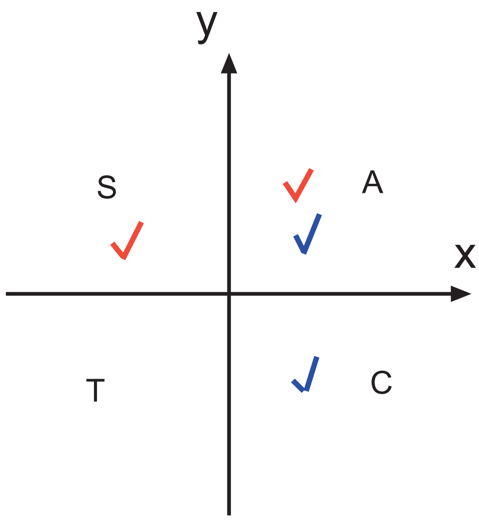
Dividing the equations ($^2/_1$) to form tan α gives:

tan α = $^4/_2$

In the first quadrant α = $\tan^{-1}(^4/_2)$ = 63° to the nearest degree.

From equation 1: sin is positive

From equation 2: cos is positive



Then the angle required lies in the first quadrant and so α = 63° to the nearest degree.

2 sin t + 4 cos t = $\sqrt{20}$ sin (t + 63)°

The waves have a maximum height of $\sqrt{20}$ = 4.47m

The waves will come over the wall by 3 - 4.47 = 1.47m at each crest.

2. At a leisure centre, the wave machine can be modelled by the formula w = sin t + cos t where w is the height of each wave and t is the time in minutes after the machine has been switched on. Find the maximum height of the waves when the formula can be expressed as k sin (t + α). At what level below the edge of the pool must the water be before the wave machine is switched on to ensure that no water will spill over when the waves are on?

Answer:

Let sin t + cos t = k sin (t + α)

sin t + cos t = k sin t cos α + k cos t sin α

Equate the coefficients of sin t: 1 = k cos α:- equation 1

Equate the coefficients of cos t: 1 = k sin α :- equation 2

Squaring and adding these two equations gives:

$1 + 1 = k^2 \cos^2\alpha + k^2 \sin^2 \alpha = k^2$

$k = \sqrt{2}$ (leave as a surd).

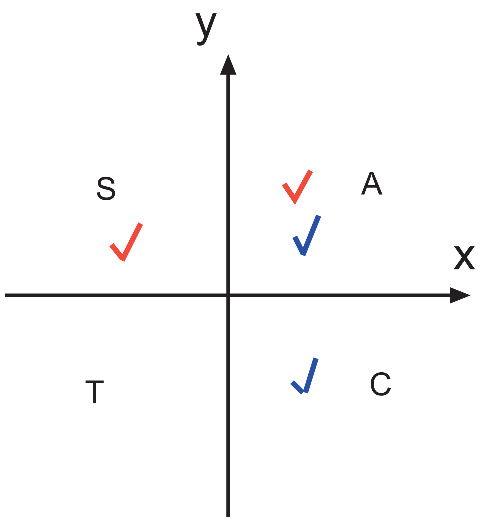
Dividing the equations ($^1/_2$) to form tan α gives:

$\tan \alpha = 1$

In the first quadrant α = tan = 45° to the nearest degree.

From equation 1: sin is positive

From equation 2: cos is positive



Then the angle required lies in the first quadrant and so α = 45° to the nearest degree.

$\sin t + \cos t = \sqrt{2} \sin (t + 45)°$

The waves have a maximum height of $\sqrt{2}$ = 1.414m

The water must be lower than 1.414m below the edge to ensure that it does not spill over.

3. The frequency of the sound waves of a siren in a factory can be modelled by the formula f = - sin t - 7 cos t + 200 where f is in hertz and t is the time in seconds after it is activated. The formula can be expressed in the form r cos (t - θ). What is the maximum frequency? The factory regulations state that the frequency must not exceed 201 Hz, after how long must the siren be switched off to ensure this the regulation is met.

Answer:

Let $-\sin t - 7 \cos t = r \cos (t - \theta)$

$-\sin x - 7 \cos x = r \cos t \cos \theta + r \sin t \sin \theta$

Equate the coefficients of sin t: $-1 = r \sin \theta$:- equation 1

Equate the coefficients of cos t: $-7 = r \cos \theta$:- equation 2

Squaring and adding these two equations gives:

$1 + 49 = r^2 \sin^2\theta + r^2 \cos^2 \theta = r^2$

$r = \sqrt{50}$

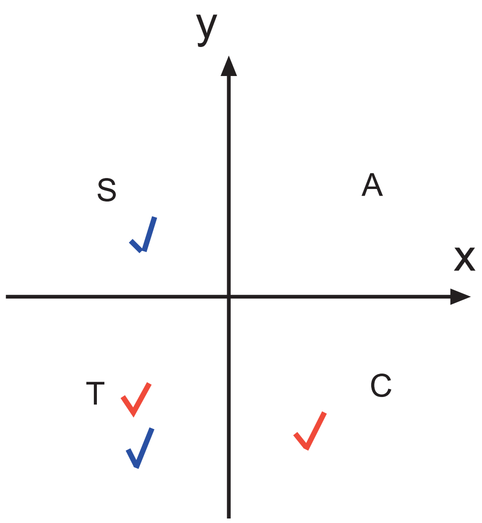
Dividing the equations ($^1/_2$) to form tan θ gives:

$\tan \theta = {}^{-1}/_{-7}$

In the first quadrant α = 8° to the nearest degree.

From equation 1: sin is negative

From equation 2: cos is negative



Then the angle required lies in the third quadrant and so θ = 188° to the nearest degree.

-sin t - 7 cos t = $\sqrt{50}$ cos (t - 188)$^\circ$

The maximum frequency is $\sqrt{50}$ + 200 = 207Hz

If $\sqrt{50}$ cos (t - 188) + 200 = 201 (to meet the regulation).

then cos (t - 188) = 1 $\div$ $\sqrt{50}$

t - 188 = 82° to the nearest degree.

t = 270 $^\circ$

But t is the time in seconds in the problem: t = 4 minutes 30 seconds

4. A generator provides an electric current modelled by the formula
I = 5 sin 300 t + 8 cos 300 t where I is in amps and t is the time in seconds after it is switched. The formula can be expressed in the form k sin (300 t - α).
What is the fluctuation in the current? The generator fails after 65 seconds. How much current is being produced at that time?

Answer:

Let 5 sin 300 t + 8 cos 300 t = k sin (300 t - α)

5 sin 300 t + 8 cos 300 t = k sin 300 t cos α - k cos 300 t sin α

Equate the coefficients of sin 300 t: 5 = k cos α:- equation 1

Equate the coefficients of cos 300 t: 8 = - k sin α $\Rightarrow$ -8 = k sin α :- equation 2

Squaring and adding these two equations gives:

25 + 64 = k^2 $\cos^2\alpha$ + k^2 $\sin^2$ α = k^2

k = $\sqrt{89}$

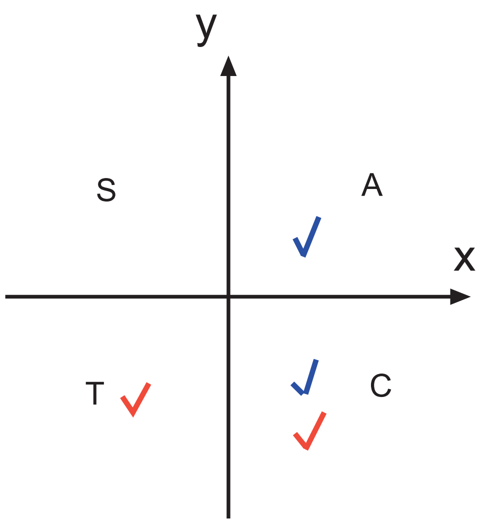
Dividing the equations ($^2/_1$) to form tan α gives:

tan α = $^8/_5$

In the first quadrant α = 58° to the nearest degree.

From equation 1: cos is positive

From equation 2: sin is negative



Then the angle required lies in the fourth quadrant and so α = 302° to the nearest degree.

5 sin 300 t + 8 cos 300 t = $\sqrt{89}$ sin (300 t - 302)°

The fluctuation in current is the difference between the maximum and the minimum.

The fluctuation is 2 $\sqrt{89}$ = 18.87 amps.

If t = 65 then $\sqrt{89}$ sin (300 t - 302)° = $\sqrt{89}$ sin (19500 - 302) = $\sqrt{89}$ sin 19198 = 8.33

The current at this point is 8.33 amps.

This is only a limited variety of the type of problem which can be set.

Problem solving for waves exercise

30 min

Q25: The depth of water in metres in a harbour at a certain time of year can be modelled by the formula d = 4 + sin t - 3 cos t where t is the time in minutes after midnight. The formula sin t - 3 cos t can be restated in terms of k sin (t - α). Find α. A cruiser wishes to moor in the harbour and needs a minimum depth of 2 m. Can the cruiser moor safely?

Q26: At an offshore engineering science laboratory, the wave simulator can be modelled by the formula w = m - 2 sin t + 5 cos t where w is the height of each wave and t is the time in minutes after the simulator has been switched on. The pool in which the simulator works must have a minimum depth of 1 metre of water when the machine is on. Find the critical value of m when the formula can be expressed as k cos (t + α).

Q27: The low drone of a processor has a frequency that can be modelled by the formula p = 2 sin 0.5 t - 10 cos 0.5 t + 15 where p is in hertz and t is the time in seconds after it is activated. The formula can be expressed in the form r cos (0.5 t + θ). What is the highest frequency it reaches? How long does it take to reach this frequency when it is switched on?

Q28: An electric current is modelled by the formula I = 5 sin 20 t + cos 20 t + 6 where I is in milliamps and t is the time in seconds after the current is switched on. The formula can be expressed in the form k sin (20 t + α). What is the minimum current and how often does it occur?

15 min

Wave functions in real situations

This activity should provide food for though for the individual student or group.

Consider, discuss and write down some of the real life situations where wave functions occur and whether the techniques of this topic can be used to solve the likely problems.

4.6 Summary

The following points and techniques should be familiar after studying this topic:

- Expressing a cos x + b sin x in the form k cos (x $\pm$ α) or k sin (x $\pm$ α).
- Solving expressions of the form a cos x + b sin x = c by using one of the combined forms.
- Finding maximum / minimum values of expressions of the form a cos x + b sin x.
- Finding the value of x for the maximum and minimum values of expressions of the form a cos x + b sin x.

4.7 Extended Information

There are links on the web which give a selection of interesting sites to visit. Browsing the web under 'trigonometry' will lead to many other good sites which cover this topic.

These links were given for the topic on trigonometric formulae. At this stage these sites can be revisited to gain further information on the work of this section.

The St. Andrews web site at
http://www-groups.dcs.st-and.ac.uk/~history/HistTopics/Trigonometric_functions.html
gives a comprehensive history of trigonometry.

Hipparchus

This Greek mathematician and astronomer greatly influenced the thinking of his time. He is generally believed to be the founder of trigonometry.

Ptolemy

He built on the work of Hipparchus and although the terminology was yet to be introduced, he discovered some of the relationships between ratios.

Aryabhata

The origin of the word 'sine' lies in the work of the Hindu mathematician and astronomer, Aryabhata who lived in the sixth century.

Fincke

It was much later that the term 'tangent' was used by the Dane, Thomas Fincke, in 1583.

Gunther

Edmund Gunther completed the trio of terms by adopting the term 'cosine' in 1620.

Pitiscus

Many astronomers and mathematicians over the centuries contributed to the work in trigonometry but the word 'trigonometry' first appeared in print in a treatise by Bartholomaeus Pitiscus in 1595. He also discovered the double angle formulae although some were known to the ancient Greek astronomers in a different format.

De Moivre and Euler

De Moivre and Euler, in the eighteenth century continued to explore trigonometry and this led to the study of trigonometry of complex variables. De Moivre's theorem and Euler's formula are well known in current day studies of complex numbers.

4.8 Review exercise

Review exercise

There is another exercise on the web if you prefer it.

25 min

Q29: Express 3 sin x + cos x in the form k sin (x + a) where k $>$ 0 and 0 $\leq$ a $<$ 360°

Q30: Express 2 sin x - 3 cos x in the form k cos (x - a) where k $>$ 0 and 0 $\leq$ a $<$ 360°

Q31: The pattern of the waves generated at a laboratory can be obtained by the equation h = 4 sin x + 3 cos x + 6 where h is the height in metres and x is the time in seconds after the wave machine is switched on.
Express 4 sin x + 3 cos x in the form k cos (x + a) where k $>$ 0 and 0 $\leq$ a $<$ 360°. Hence find the maximum height in metres of the waves generated.

Q32: The frequency pattern of the sound waves generated by a siren can be obtained by the formula -3 sin x + 2 cos x . Express this formula in the form k sin (x - a) where k $>$ 0 and 0 $\leq$ a $<$ 360°. What is the amplitude of the wave?

4.9 Advanced review exercise

Advanced review exercise

25 min

There is another exercise on the web if you prefer it.

Q33: Express -5 sin θ - 3 cos θ in the form r cos (θ - α) where r $>$ 0 and 0 $\leq \alpha <$ 360°
Hence solve -5 sin θ - 3 cos θ = 3, 0 $\leq \theta <$ 360°

Q34: f (x) = 4 cos x° - 2 sin x°

a) Express f (x) in the form k sin (x - α)° where k $>$ 0 and 0 $\leq \alpha <$ 360°
b) Hence solve f (x) - 0.3 = 0, 0 $\leq$ x $<$ 360°
c) Find the x-coordinate of the point nearest to the origin where the graph of f (x) = 4 cos x° - 2 sin x° cuts the x-axis for 0 $\leq$ x $<$ 360°

Q35: f (x) = -2 cos x° + 5 sin x°

a) Express f (x) in the form k cos (x - α)° where k $>$ 0 and 0 $\leq \alpha <$ 360°
b) Hence solve f (x) - 1 = 0, 0 $\leq$ x $<$ 360°
c) Find the x-coordinate of the point nearest to the origin where the graph of f (x) = 4 cos x° - 2 sin x° cuts the x-axis for 0 $\leq$ x $<$ 360°

Q36: The sound waves of a siren are modelled by the equation
cos x = 3 sin x. Find the phase shift and the amplitude of the waves.

4.10 Set review exercise

Set review exercise

30 min

Work out your answers and then access the web to input your answers in the online test called set review exercise.

Q37: Express sin x + cos x in the form k sin (x - a) where k $>$ 0 and 0 $\leq$ a $<$ 360°

Q38: Solve 6 sin x + 8 cos x = 5 by first expressing in the form k cos (x - a) where k $>$ 0 and 0 $\leq$ a $<$ 360°. State the minimum value of the expression 6 sin x + 8 cos x

Q39: Find the amplitude and phase shift of the graph given by y = -2 sin x - 2 cos x in relation to a sine wave of the form k sin (x + a)

Q40: Find the maximum and minimum values of f (x) = 5 - (4 cos x + 2 sin x) using the form k sin (x + a)

Glossary

The chain rule

Function notation

h' (x) = g' (f (x)) $\times$ f' (x)

Leibniz notation

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Collinearity

All points which lie on a straight line are said to be collinear.

If $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are two vectors the following two vector conditions determine collinearity:

1. k is a scalar such that $\overrightarrow{AB} = k\overrightarrow{BC}$
2. the point B is common to both vectors

Equation of the form a sin x + b cos x

An **equation of the form a sin x + b cos x** can be expressed in one of the following four forms:

1. k sin (x + α)
2. k sin (x - α)
3. k cos (x + α)
4. k cos (x - α)

where k is the amplitude of the function and α is the phase angle.

Exponential function

A function of the form f (x) = a^x where a $>$ 0 and a $\neq$ 1 is called an exponential function

Length of a vector in three dimensions

Let **p** be the vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ then the length of **p** is defined as $|\mathbf{p}| = \sqrt{a^2 + b^2 + c^2}$

Length of a vector in two dimensions

Let **p** be the vector $\begin{pmatrix} a \\ b \end{pmatrix}$ then the length of **p** written as $|\mathbf{p}|$ is defined as $|\mathbf{p}| = \sqrt{a^2 + b^2}$

Log and exp relationship

$$y = a^x \Leftrightarrow \log_a y = x$$

where a $>$ 1 and y $>$ 0

Log Laws

Law 1:

$\log_a 1 = 0$

Law 2:

$\log_a a = 1$

Law 3:

$\log_a (bc) = \log_a b + \log_a c$

Law 4:

$\log_a (^b/_c) = \log_a b - \log_a c$

Law 5:

$\log_a b^n = n \log_a b$

Parallel vectors

If two vectors are parallel they have the same direction but their magnitudes are scalar multiples of each other.

Position vector

A position vector is a vector which starts at the origin.

Scalar product in component form (three dimensions)

If $\mathbf{p} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} d \\ e \\ f \end{pmatrix}$

then the scalar product is the number $\mathbf{p} \bullet \mathbf{q} = ad + be + cf$

Scalar product in component form (two dimnesions)

If $\mathbf{p} = \begin{pmatrix} a \\ b \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} d \\ e \end{pmatrix}$

then the scalar product is the number $\mathbf{p} \bullet \mathbf{q} = ad + be$

Scalar product in geometric form

The scalar product of two vectors **a** and **b** is defined as

$\mathbf{a} \bullet \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$ where θ is the angle between **a** and **b**, $0 \leq \theta \leq 180°$

Section formula

If **p** is the position vector of the point P which divides AB in the ratio m : n then

$\mathbf{p} = \frac{n}{m+n}\mathbf{a} + \frac{m}{m+n}\mathbf{b}$

Vector quantity

A vector quantity is a quantity which has both direction and magnitude.

Hints for activities

Topic 4: Further trigonometric relationships

Calculator investigation

Hint 1:

QUESTION1: The connection between the coefficients and the maximum value is best seen with the graph of 3 sin x + 4 cos x where the maximum value is 5.
The connection is that the maximum value of the function is given by the square root of the sums of the squares of the coefficients. (Think of Pythagoras). Check example 1 to help with the connection now.

QUESTION 2: The connection between the coefficients and the phase angle is easiest to find from the original example where the angle was 45° and the tan of 45° = 1. However look at examples 4 and 5 which will help to solve the problem.

Answers to questions and activities

1 Vectors in three dimensions

Revision exercise (page 2)

Q1: The distance formula gives $\sqrt{(3 - (-3))^2 + (4 - 2)^2} = 6.32$

Q2: The midpoint formula gives $\left(\frac{2 + (-4)}{2}, \frac{5 + 1}{2}\right) = (-1, 3)$

Q3: cos BAC = adjacent $\div$ hypotenuse = $^4/_7$

Thus angle BAC = 55.15^0 correct to two decimal places.

Answers from page 6.

Q4:

1. $\overrightarrow{CD} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$
2. $\overrightarrow{PQ} = \begin{pmatrix} -2 \\ -5 \end{pmatrix}$
3. $\overrightarrow{RS} = \begin{pmatrix} 8 \\ 1 \end{pmatrix}$
4. $\overrightarrow{SR} = \begin{pmatrix} -8 \\ -1 \end{pmatrix}$
5. $\overrightarrow{QP} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$
6. $\overrightarrow{DC} = \begin{pmatrix} -4 \\ 1 \end{pmatrix}$

Answers from page 6.

Q5: The vector from O to the point Q (c, d) can be expressed as **q**, $\underline{q}$, $\overrightarrow{OQ}$ or $\begin{pmatrix} c \\ d \end{pmatrix}$

Q6: The directed line segments representing vectors are:

$\overrightarrow{BA}, \overrightarrow{BC}, \overrightarrow{CD}, \overrightarrow{AD}, \overrightarrow{DB}, \overrightarrow{CA}$

In the same order the component forms are:

$\begin{pmatrix} -2 \\ 3 \end{pmatrix}, \begin{pmatrix} 6 \\ 1 \end{pmatrix}, \begin{pmatrix} -4 \\ 5 \end{pmatrix}, \begin{pmatrix} 4 \\ 3 \end{pmatrix}, \begin{pmatrix} -2 \\ -6 \end{pmatrix}, \begin{pmatrix} -8 \\ 2 \end{pmatrix}$

Q7:

a)

$\overrightarrow{OB} = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$ $\overrightarrow{OD} = \begin{pmatrix} 0 \\ 1 \\ -4 \end{pmatrix}$ and $\overrightarrow{OR} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$

b)

$\overrightarrow{OB} = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$ $\overrightarrow{OD} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$ and $\overrightarrow{OR} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

Q8: $\overrightarrow{OA} = \begin{pmatrix} -1 \\ -2 \\ 4 \end{pmatrix}$, $\overrightarrow{OB} = \begin{pmatrix} 3 \\ -5 \\ -2 \end{pmatrix}$, $\overrightarrow{OC} = \begin{pmatrix} 4 \\ 2 \\ -5 \end{pmatrix}$, $\overrightarrow{OD} = \begin{pmatrix} 2 \\ 5 \\ 4 \end{pmatrix}$

Answers from page 8.

Q9: The length is 5

$(\sqrt{(5-2)^2 + (4-8)^2} = \sqrt{25} = 5)$

Vector length exercise (page 10)

Q10:

a) $|\mathbf{a}| = 3.16$
b) $|\mathbf{b}| = 6.40$
c) $|\mathbf{c}| = 4.24$

Q11:

a) $\overrightarrow{XY} = \sqrt{(3-(-3))^2 + (2-1)^2} = 6.08$

b) $\overrightarrow{XZ} = \sqrt{(-2-(-3))^2 + (-2-1)^2} = 3.16$

c) $\overrightarrow{ZW} = \sqrt{(5-(-2))^2 + (-1-(-2))^2} = 7.07$

d) $\overrightarrow{WY} = \sqrt{(3-5)^2 + (2-(-1))^2} = 3.61$

e) $\overrightarrow{XW} = \sqrt{(5-(-3))^2 + (-1-1)^2} = 8.25$

f) $\overrightarrow{YZ} = \sqrt{(-2-3)^2 + (-2-2)^2} = 6.40$

Q12:

a) $\overrightarrow{PS} = \sqrt{(-4-1)^2 + (-3-2)^2 + (-2-3)^2} = 8.66$

b) $\overrightarrow{RS} = \sqrt{(-4-(-3))^2 + (-3-1)^2 + (-2-2)^2} = 5.74$

c) $\overrightarrow{SQ} = \sqrt{(-2-(-4))^2 + (-4-(-3))^2 + (1-(-2))^2} = 3.74$

d) $\overrightarrow{RP} = \sqrt{(1-(-3))^2 + (2-1)^2 + (3-2)^2} = 4.24$

e) $\overrightarrow{QP} = \sqrt{(1-(-2))^2 + (2-(-4))^2 + (3-1)^2} = 7.00$

f) $\overrightarrow{QR} = \sqrt{(-3-(-2))^2 + (1-(-4))^2 + (2-1)^2} = 5.20$

Q13: The distance between the two points is the same as the length of the vector between them (either $\overrightarrow{ST}$ or $\overrightarrow{TS}$)

$\overrightarrow{ST} = \sqrt{(-2-1)^2 + (5-2)^2 + (3-(-4))^2} = 8.19$

Adding vectors exercise (page 16)

Q14:

a)

$$\mathbf{a} + \mathbf{c} = \begin{pmatrix} 0 \\ -3 \\ -3 \end{pmatrix}$$

b)

$$\mathbf{b} + \mathbf{a} = \begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix}$$

c)

$\mathbf{c}$ + $\mathbf{-c}$ = the zero vector $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Q15:

1.

$\overrightarrow{AB}$ and $\overrightarrow{AC}$ = $\begin{pmatrix} 4 \\ -6 \\ -1 \end{pmatrix} + \begin{pmatrix} -1 \\ -3 \\ -5 \end{pmatrix} = \begin{pmatrix} 3 \\ -9 \\ -6 \end{pmatrix}$

2. $\overrightarrow{AB}$ and $\overrightarrow{BA}$ = zero vector since $\overrightarrow{AB} = -\overrightarrow{BA}$
3.

$\overrightarrow{BC}$ and $\overrightarrow{AC}$ = $\begin{pmatrix} -3 \\ 3 \\ -4 \end{pmatrix} + \begin{pmatrix} 1 \\ -3 \\ -5 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ -9 \end{pmatrix}$

Subtracting vectors (page 19)

Q16:

a)

$$\mathbf{a} - \mathbf{b} = \begin{pmatrix} 3 \\ -4 \\ 3 \end{pmatrix}$$

b)

$$\mathbf{c} - \mathbf{a} = \begin{pmatrix} -2 \\ 6 \\ -3 \end{pmatrix}$$

c)

$$\mathbf{b} - \mathbf{c} = \begin{pmatrix} -1 \\ -2 \\ 0 \end{pmatrix}$$

Q17:

a)

$$\overrightarrow{AD} - \overrightarrow{BD} = \begin{pmatrix} -2 \\ 0 \\ 3 \end{pmatrix} - \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}$$

b)

$$\overrightarrow{CD} - \overrightarrow{BA} = \begin{pmatrix} -4 \\ 4 \\ 7 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -5 \\ 4 \\ 6 \end{pmatrix}$$

c)

$$\overrightarrow{AC} - \overrightarrow{CD} = \begin{pmatrix} 2 \\ -4 \\ -4 \end{pmatrix} - \begin{pmatrix} -4 \\ 4 \\ 7 \end{pmatrix} = \begin{pmatrix} 6 \\ -8 \\ -11 \end{pmatrix}$$

d)

$$\overrightarrow{BD} - \overrightarrow{BA} = \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ 3 \end{pmatrix}$$

Multiplication by a scalar exercise (page 20)

Q18:

$$-3\mathbf{a} = \begin{pmatrix} -6 \\ -9 \\ 0 \end{pmatrix}, \ -3\mathbf{b} = \begin{pmatrix} 3 \\ 9 \\ -3 \end{pmatrix} \text{ and } -3\mathbf{c} = \begin{pmatrix} -9 \\ 12 \\ -6 \end{pmatrix}$$

Q19: Let the vector **a** be $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$

then ka = 8, kb = -6 and kc = 0

so c = 0 and k is a common factor of 8 and -6

The only value is k = 2 and the vector **a** = $\begin{pmatrix} 4 \\ -3 \\ 0 \end{pmatrix}$

Properties exercise (page 23)

Q20: Let the vector be $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$

Then $\begin{pmatrix} a \\ b \\ c \end{pmatrix} = m \begin{pmatrix} -5 \\ 4 \\ 1 \end{pmatrix}$

Thus a = -5m, b = 4m and c = m

But c = -2 and so m = -2

The vector parallel to the position vector through the point (-5, 4, 1) with z-component equal to -2 is $\begin{pmatrix} 10 \\ -8 \\ -2 \end{pmatrix}$

Q21:

a)

$\overrightarrow{AB} = \begin{pmatrix} 2 \\ 2 \\ -2 \end{pmatrix}$ and $\overrightarrow{BC} = \begin{pmatrix} 2 \\ 2 \\ -2 \end{pmatrix}$

$\overrightarrow{AB} = \overrightarrow{BC}$ and B is a common point to the vectors. The points are collinear. (note that in this case, B is actually the mid-point of the line AC).

b)

$\overrightarrow{XY} = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}$ and $\overrightarrow{YZ} = \begin{pmatrix} -4 \\ 4 \\ 3 \end{pmatrix}$

$\overrightarrow{XY} \neq k\overrightarrow{YZ}$ for any scalar value k and so the points are not collinear.

Q22: Using the section formula with m = 3, n = 5 gives

$\mathbf{p} = \frac{5}{8}\mathbf{a} + \frac{3}{8}\mathbf{b} = \begin{pmatrix} -5 \\ 10 \end{pmatrix} + \begin{pmatrix} 9 \\ 12 \end{pmatrix} = \begin{pmatrix} 4 \\ 22 \end{pmatrix}$

Q23: $\overrightarrow{ST} = \begin{pmatrix} -4 \\ 4 \end{pmatrix}$

The ratio of SR : ST = 1 : 4

$\overrightarrow{SR} = \frac{1}{4}\overrightarrow{ST} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$\overrightarrow{SR} = \mathbf{r} - \mathbf{s} = \mathbf{r} - \begin{pmatrix} -4 \\ 16 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

so $\mathbf{r} = \begin{pmatrix} -5 \\ 17 \end{pmatrix}$

Arithmetic on vectors using standard basis exercise (page 25)

Q24: 3**i** - 3**k**

Q25: -5**i** + 2**j** - 5**k**

Q26: -2 (2**i** - **j** + 3**k**) = -4**i** + 2**j** - 6**k**

These are scalar multiples of each other. The vectors are therefore parallel.

Q27: Let the vector be a**i** + b**j** + c**k**

Then a**i** + b**j** + c**k** = m(2**i** - 4**j** + **k**)

Thus a = 2m, b = -4m and c = m

But c = 3 and so m = 3

The vector parallel to 2**i** - 4**j** + **k** with z = 3 is 3(2**i** - 4**j** + **k**) = 6**i** - 12**j** + 3**k**

Algebraic scalar product exercise (page 26)

Q28: $\mathbf{a} \bullet \mathbf{b} = (2 \times -3) + (3 \times 1) = -3$

Q29: $\mathbf{a} \bullet \mathbf{b} = -6 + 6 - 12 = -12$

Q30: 2 - 9 + 2 = -5

Q31: The scalar product is -16

Answers from page 27.

Q32: **b + c** = -4**i** + **j** + 2**k**

$\mathbf{a} \bullet (\mathbf{b} + \mathbf{c}) = -8 - 2 = -10$

$\mathbf{a} \bullet \mathbf{b} = -6 - 3 = -9$ and $\mathbf{a} \bullet \mathbf{c} = -2 + 1 = -1$

so $\mathbf{a} \bullet \mathbf{b} + \mathbf{a} \bullet \mathbf{c} = -10$ as required.

Q33: $\mathbf{a} \bullet \mathbf{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = a_1b_1 + a_2b_2 + a_3b_3$ and

$\mathbf{b} \bullet \mathbf{a} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = b_1a_1 + b_2a_2 + b_3a_3$

But $a_1b_1 + a_2b_2 + a_3b_3 = b_1a_1 + b_2a_2 + b_3a_3$ by the laws of algebra.

Thus $\mathbf{a} \bullet \mathbf{b} = \mathbf{b} \bullet \mathbf{a}$

Geometric scalar product exercise (page 28)

Q34: $\mathbf{a} \bullet \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta = 4 \times 8 \cos 45^0 = 22.627$

Q35: $\mathbf{a} \bullet \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta = 2.5 \times 5 \cos 150^0 = -10.825$

Q36: $-6 = 4 \times 3 \times \cos\theta$

So $\cos\theta = -{}^1/_2$ which gives $\theta = 120°$

But the acute angle is normally stated, so the angle is 60°

Q37: let **a** = 3**i** - 4**j** and **b** = 12**i** + 5**j**

a •**b** = 16

a has length = 5

b has length = 13

So cos θ = 16 / (5 $\times$ 13) = $^{16}/_{65}$

Thus θ = 75.75°

Q38: $\overrightarrow{CA}$ = **i** - 2**j** + 2**k**

$\overrightarrow{CB}$ = 3**i** - 4**k**

$$\cos ACB = \frac{\overrightarrow{CA} \bullet \overrightarrow{CB}}{\left|\overrightarrow{CA}\right| \times \left|\overrightarrow{CB}\right|} = \frac{-5}{3 \times 5} = \frac{-1}{3} \text{ so angle ACB} = 109.47^\circ$$

The acute angle is 70.53°

Perpendicular vectors exercise (page 29)

Q39: (c**i** + 2**j** - **k**)•(**i** - 3**k**) = c + 3

but c + 3 = 0 if the vectors are perpendicular so c = -3

Q40: c = 9

Q41: **a** • **b** = (-4 $\times$ 2) + (-3 $\times$ -2) + (2 $\times$ 1) = 0

Therefore the vectors are perpendicular.

Review exercise (page 32)

Q42:

a) $\overrightarrow{AC} = \begin{pmatrix} -9 \\ -3 \\ 9 \end{pmatrix}$

b) $\overrightarrow{BC} = \begin{pmatrix} -6 \\ -2 \\ 6 \end{pmatrix}$ and so $\overrightarrow{AC} = \frac{3}{2}\overrightarrow{BC}$

Thus the vectors are parallel but C is a common point on the two vectors
So the three points must be collinear.

Q43: **h** = $^1/_3$**p** + $^2/_3$**q**

$$h = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 4 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

The point H has coordinates (1, -1, 2)

Q44:

a) $\overrightarrow{CA} \bullet \overrightarrow{CB}$ = (-2 $\times$3) + (1 $\times$ -2) + (3 $\times$ 4) = 4

b) $\left|\overrightarrow{CA}\right| = \sqrt{14}$ and $\left|\overrightarrow{CB}\right| = \sqrt{29}$
so cos $\theta = 4 \div (\sqrt{14} \times \sqrt{29}) = 0.1985$
$\theta = 78.55°$

Advanced review exercise (page 33)

Q45:

a) **p** = **q** = **r** = 6 (equilateral triangles and base = 6cm)
$\mathbf{p} \bullet \mathbf{r} = |p|\ |q|\ \cos 60^0 = 36 \cos 60 = 18$

b) $\mathbf{p} \bullet (\mathbf{r} + \mathbf{q}) = \mathbf{p} \bullet \mathbf{r} + \mathbf{p} \bullet \mathbf{q}$
But $\mathbf{p} \bullet \mathbf{r} = \mathbf{p} \bullet \mathbf{q} = 18$
so $\mathbf{p} \bullet (\mathbf{r} + \mathbf{q}) = 36$

c) $\overrightarrow{AD} = \overrightarrow{AC} + \overrightarrow{CD}$
But $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$ and $\overrightarrow{CD} = \overrightarrow{BE}$
So $\overrightarrow{AD} = -\overrightarrow{BA} + \overrightarrow{BC} + \overrightarrow{BE} = -\mathbf{p} + \mathbf{r} + \mathbf{q}$

d) cos DAB $= \dfrac{-\mathbf{p} \bullet (-\mathbf{p} + \mathbf{q} + \mathbf{r})}{36} = \dfrac{(-\mathbf{p} \bullet -\mathbf{p}) - \mathbf{p} \bullet \mathbf{q} - \mathbf{p} \bullet \mathbf{r}}{36} = \dfrac{0}{36} = 0$
Thus angle DAB = $\cos^{-1} 0 = 90°$

Q46:

1.
   a) $\mathbf{a} \bullet \mathbf{a} = 3 \times 3 = 9$
   b) $\mathbf{b} \bullet \mathbf{a} = 4 \times 3 \cos 60^0 = 6$
   c) $\mathbf{b} \bullet \mathbf{b} = 4 \times 4 = 16$

2. $\mathbf{v} \bullet \mathbf{v} = (3\mathbf{a} - 2\mathbf{b})(3\mathbf{a} - 2\mathbf{b})$
$= 9\mathbf{a} \bullet \mathbf{a} - 6\mathbf{b} \bullet \mathbf{a} - 6\,\mathbf{a} \bullet \mathbf{b} + 4\mathbf{b} \bullet \mathbf{b}$
$= 81 - 36 - 36 + 64 = 73$ and the length of **v** is $\sqrt{73}$

Q47:

a) The midpoint M = ((4 + 8)/2, (-1 + 3)/2, (3 - 1)/2) = (6, 1, 1)

b)
$$\overrightarrow{CT} = \frac{2}{3}\overrightarrow{CM} = \frac{2}{3}\begin{pmatrix} 6 \\ -3 \\ -3 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \\ -2 \end{pmatrix}$$
Since $\mathbf{c} = \begin{pmatrix} 0 \\ 4 \\ 4 \end{pmatrix}$ and $\overrightarrow{CT} = \begin{pmatrix} 4 \\ -2 \\ -2 \end{pmatrix} = \mathbf{t} - \mathbf{c}$ then $\mathbf{t} = \begin{pmatrix} 4 \\ 2 \\ 2 \end{pmatrix}$
The coordinates of the point T are (4, 2, 2)

c)
$$\overrightarrow{BT} = \begin{pmatrix} -4 \\ -1 \\ 3 \end{pmatrix} \text{ and } \overrightarrow{TD} = \begin{pmatrix} -8 \\ -2 \\ 6 \end{pmatrix}$$

Thus the vector TD is 2 $\times$ the vector BT: they are parallel

but B is a common point and so B, T, D are collinear points and the ratio of in which T divides BD is 1 : 2

Q48:

a) $\left|\overrightarrow{BR}\right| = \sqrt{(7 - 5)^2 + (2 - (-5))^2 + (3 - (-1))^2} = 8.31$

The distance from B to R is therefore 2 $\times$ 8.31km = 16.62 km

b) $\overrightarrow{TC} = \begin{pmatrix} 12 \\ -4 \\ 1 \end{pmatrix}$ and $\overrightarrow{BR} = \begin{pmatrix} 2 \\ 7 \\ 4 \end{pmatrix}$

If the vectors are perpendicular then the scalar product is zero

$\overrightarrow{TC} \bullet \overrightarrow{BR} = 24 - 28 + 4 = 0$

The direction of the beam TC is perpendicular to the direction of the beam BR

c)

The answer is 36.72°

$\overrightarrow{CT} = \begin{pmatrix} -12 \\ 4 \\ -1 \end{pmatrix}$, $\overrightarrow{CR} = \begin{pmatrix} -5 \\ 6 \\ 2 \end{pmatrix}$,

$\left|\overrightarrow{CT}\right| = \sqrt{(144 + 16 + 1)} = 12.69$, $\left|\overrightarrow{CR}\right| = \sqrt{(25 + 36 + 4)} = 8.06$

$\cos TCR = \dfrac{\overrightarrow{CT} \bullet \overrightarrow{CR}}{\left|\overrightarrow{CT}\right| \times \left|\overrightarrow{CR}\right|} = \dfrac{82}{102.3} = 0.802$

Angle TCR = 36.72°

Set review exercise (page 34)

Q49: This answer is only available on the web.

Q50: This answer is only available on the web.

Q51: This answer is only available on the web.

Q52: This answer is only available on the web.

2 Further differentiation and integration

Revision exercise (page 38)

Q1:

a) f' (x) = $6x - \frac{1}{2x^2}$

b) f' (x) = $4x - 5$

c) f' (x) = $\frac{5}{2}\sqrt{x^3} - \frac{3}{\sqrt{x}}$

d) f' (x) = $8x + \frac{3}{x^2}$

Q2:

a) $2x^3 - \frac{3}{4}x^{4/3} + C$

b) $\frac{1}{2}x^2 - \frac{4}{5}x^5 + C$

c) $\frac{2}{5}\sqrt{z^5} - \frac{2}{3}\sqrt{z} + C$

Q3:

a) 21

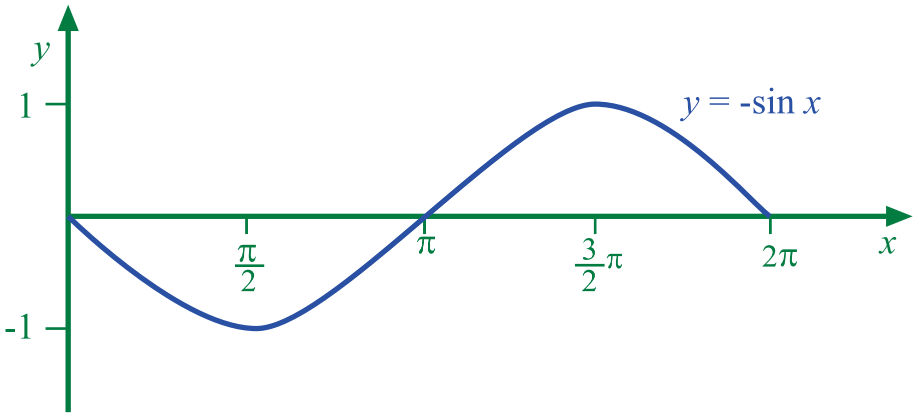
b) 9

c) $1\frac{1}{3}$

Answers from page 40.

Q4:

x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
m_T	0	- 1	0	1	0



Q5: c) - sin x

Exercise 2 (page 42)

Q6:

a) f ' (x) = 2 cos x

b) f ' (x) = -3 sin x

c) f ' (x) = - 5 cos x

d) f ' (x) = 4 sin x

e) f ' (x) = 5 cos x + sin x

f) f ' (x) = - 3 sin x + 2 cos x

g) f ' (x) = 6 x + 5 sin x

h) f ' (x) = 7 cos x - sin x

i) f' (x) = 2 cos x + $\frac{5}{x^2}$

j) f' (x) = - $\frac{2}{x^{3/2}}$ + 3 sin x

Q7:

a) - 3 sin x

b) 5 cos t + 2 sin t

c) - $\frac{10}{u}$ - 3 cos u

d)

Q8:

a) 2

b) $\sqrt{2}$

c) 1

d) 0

Q9:

a) 3

b) $\frac{3\sqrt{2} + 2}{2}$

c) $\sqrt{2}$

d) - $\sqrt{2}$

Q10: f ' (x) = 3 cos x = 0 at turning points and thus x = $\frac{\pi}{2}$ or $\frac{3\pi}{2}$

When x = $\frac{\pi}{2}$ then f (x) = 3 sin $\left(\frac{\pi}{2}\right)$ = 3

and when x = $\frac{3\pi}{2}$ then f (x) = 3 sin $\left(\frac{3\pi}{2}\right)$ = - 3

Q11: $y = x + \sqrt{3} - \frac{\pi}{3}$

Q12: $y = -\sqrt{2}\,x + \sqrt{2}\left(\frac{\pi + 8}{4}\right)$

Q13: $\frac{dy}{dx} = 1 + \cos x$ and since $-1 \leq \cos x \leq 1$ then $0 \leqslant \frac{dy}{dx} \leqslant 2$

Thus $\frac{dy}{dx} \geqslant 0$ and so $y = x + \sin x$ is never decreasing.

Q14:

a) When x = 0 the gradient is 2
   When $x = \frac{\pi}{2}$ the gradient is π

b) 8.9 $^{\circ}$

Q15: $y = 6x - 2\pi$

Answers from page 44.

Q16:

1. $f'(x) = 2(x + 3)$
2. $f'(x) = 3(x + 3)^2$
3. $f'(x) = 2(x - 2)$
4. $f'(x) = 3(x - 2)^2$

Answers from page 44.

Q17:

a) f ' (x) = 4 (x + 3) $^3 f'(x) = 4\,(x + 3)^3$

b) f ' (x) = 5 (x + 3) $^4 f'(x) = 5\,(x + 3)^4$

c) f ' (x) = 6 (x + 3) $^5 f'(x) = 6\,(x + 3)^5$

d) $\frac{dy}{dx} = 4\,(x - 2)^3$

e) $\frac{dy}{dx} = 5\,(x - 2)^4$

f) $\frac{dy}{dx} = 6\,(x - 2)^5$

Exercise 3 (page 45)

Q18:

a) $8(x + 6)^7$

b) $5(x - 2)^4$

c) $-3(x + 8)^{-4}$

d) $\frac{3}{2}(x-4)^{1/2}$

e) $-\frac{2}{(x+3)^3}$

f) $\frac{5}{2}\sqrt{(x-6)^3}$

g) $-\frac{1}{(x-1)^2}$

h) $\frac{2}{3\sqrt[3]{x+5}}$

Q19: f ' (0) = - 4 and f ' (1) = 0

Q20: 2x + 27y - 7 = 0

Q21:

a) $\cos x + 8(x+9)^7$

b) $-\frac{1}{(x-5)^2} + 3\sin x$

c) $4\cos x + \frac{1}{4(x-2)^{3/4}}$

d) $24x^7 - 5(x+4)^4$

Answers from page 47.

Q22:

a) $\frac{dy}{dx} = 6(3x+1) = 2(3x+1) \times 3$

b) $\frac{dy}{dx} = 10(5x-2) = 2(5x-2) \times 5$

c) $\frac{dy}{dx} = 6(2x+5)^2 = 3(2x+5)^2 \times 2$

d) $\frac{dy}{dx} = 12(4x-1)^2 = 3(4x-1)^2 \times 4$

Answers from page 47.

Q23:

a) $\frac{dy}{dx} = 8(2x+1)^3$

b) $\frac{dy}{dx} = 12(3x+1)^3$

c) $\frac{dy}{dx} = 20(5x+1)^3$

d) $\frac{dy}{dx} = 40(5x-2)^7$

e) $\frac{dy}{dx} = 28(7x+5)^3$

Exercise 4 (page 48)

Q24:

1. $\frac{dy}{dx} = 21\ (3x + 5)^6$
2. $\frac{dy}{dx} = -12\ (1 - 4x)^2$
3. $\frac{dy}{dx} = 3\ (6x + 1)^{-1/2}$
4. $\frac{dy}{dx} = -12\ (2 - 3x)^3$
5. $\frac{dy}{dx} = -10\ (2x + 1)^{-6}$
6. $\frac{dy}{dx} = -\frac{28}{(7x + 2)^5}$
7. $\frac{dy}{dx} = \frac{2}{\sqrt{(5 - 4x)^3}}$
8. $\frac{dy}{dx} = \frac{-5}{(5x + 6)^2}$

Exercise 5 (page 52)

Q25:

a) $h'(x) = 6\ (2x - 5)^2$

b) $h'(x) = 7\ (3x^2 + 2)\ (x^3 + 2x)^6$

c) $h'(x) = -30x\ (4 - 3x^2)^4$

d) $h'(x) = \frac{16x}{\sqrt{16x^2 + 9}}$

e) $h'(x) = \frac{-15x^4}{\left(3x^5 + 2\right)^2}$

f) $h'(x) = 5\left(1 - \frac{1}{x^2}\right)\left(\frac{1}{x} + x\right)^4$

Q26:

a) $\frac{dy}{dx} = 2\cos\ (2x)$

b) $\frac{dy}{dx} = -\frac{1}{2}\sin\left(\frac{1}{2}x\right)$

c) $\frac{dy}{dx} = -2\sin(2x - 3)$

d) $\frac{dy}{dx} = -\frac{1}{2}\cos\left(5 - \frac{x}{2}\right)$

e) $\frac{dy}{dx} = 2x \cos\left(x^2 + 4\right)$

f) $\frac{dy}{dx} = -6 \sin\left(6x + \frac{\pi}{4}\right)$

Q27:

a) $f'(x) = 2 \sin x \cos x = \sin 2x$

b) $f'(x) = -5 \sin x \cos^4 x$

c) $f'(x) = -\frac{3 \cos x}{\sin^4 x}$

d) $f'(x) = \frac{\cos x}{2\sqrt{\sin x}}$

Q28: $f'(x) = 4 \sin x (1 - 2 \cos x)$ therefore $f'\left(\frac{\pi}{6}\right) = 2\left(1 - \sqrt{3}\right)$ and $f'\left(\frac{\pi}{3}\right) = 0$

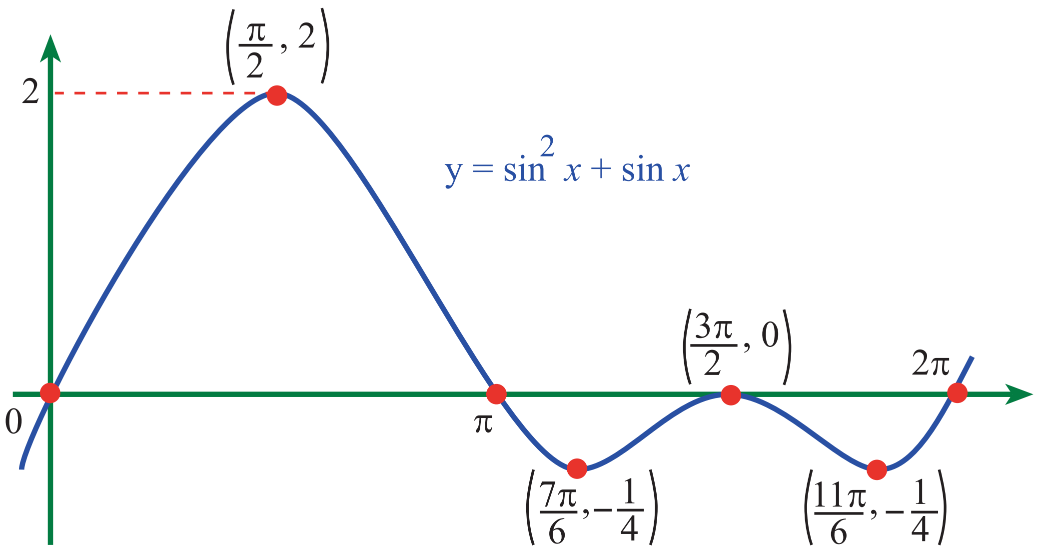
Q29: $4x + 9y - 7 = 0$

Q30:

a) Maximum turning points at $\left(\frac{\pi}{2}, 2\right)$ and $\left(\frac{3\pi}{2}, 0\right)$

Minimum turning points at $\left(\frac{7\pi}{6}, -\frac{1}{4}\right)$ and $\left(\frac{11\pi}{6}, -\frac{1}{4}\right)$

b)



Q31: $\frac{dy}{dx} = 5 + 2 \cos 2x$ and since $-1 \leq \cos 2x \leq 1$ then $3 \leqslant \frac{dy}{dx} \leqslant 7$

Thus $\frac{dy}{dx} \geqslant 0$ and so $y = 5x + \sin 2x$ is never decreasing.

Exercise 6 (page 54)

Q32:

a) $-3 \cos x + C$

b) $\sqrt{5}$ sin x + C

c) π cos θ + C

d) 8u - $\sqrt{2}$ sin u + C

e) $\frac{2\sqrt{x^3}}{3}$ - 3 cos x + C

f) - 6 cos t - sin t + C

g) $6\sqrt{\omega}$ - 7 cos ω + C

h) $\frac{1}{3}$ sin x + $\sqrt{5}$ sin x + 2 π x + C

Q33:

a) 2

b) $\sqrt{2}$ - 1

c) $\frac{1}{2}\left(\sqrt{3} - 5\right)$

d) π - $2\frac{1}{2}$

e) $\frac{\pi^2}{9} + \frac{3\sqrt{3}}{2}$

f) 10 + π^2

Q34:

a) 2

b) -2

c) 0

d) 4 square units

Q35:

a) At the points of intersection

sin x = cos x

$\frac{\sin x}{\cos x} = 1 \quad (\cos x \neq 0)$

tan x = 1

x = $\frac{\pi}{4}$ or $\frac{5\pi}{4}$

Thus$a$ has coordinates $\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$ and b has coordinates $\left(\frac{5\pi}{4}, \frac{1}{\sqrt{2}}\right)$

b) Shaded area = $2\sqrt{2}$ square units

Q36:

a) s (t) = 5t - 2 cost + 2

b) s$\left(\frac{\pi}{3}\right)$ = $\frac{5\pi}{3}$ + 1

Exercise 7 (page 57)

Q37:

a) $\frac{1}{16}(2x + 5)^8 + C$

b) $\frac{1}{15}(3t - 1)^5 + C$

c) $-\frac{1}{4}(4x - 3)^{-1} + C$

d) $\frac{1}{15}(2 - 3x)^{-5} + C$

e) $\frac{1}{6}(4r + 1)^{3/2} + C$

f) $5(x + 6)^{1/5} + C$

Q38:

a) $\frac{2}{3}\sqrt{(x + 9)^3} + C$

b) $-\frac{1}{4(2x - 1)^2} + C$

c) $\frac{2}{3}\sqrt{3x + 4} + C$

d) $-\frac{1}{10}\sqrt{(1 - 4x)^5} + C$

e) $\frac{3}{10}\sqrt[3]{(5x - 1)^2} + C$

f) $-\frac{1}{6\sqrt{(4x + 7)^3}} + C$

Q39:

a) $y = -\frac{1}{10}(1 - 2x)^5 + C$

b) $y = -\frac{1}{4(4t + 3)} + C$

Q40:

a) 14

b) $-\frac{5}{4}$

c) 2

d) $\frac{1}{9}$

e) 18

f) -1

Q41:

a) $u = -\frac{2}{3}\sqrt{(5 - t)^3}$

b) $x = (3t + 1)^{1/3} + 4$

Q42:

a) 21

b) $\sqrt{7} - 1 \approx 1.65$

Exercise 8 (page 59)

Q43:

a) $\frac{1}{2} \sin (2x - 1) + C$

b) $-\frac{1}{3} \cos (3x + 5) + C$

c) $3 \sin (2x - 3) + C$

d) $-\frac{1}{4} \cos \left(4x + \frac{\pi}{3}\right) + C$

e) $3 \sin \left(\frac{1}{3}x\right) + C$

f) $-10 \cos \left(\frac{1}{2}x + \frac{\pi}{3}\right) + C$

Q44:

a) $\frac{1}{2}$

b) $\frac{3}{4}$

c) $\frac{6}{\sqrt{2}} = 3\sqrt{2}$

d) $-\frac{1}{3\sqrt{2}} = -\frac{\sqrt{2}}{6}$

Q45:

a) 1 square unit.

b) $\frac{\sqrt{3}+1}{4}$ square units.

Q46:

a) At the points of intersection $\sqrt{3} \sin 2x = \cos 2x$ hence

$$\sqrt{3}\frac{\sin 2x}{\cos 2x} = \frac{\cos 2x}{\cos 2x} \quad (\cos 2x \neq 0, \text{ at the points of intersection})$$

$$\sqrt{3}\tan 2x = 1$$

$$\tan 2x = \frac{1}{\sqrt{3}}$$

$$\Rightarrow 2x = \frac{\pi}{6} \text{ or } \frac{7\pi}{6}$$

$$\Rightarrow x = \frac{\pi}{12} \text{ or } \frac{7\pi}{12}$$

b) 2 square units

Q47:

a)

$$\cos 2x = 2\cos^2 x - 1$$

$$\cos 2x + 1 = 2\cos^2 x$$

$$\frac{1}{2}\cos 2x + \frac{1}{2} = \cos^2 x$$

$$\cos^2 x = \frac{1}{2}\cos 2x + \frac{1}{2}$$

b) $\frac{1}{4}\sin 2x + \frac{1}{2}x + C$

Q48:

a) $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

$\frac{1}{2}x - \frac{1}{4}\sin 2x + C$

Q49: $-\frac{1}{4}\cos 2x + C$

Q50:

a) $-\frac{1}{2}\sin 2x + C$

b) $\frac{1}{12}\sin 6x + \frac{1}{2}x + C$

c) $-\frac{1}{5}\cos 10x + C$

d) $\frac{1}{2}x^2 - \frac{3}{2}x + \frac{3}{8}\sin 4x + C$

Review exercise in further diff and int (page 62)

Q51:

a) $3\sin x$

b) $\frac{dy}{dx} = 4\cos x$

Q52: $f'(x) = -4(x-5)^{-5}$

Q53:

a) $-\frac{2}{3}\cos x + C$

b) $-4\sin x + C$

c) $3\frac{3}{4}$

Advanced review exercise in further diff and int (page 63)

Q54:

a) $-2\sin x\cos x - \frac{6}{5x} = -\sin 2x - \frac{6}{5x^3}$

b) 24

c) $\frac{2}{x^{1/3}} + 15\cos 3x$

d) $\frac{3}{\sqrt{x}} + 2\sin x\cos x = \frac{3}{\sqrt{x}} + \sin 2x$

e) $-\frac{\cos x}{2\sqrt{1-\sin x}}$

Q55:

a) $x^4 - \frac{2}{3}x^{3/2} + \cos x + C$

b) 1

c) $-\frac{5}{4}$

d) $\int \sqrt{1+2x}\,dx = \frac{1}{3}(1+2x)^{3/2} + C$ and $\int_0^4 \sqrt{1+2x}\,dx = 8\frac{2}{3}$

Q56:

a)

$$\begin{aligned}(\cos x + \sin x)^2 &= \cos^2 x + 2\cos x\sin x + \sin^2 x\\ &= \left(\cos^2 x + \sin^2 x\right) + 2\cos x\sin x\\ &= 1 + \sin 2x\end{aligned}$$

b) $\frac{\pi}{4} + \frac{1}{2}$

Q57:

$\sin^5 x = 5\sin^4 x\cos x$

$\int \sin^4 x\cos x\,dx = \frac{1}{5}\sin^5 x + C$

Q58:

$$\begin{aligned}\cos 3x &= \cos(2x + x)\\ &= \cos 2x \cos x - \sin 2x \sin x\\ &= \left(2\cos^2 x - 1\right)\cos x - 2\sin x \cos x \sin x\\ &= 2\cos^3 x - \cos x - 2\sin^2 x \cos x\\ &= 2\cos^3 x - \cos x - 2\left(1 - \cos^2 x\right)\cos\\ &= 2\cos^3 x - \cos x - 2\cos x + 2\cos^3 x\\ &= 4\cos^3 x - 3\cos x\end{aligned}$$

$$\begin{aligned}\int \cos^3 x\,dx &= \frac{1}{4}\int (\cos 3x + 3\cos x)\,dx\\ &= \frac{1}{12}\sin 3x + \frac{3}{4}\sin x + C\end{aligned}$$

Q59: $1 - \frac{\sqrt{3}}{4}$ square units

Q60:

a) Note that

$$\begin{aligned}x^2 + y^2 &= 20\\ y^2 &= 20 - x^2\\ y &= \sqrt{20 - x^2}\end{aligned}$$

Hence

$$\begin{aligned}T &= 2x + y\\ &= 2x + \sqrt{20 - x^2}\end{aligned}$$

b) When

$$\begin{aligned}T &= 2x + \sqrt{20 - x^2}\\ &= 2x + \left(20 - x^2\right)^{1/2}\end{aligned}$$

then

$$\begin{aligned}\frac{dT}{dx} &= 2 + \frac{1}{2}\left(20 - x^2\right)^{1/2}(-2x)\\ &= 2 - \frac{x}{\sqrt{20 - x^2}}\end{aligned}$$

Since $\frac{dT}{dx} = 0$ at a stationary value then

$$\begin{aligned}2 - \frac{x}{\sqrt{20 - x^2}} &= 0\\ \frac{x}{\sqrt{20 - x^2}} &= 2\\ x &= 2\sqrt{20 - x^2}\end{aligned}$$

c)

$$\begin{aligned} x &= 2\sqrt{20 - x^2} \\ x^2 &= 4\left(20 - x^2\right) \\ x^2 &= 80 - 4x^2 \\ 5x^2 &= 80 \\ x^2 &= 16 \\ x &= 4 \text{ (only positive values are valid)} \end{aligned}$$

When $x = 4$ then $y = 2$ (from $x^2 + y^2 = 20$)
Thus the greatest length of the arms of the "L" is

$$\begin{aligned} T &= 2x + y \\ &= 10 \text{ units.} \end{aligned}$$

Set review exercise in further diff and int (page 65)

Q61: This answer is only available on the course web site.

Q62: This answer is only available on the course web site.

Q63: This answer is only available on the course web site.

3 Logarithmic and exponential functions

Revision exercise (page 68)

Q1:

a) a^7 Remember to add the indices.

b) b^6 Remember to subtract the indices.

c) a^6 For a power of a power - multiply the indices.

d) $\frac{a^5b^4c}{abc} \times \frac{b^2c}{a^2} \div \frac{ab}{c^2} = \frac{a^2b^5c}{1} \times \frac{c^2}{ab} = ab^4c^3$

Q2: When x = 0, the function crosses the y-axis.

That is, when y = 2

The point where it crosses the y-axis is (0, 2)

When y = 0, the function crosses the x-axis.

That is, when $x^2 - 3x + 2 = 0 \Rightarrow (x - 2)(x - 1) = 0$

It crosses the x-axis at the two points (2, 0) and (1, 0)

Q3: Using the point (0, 3) in the equation gives

$3 = a^0 + k \Rightarrow 3 = 1 + k \Rightarrow k = 2$

Using the point (2, 11) and k = 2 in the equation gives

$11 = a^2 + 2 \Rightarrow a^2 = 9 \Rightarrow a = 3$

(remember that a is positive)

The equation of the function is $f(x) = 3^x + 2$

Q4:

A) For the related exponential the formula will take the form $y = 2^x + k$
Reflection in the line y = x shows that $y = 2^x + k$ will pass through the point (0, 3) and this gives $3 = 2^0 + k \Rightarrow k = 2$
The equation of the exponential graph is $y = 2^x + 2$ and the logarithmic function is $f(x) = \log_2 x + 2$

B) In a similar manner to graph A, k can be found and is equal to -3
The equation is $f(x) = \log_3 x - 3$

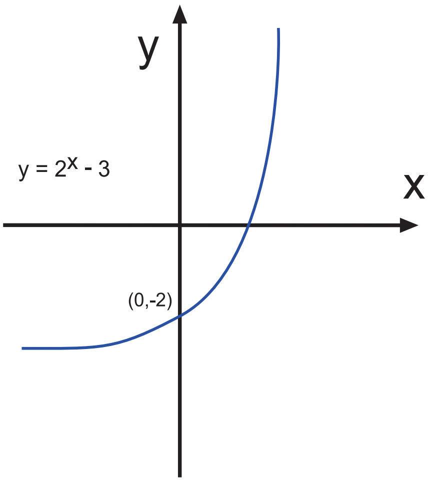
C) The related exponential passes through the points (0, -2) and (2, 1)
The point (0, -2) when substituted into $y = a^x + k$ gives k = -3
Then using (2, 1) gives $1 = a^2 - 3 \Rightarrow a = 2$
Thus the equation of the log graph is $f(x) = \log_2 x - 3$

D) The related exponential graph, $y = a^x + k$ passes through the points (0, -1) and (3, 6)
Using (0, -1) gives k = -2 and subsequently using (3, 6) gives $6 = a^3 - 2 \Rightarrow a = 2$
The equation of the logarithmic graph is $f(x) = \log_2 x - 2$

Q5: There are two ways to obtain the answer.

Either use coordinates and the shape of an exponential graph. When x = 0, y = -2 and plot.

Or use the rules for related graphs, then this graph is 3 units down the y-axis from the graph of $y = 2^x$



Calculator button exercise (page 73)

Q6:

a) 0
b) 0
c) 1
d) 1
e) 2
f) -3
g) -6.91
h) 1.10
i) 2

Q7:

a) 2.72
b) 1
c) 2.72
d) 31.62
e) 0.57
f) 0.99
g) 80.34
h) 148.41
i) 6.58

Relationship exercise (page 74)

Q8:

a) $\log_a x = y$
b) $\log_a 3 = x$
c) $\log_2 9 = x$
d) $\log_b 3 = 5$
e) $\log_6 p = 2$
f) $\log_2 y = -3$

Q9:

a) $a^y = x$
b) $a^9 = x$
c) $4^2 = x$
d) $x^k = 2$
e) $x^2 = y$
f) $2^4 = s$
g) $3^5 = b$

Q10:

a) $y = 0.69897$ ($\log_{10} 5$)
b) $x = 7.389$ (e^2 in the calculator)
c) $k = 1.0986$ (calculator)
d) $x = 3.2623$ ($10^{0.5}$)
e) $a = 2$ ($a^3 = 8$)
f) $b = 3$ ($b^4 = 81$)
g) $x = 9$ ($x = 3^2$)
h) $y = 32$ (2^5)

Log laws exercise (page 78)

Q11:

a) $3 \log_2 6 - 3 \log_2 3 + 3 \log_2 2$
$3 (\log_2 2 + \log_2 3) - 3 \log_2 3 + 3 \log_2 2$
$= 3 + 3 \log_2 3 - 3 \log_2 3 + 3$
$= 6$
b) Since $\log_e 1 = 0$ the answer is 0
c) $\log_5 625 - \log_4 64$
$= 4 \log_5 5 - 3 \log_4 4$
$= 1$

d) $2 \log_x x^2 + \log_x 1$
$= 4 \log_x x + 0 = 4$

e) $\log_e 32 - 5 \log_e 2$
$= 5 \log_e 2 - 5 \log_e 2 = 0$

Q12:

a) $2 \log_x 5 + \log_x 4 - \log_x 50$
$= \log_x (25 \times 4 \div 50)$
$= \log_x 2$

b) $\log_3 12 - \log_3 16$
$= \log_3 (^{12}/_{16}) = \log_3 (^3/_4) = \log_3 3 - \log_3 4 = 1 - \log_3 4$

c) $2 \log_{10} x + \log_{10} (x - 2)$
$= \log_{10} x^2 + \log_{10} (x - 2)$
$= \log_{10} x^2(x - 2) = \log_{10} (x^3 - 2x^2)$

d) $\log_a 4 + \log_b 25 - \log_b 5 - \log_a 2$
$= \log_a (^4/_2) + \log_b (^{25}/_5)$
$= \log_a 2 + \log_b 5$

e) $\log_{10} 30 - \log_2 20$
$= \log_{10} 3 + \log_{10} 10 - \log_2 2 - \log_2 10$
$= \log_{10} 3 - \log_2 10$

Q13:

a) $2 \log_2 6 - \log_2 9 = x \Rightarrow$
$\log_2 (^{36}/_9) = x \Rightarrow$
$\log_2 4 = x \Rightarrow$
$2 \log_2 2 = x$
$x = 2$

b) $-2 + \log_2 x = 0 \Rightarrow$
$\log_2 x = 2 \Rightarrow$
$x = 2^2 \Rightarrow$
$x = 4$

c) $\log_e x^2 - \log_e 36 = \log_e 1 \Rightarrow$
$\log_e x^2 - \log_e 36 = 0 \Rightarrow$
$\log_e (x^2 \div 36) = 0$
$x^2 \div 36 = e^0 \Rightarrow$
$x^2 = 36 \Rightarrow x = 6$

d) $3 \log_3 x - \log_x 27 = 0 \Rightarrow$
$\log_3 x^3 = \log_x 3^3 \Rightarrow$
$x = 3$

Equation solving exercise (page 80)

Q14: $\log_{15}(x - 3) + \log_{15}(x - 1) = 1$

$\log_{15}(x - 3)(x - 1) = 1 \Rightarrow$

$(x - 3)(x - 1) = 15^1 = 15 \Rightarrow$

$x^2 - 4x + 3 - 15 = 0$

$(x - 6)(x + 2) = 0$

$x = 6$ or $x = -2$

-2 is a solution to the quadratic but is not a solution to the original log equation since for $x = -2$, $\log_{15}(x - 1)$ is undefined and that solution must be rejected

$x = 6$ is the only solution

Q15: $\log_3(x^2 - 4) - \log_3(x - 2) - 2 = 0$

$\log_3(x^2 - 4) - \log_3(x - 2) = 2\log_3 3 = \log_3 9 \Rightarrow$

$^{((x + 2)(x - 2))}/_{(x - 2)} = 9 \Rightarrow$

$x + 2 = 9$

$x = 7$

Q16: $\log_{10} 2^x = \log_{10} 7$

$x \log_{10} 2 = \log_{10} 7$

$x = \log_{10} 7 \div \log_{10} 2 = 2.807$ correct to 3 d.p.

Q17: $\log_e 3^x = \log_e e$

$x \log_e 3 = 1$

$x = 1 \div \log_e 3 = 0.910$ correct to 3 d.p.

Q18: $x^3 = 125$

$x = 5$

Q19: $\log_5(x - 3)^2 = 2$

$\log_5(x - 3)(x - 3) = 2 \Rightarrow$

$(x - 3)(x - 3) = 5^2 = 25 \Rightarrow$

$x^2 - 6x + 9 - 25 = 0$ i.e. $x^2 - 6x - 16 = 0$

$(x - 8)(x + 2) = 0$

$x = 8$ or $x = -2$

Q20: $\log_3 3 + \log_3 2 = x \Rightarrow$

$\log_3 6 = x \Rightarrow$

$3^x = 6 \Rightarrow$

$\log_{10} 3^x = \log_{10} 6 \Rightarrow$

$x \log_{10} 3 = \log_{10} 6 \Rightarrow$

$x = \log_{10} 6 \div \log_{10} 3 = 1.63$ correct to 2 d.p.

Q21: $\log_2 (x - 2)(x + 2) - \log_2 (x - 2) = 5$

$\log_2 (x + 2) = 5 \Rightarrow$

$(x + 2) = 2^5 = 32 \Rightarrow$

$x = 30$

Log and exp problems (page 83)

Q22:

a) $500 = 50\ e^{40k} \Rightarrow$
   $\log_e 10 \div 40 = k$
   $k = 0.0576$

b) $2 = e^{0.0576t} \Rightarrow$
   $t = \log_e 2 \div 0.0576 = 12.03$ seconds
   It takes 12 seconds to double in volume.

c) $1000000 = 75\ e^{0.0576t} \Rightarrow$
   $\log_e ({}^{1000000}/_{75}) \div 0.0576 = t$
   $t = 164.896$ seconds
   The maximum time that the experiment can run is 164 seconds, assuming that the volume of the gas is 75 grams at the beginning of the experiment.

Q23: $M_1 = 0.5\ M_0$

$0.5 = e^{-0.019t}$

$\log_e 0.5 = \log_e e^{-0.019t}$

$\log_e 0.5 = -0.019t$

$t = \log_e 0.5 \div (-0.019) = 36.48$

The half life is 36.48 years.

$5 = 35\ e^{-0.019t} \Rightarrow t = 102.42$ years.

Q24:

a) 150 since $e^{1.02t} = 1$ when $t = 0$

b) $2 = e^{1.02t} \Rightarrow$
   $\log_e 2 = 1.02t \Rightarrow$
   $t = 0.6796$ hours = 40.77 minutes.

c) After 4 hours there are N bacteria where $N = 150\ e^{4.08} = 8871.8$ bacteria.
   $3000 = 8871.8\ e^{-0.03t} \Rightarrow$
   $t = \log_e ({}^{3000}/_{8871.8}) \div -0.03 \Rightarrow$
   $t = 36.142$ hours = 36 hours and 9 minutes.

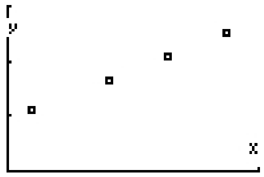
Q25:

a) $1 = 1000\ e^{-20k}$
   $\log_e 0.001 = \log_e e^{-20k}$
   $\log_e 0.001 = -20k$
   $k = \log_e 0.001 \div (-20) = 0.3454$ correct to 4 d.p.

b) $45 = E_0\ e^{-0.3454(0.5)} \Rightarrow$
$45 \div e^{-0.3454(0.5)} = 53.48$ ml
There was initially 53 ml present.

Q26: The table of values for the $\log_e$ of each variable is constructed and the graph plotted to verify that it is close to a straight line.

$\log_e x$	0.095	0.405	0.642	0.875
$\log_e y$	1.099	1.649	2.067	2.493



Choose two points on the line and form two equations of the form Y = mX + c.

Choose, say, (0.095, 1.099) and (0.642, 2.067)

1.099 = 0.095m + c: call equation1

2.067 = 0.642m + c: call this equation 2

Subtract 1 from 2 to give

$0.968 = 0.547m \Rightarrow m = n = 1.77$

Substituting in equation 1 gives

$1.099 = 0.095 \times 1.77 + c \Rightarrow c = 0.93$

Since $c = \log_e a$ then $0.93 = \log_e a \Rightarrow$

$a = e^{0.93} = 2.53$

The equation is $y = 2.53\ x^{1.77}$

Q27:

a) $30 = 80\ e^{-7k} \Rightarrow \log_e ({}^{30}/_{80}) \div -7 = 0.14$

b) $95 = 150\ e^{-0.14t} \Rightarrow$
$t = \log_e ({}^{95}/_{150}) \div -0.14$
t = 3.26 hours = 3 hours 15.6 minutes.
The experiment can last for 3 hours and 15 minutes in safety to the nearest safe minute.

Q28: Take two points, say, (0.04, 0.31) and (0.2, 0.37)

These give equations

$0.31 = 0.04m + c$ and $0.37 = 0.2m + c \Rightarrow$

$0.06 = 0.16m \Rightarrow m = 0.375$ (= k) which in turn gives c = 0.295

$\log s = c \Rightarrow s = 10^{0.295} = 1.97$

Thus $T = 0.375\ Q^{1.97}$

Review exercise (page 87)

Q29:

a) $\log_2 (^{16}/_4) = \log_2 4 = 2 \log_2 2 = 2 \times 1 = 2$
b) $\log_e e^2 = 2 \log_e e = 2 \times 1 = 2$
c) $\log_7 (^5/_1) = \log_7 5$

Q30: $\log_x 36 = 2 \Rightarrow x^2 = 36 \Rightarrow x = 6$

The base has to be positive.

Q31: $\log_5 (2x^2 + 5x + 2) = 1$

$2x^2 + 5x + 2 = 5^1$

$2x^2 + 5x - 3 = 0$

$(2x - 1)(x + 3) = 0 \Rightarrow x = -3$ or $x = {}^1/_2$

For $x = -3$ neither $\log_5 (2x + 1)$ nor $\log_5 (x + 2)$ are defined and that solution must be rejected

Thus $x = {}^1/_2$ is the only solution

Q32: $x = e^1 = e = 2.72$ correct to two d.p.

Q33: $\log_{10} 3 = t \Rightarrow t = 1.10$ to 2 d.p.

Q34:

a) There are 1,096,633 whole bacteria. ($e^7 = 1096.6332$)
b) $500000 = e^{1.4t} \Rightarrow \log_e 50000 = 1.4t$
   $t = \log_e 500000 \div 1.4 = 9.373$ hours = 9 hours and 22 minutes.

Q35: $\log_3 (25 \times 4) - \log_3 (2 \times 10) =$

$\log_3 (^{100}/_{20}) = \log_3 5$

Advanced review exercise (page 87)

Q36:

a) 3000 since $e^{1.28t} = 1$ when $t = 0$
b) This is best kept in minutes
   $$N = 3e^{\frac{1.28 \times 80}{60}} = 16.5317$$
   There are 16531 whole bacteria.
c) After two hours using the original formula there will be $3e^{2.56} = 38807$ whole bacteria.
   The new formula assumes a starting number of 1 thousand
   (since at $t = 0$, $N = e^0 = 1$)
   After 2 more hours, this formula produces $e^{4.8}$ thousand bacteria for each one thousand present after two hours.

The full number of bacteria present is $38.80745195 \times e^{4.8} = 4715.5097$
The number after three hours is 4,715,509 whole bacteria

Q37: $w = a\,t^b \Rightarrow \log_{10} w = \log_{10} (at^b)$

$\log_{10} w = \log_{10} a + b \log_{10} t$

This is in the form $y = mx + c$

Using the two points gives two equations to solve simultaneously:

$3.7 = 1.5b + \log_{10} a$: equation 1

$5.7 = 2b + \log_{10} a$: equation2

Subtracting gives $2 = 0.5$: $b \Rightarrow b = 4$

Substituting in equation 2 gives $5.7 = 8 + \log_{10} a$

$-2.3 = \log_{10} a \Rightarrow a = 10^{-2.3} \Rightarrow a = 0.005$

The equation is $w = 0.005\,t^4$

Q38: $M_1 = 0.5\,M_0$

$0.5 = e^{-0.003t}$

$\log_e 0.5 = \log_e e^{-0.003t\log_e 0.5} = -\,0.003t$

$t = \log_e 0.5 \div (-0.003) = 231$

The half life is 231 years.

Q39: When $y = 0$, $0 = 3 \log_5 (2x + 5) \Rightarrow$

$\log_5 (2x + 5) = 0 \Rightarrow$

$2x + 5 = 5^0 = 1 \Rightarrow$

$2x = -4 \Rightarrow x = -2$

The point A has coodinates (-2, 0)

The intersection with $y = 6$ gives

$6 = 3 \log_5 (2x + 5) \Rightarrow$

$2 = \log_5 (2x + 5) \Rightarrow$

$5^2 = 2x + 5 \Rightarrow 2x = 20 \Rightarrow x = 10$

The x-coordinate of point B is 10

Q40:

a) $P_1 = P_0\,e^{-kt}$
If $P_1 = 100$, $P_0 = 150$ and $t = 8$ then
$100 = 150\,e^{-8k} \Rightarrow \log_e ({}^{100}/_{150}) = -8k \Rightarrow$
$k = \log_e ({}^{100}/_{150}) \div -8 = 0.0507$ correct to 4 d.p.

b) With $k = 0.0507$, $P_0 = 200$ and $P_1 = 75$ the equation is
$75 = 200\,e^{-0.0507t} \Rightarrow \log_e ({}^{75}/_{200}) = -0.0507t$
Thus t = 19.3457 hours = 19 hours and 20 minutes correct to the nearest safe minute.

Set review exercise (page 88)

Q41: This answer is only available on the web.

Q42: This answer is only available on the web.

Q43: This answer is only available on the web.

Q44: This answer is only available on the web.

Q45: This answer is only available on the web.

Q46: This answer is only available on the web.

Q47: This answer is only available on the web.

Q48: This answer is only available on the web.

4 Further trigonometric relationships

Revision exercise (page 92)

Q1:

a) cos 2x cos y + sin 2x sin y
b) sin a cos b + cos a sin b
c) cos m cos n - sin m sin n
d) sin p cos q - cos p sin q

Q2: 155°

Q3: x can lie in the second and fourth quadrants.

Q4: 195°

Q5: $0 \leq x < 180° \Rightarrow 0 \leq 2x < 360°$

cos 2x = 0.5 $\Rightarrow \alpha$ = 60°

Since cos is positive the angle is in either quadrants one or four.

2x = 60° or 300° and there are no further values in the range.

x = 30° or x = 150°

Answers from page 93.

Q6:

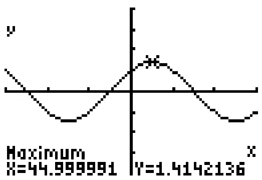
A) 3 sin 2(x + 30)
B) -6 sin 4x
C) -2 cos 0.5x
D) cos 6x

Calculator investigation (page 94)

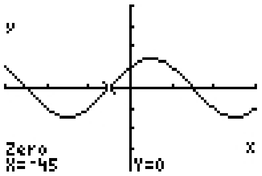
Q7: 2.828 or $2\sqrt{2} = \sqrt{8}$

Q8: 45°

Q9:



Note the maximum value of 1.414



Note the phase angle of -45°

Q10: If the equation under investigation is f (x) = a sin x + b cos x then the maximum value m of f (x) is $\sqrt{a^2 + b^2}$

Q11: If the equation under investigation is f (x) = a sin x + b cos x and the phase angle is say, α, then tan α = $^b/_a$ when related to a combined sine graph.

Wave function exercise (page 99)

Q12: Let 4 sin x + cos x = k cos (x - α) = k cos x cos α + k sin x sin α

Equate the coefficients of sin x: 4 = k sin α :- equation 1

Equate the coefficients of cos x: 1 = k cos α:- equation 2

Squaring and adding these two equations gives:

$16 + 1 = k^2 \sin^2\alpha + k^2 \cos^2 \alpha = k^2$

$k = \sqrt{17}$

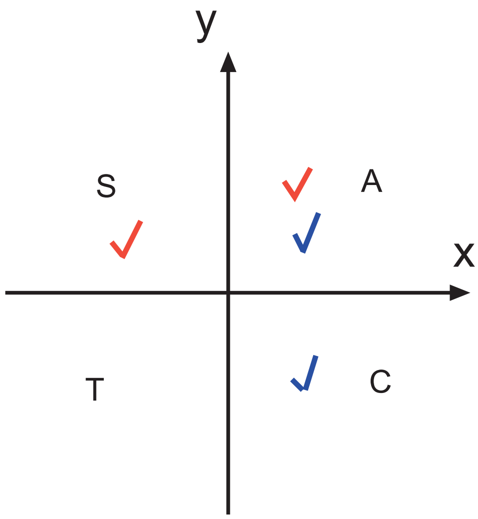
Dividing the equations ($^1/_2$) to form tan α gives:

$^4/_1$ = tan α (note the k cancels and $^{\sin}/_{\cos}$ = tan)

In the first quadrant $\alpha = \tan^{-1} 4 = 75.96°$

From equation 1: sin is positive

From equation 2: cos is positive



Then the angle required lies in the first quadrant and so α = 76° to the nearest degree.

$4 \sin x + \cos x = \sqrt{17} \cos (x - 76)°$

Q13: Let 6 sin x + 3 cos x = k cos (x + α) = k cos x cos α - k sin x sin α

Equate the coefficients of sin x: 6 = - k sin α $\Rightarrow$ -6 = k sin α :- equation 1

Equate the coefficients of cos x: 3 = k cos α:- equation 2

Squaring and adding these two equations gives:

$36 + 9 = k^2 \sin^2\alpha + k^2 \cos^2 \alpha = k^2$

$k = \sqrt{45}$

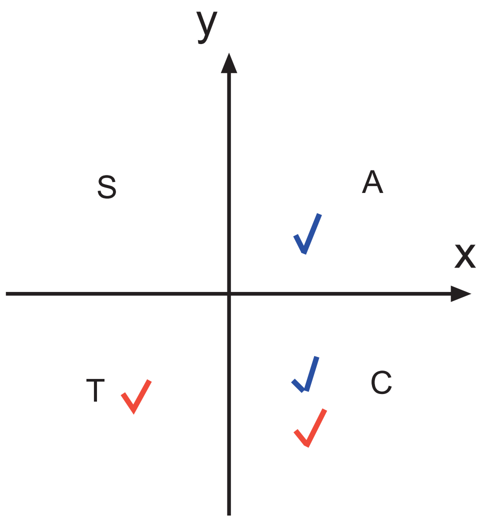
Dividing the equations ($^1/_2$) to form tan α gives:

$-^6/_3 = \tan \alpha$ (note the k cancels and $^{\sin}/_{\cos}$ = tan)

In the first quadrant $\alpha = \tan^{-1} 2 = 63.43°$

From equation 1: sin is negative

From equation 2: cos is positive



Then the angle required lies in the fourth quadrant and so
$\alpha = 360 - 63.43° = 296.56 = 297°$ to the nearest degree.

$6 \sin x + 3 \cos x = \sqrt{45} \cos (x + 297)°$

Q14: Let $3 \sin x - 2 \cos x = k \sin (x - \alpha) = k \sin x \cos \alpha - k \cos x \sin \alpha$

Equate the coefficients of sin x: 3 = k cos α :- equation 1

Equate the coefficients of cos x: $-2 = -k \sin \alpha \Rightarrow 2 = k \sin \alpha$:- equation 2

Squaring and adding these two equations gives:

$9 + 4 = k^2 \cos^2\alpha + k^2 \sin^2 \alpha = k^2$

$k = \sqrt{13}$

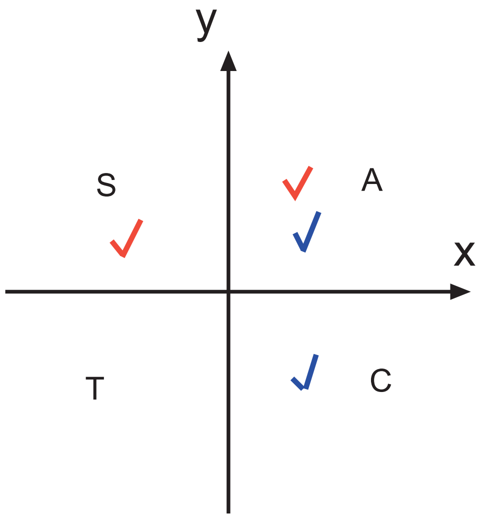
Dividing the equations ($^2/_1$) to form tan α gives:

$^2/_3 = \tan \alpha$ (note the k cancels and $^{\sin}/_{\cos}$ = tan)

In the first quadrant $\alpha = \tan^{-1} (^2/_3) = 33.69°$

From equation 1: sin is positive

From equation 2: cos is positive



Then the angle required lies in the first quadrant and so α = 34° to the nearest degree.

3 sin x - 2 cos x = $\sqrt{13}$ sin (x - 34)°

Q15: Let -sin x - 6 cos x = k sin (x + α) = k sin x cos α + k cos x sin α

Equate the coefficients of sin x: -1 = k cos α :- equation 1

Equate the coefficients of cos x: -6 = k sin α:- equation 2

Squaring and adding these two equations gives:

$1 + 36 = k^2 \cos^2\alpha + k^2 \sin^2 \alpha = k^2$

k = $\sqrt{37}$

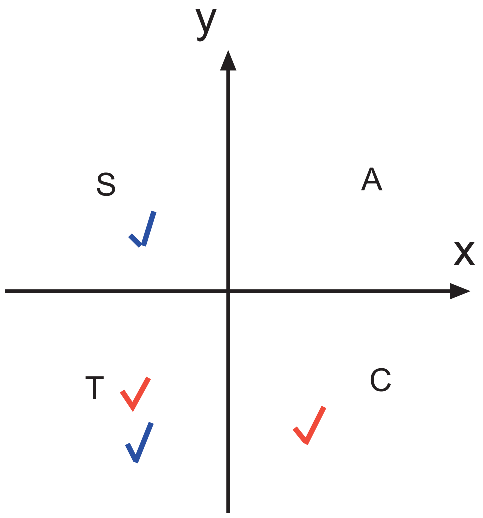
Dividing the equations ($^2/_1$) to form tan α gives:

$^{-6}/_{-1}$ = tan α (note the k cancels and $^{\sin}/_{\cos}$ = tan)

In the first quadrant $\alpha = \tan^{-1} 6 = 80.54°$

From equation 1: cos is negative

From equation 2: sin is negative



Then the angle required lies in the third quadrant and so α = 180 + 81° = 261° to the nearest degree.

-sin x - 6 cos x = $\sqrt{37}$ sin (x + 261)°

Q16: Let 2 cos x - 4 sin x = k cos (x - α) = k cos x cos α + k sin x sin α

Equate the coefficients of sin x: -4 = k sin α:- equation 1

Equate the coefficients of cos x: 2 = k cos α:- equation 2

Squaring and adding these two equations gives:

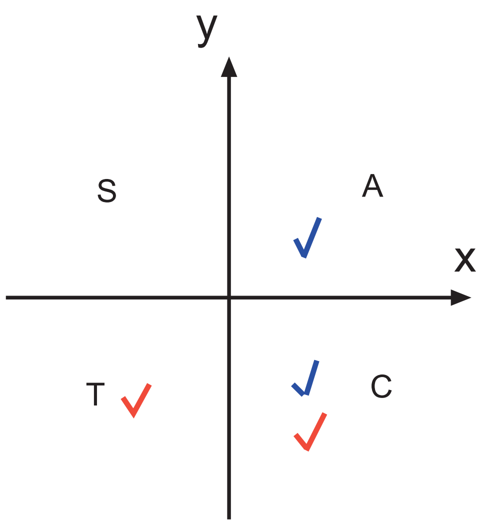
$16 + 4 = k^2 \sin^2\alpha + k^2 \cos^2 \alpha = k^2$

$k = \sqrt{20}$

Dividing the equations ($^1/_2$) to form tan α gives:

$^{-4}/_2 = \tan \alpha$ (note the k cancels and $^{\sin}/_{\cos}$ = tan)

In the first quadrant $\alpha = \tan^{-1} 2 = 63.43°$



From equation 1: sin is negative

From equation 2: cos is positive

Then the angle required lies in the fourth quadrant and so α = 360 - 63° = 297° to the nearest degree.

$2 \cos x - 4 \sin x = \sqrt{20} \cos (x - 297)°$

Q17: Let $-\sqrt{3}$ sin x + 2 cos x = k sin (x - α) = k sin x cos α - k cos x sin α

Equate the coefficients of sin x: $-\sqrt{3} = k \cos \alpha$:- equation 1

Equate the coefficients of cos x: $2 = -k \sin \alpha \Rightarrow -2 = k \sin \alpha$:- equation 2

Squaring and adding these two equations gives:

$3 + 4 = k^2 \cos^2\alpha + k^2 \sin^2 \alpha = k^2$

$k = \sqrt{7}$

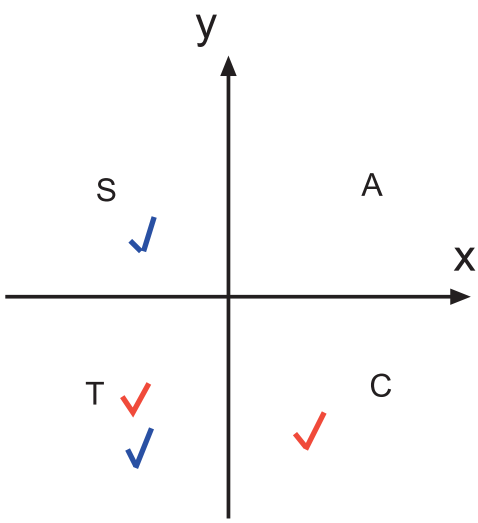
Dividing the equations ($^2/_1$) to form tan α gives:

$\frac{-2}{-\sqrt{3}} = \tan \alpha$ (note the k cancels and $^{\sin}/_{\cos}$ = tan)

In the first quadrant $\alpha = \tan^{-1} \frac{2}{\sqrt{3}} = 49.11°$

From equation 1: cos is negative

From equation 2: sin is negative



Then the angle required lies in the third quadrant and so α = 180 + 49° = 229° to the nearest degree.

- $\sqrt{3}$ sin x + 2 cos x = $\sqrt{7}$ sin (x - 229)°

Maximum and minimum exercise (page 101)

Q18:

a) The maximum is 6 (when sin (x - 300)° = -1)
The minimum is -2 (when sin (x - 300)° = 1)

b) The maximum is -4 and the minimum is -6

c) The maximum is 10 and the minimum is 0

d) 2 sin x - 4 cos x can be expressed in one of four ways:
$\sqrt{20}$ sin (x + 297)°, $\sqrt{20}$ sin (x - 63)°, $\sqrt{20}$ cos (x + 207)°, $\sqrt{20}$ cos (x - 153)°
In all of these expressions k = $\sqrt{20}$ and gives the maximum as 3 + $\sqrt{20}$ and the minimum as 3 - $\sqrt{20}$

e) Since all that is required is the maximum and minimum values, use a shortcut.
k = $\sqrt{(a^2 + b^2)}$ = $\sqrt{2}$
Let the new single trig expression be X in any of the four forms for sin x - cos x then
-1 + $\sqrt{2}$X can be a maximum when X = 1: the maximum is $\sqrt{2}$ - 1
or a minimum when X = -1: the minimum value is -1 - $\sqrt{2}$

f) Using the shortcut gives k = $\sqrt{5}$
The maximum value is $\sqrt{5}$ - 2 and the minimum value is -2 - $\sqrt{5}$

Solving equations exercise (page 103)

Q19: 7 sin x + 3 cos x can be expressed as say, k sin (x + α)

7 sin x + 3 cos x = k sin x cos α + k cos x sin α

Equate the coefficients of sin x: 7 = k cos α:- equation 1

Equate the coefficients of cos x: 3 = k sin α:- equation 2

Squaring and adding gives 49 + 9 = $k^2(\cos^2\alpha + \sin^2\alpha) = k^2$

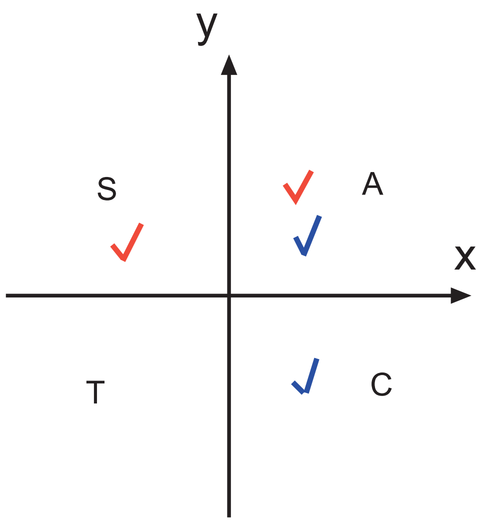
k = $\sqrt{58}$

Dividing gives tan α = $^3/_7$

In the first quadrant this solves to give α = 23° to the nearest degree.

from equation 1: cos is positive

from equation 2: sin is positive



The angle lies in quadrant one.

α = 23°

7 sin x + 3 cos x can be expressed as $\sqrt{58}$ sin (x + 23)°

So 7 sin x + 3 cos x = $\sqrt{58}$ sin (x + 23)° $\Rightarrow$

maximum value is $\sqrt{58}$ when sin (x + 23)° = 1 $\Rightarrow$

x + 23 = 90° since sin is positive.

$0 \leq x < 360° \Rightarrow 23 \leq x + 23 < 383$

Check that the solutions lie within the range and that there are no others.

Thus x + 23 = 90°$\Rightarrow$ x = 67°

Q20: 5 sin x + 2 cos x can be expressed as say, k sin (x + α)

5 sin x + 2 cos x = k sin x cos α + k cos x sin α

Equate the coefficients of sin x: 5 = k cos α:- equation 1

Equate the coefficients of cos x: 2 = k sin α:- equation 2

Squaring and adding gives 25 + 4 = $k^2(\cos^2\alpha + \sin^2\alpha) = k^2$

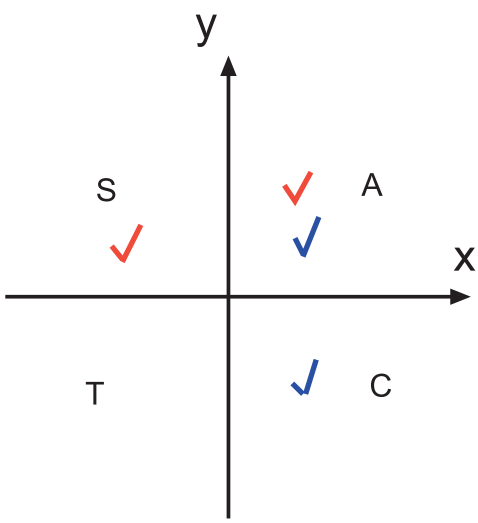
k = $\sqrt{29}$

Dividing gives tan α = $^2/_5$

In the first quadrant this solves to give α = 22° to the nearest degree.

from equation 1: cos is positive

from equation 2: sin is positive



The angle lies in quadrant one.

α = 22°

5 sin x + 2 cos x can be expressed as $\sqrt{29}$ sin (x + 22)°

So 5 sin x + 2 cos x = $\sqrt{29}$ sin (x + 22)° $\Rightarrow$

minimum value is -$\sqrt{29}$ when sin (x + 22)° = -1 $\Rightarrow$

x + 22 = 270° since sin is positive.

$0 \leq x < 360° \Rightarrow 22 \leq x + 22 < 382$

Check that the solutions lie within the range and that there are no others.

Thus x + 22 = 270°$\Rightarrow$ x = 248°

Q21: - sin x - 4 cos x can be expressed as say, k sin (x + α)

- sin x - 4 cos x = k sin x cos α + k cos x sin α

Equate the coefficients of sin x: -1 = k cos α:- equation 1

Equate the coefficients of cos x: - 4 = k sin α:- equation 2

Squaring and adding gives 1 + 16 = $k^2(\cos^2\alpha + \sin^2\alpha) = k^2$

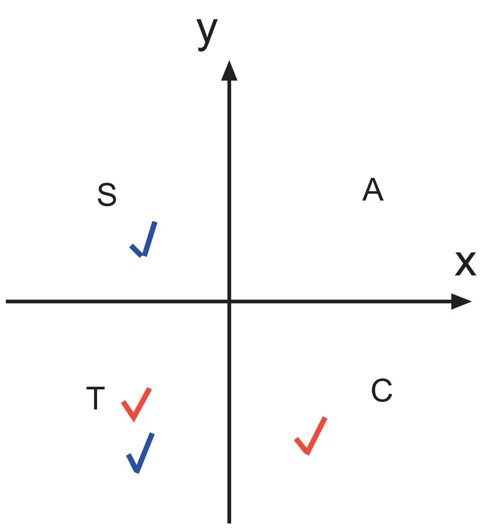
k = $\sqrt{17}$

Dividing gives tan α = $^{-4}/_{-1}$

In the first quadrant this solves to give α = 76° to the nearest degree.

from equation 1: cos is negative

from equation 2: sin is negative



The angle lies in quadrant three.

α = 180 + 76° = 256°

- sin x - 4 cos x can be expressed as $\sqrt{17}$ sin (x + 256)°

So - sin x - 4 cos x = $\sqrt{17}$ sin (x + 256)° = 3 $\Rightarrow$

sin (x + 256)° = 3 $\div$ $\sqrt{17}$

Thus the first quadrant angle α = 47° to the nearest degree.

Since sin is positive the solutions lie in quadrants one and two.

x + 256 = 47° or x + 256 = 180 - 47° = 133°

$0 \leq x < 360° \Rightarrow 256 \leq x + 256 < 616$

Check that the solutions lie within the range and whether there are others.

Thus x + 256 = 47 + 360 = 407° and x + 256 = 493°

x = 151° and x = 237°

Q22: 2 sin x - 4 cos x can be expressed as say, k sin (x + α)

2 sin x - 4 cos x = k sin x cos α + k cos x sin α

Equate the coefficients of sin x: 2 = k cos α:- equation 1

Equate the coefficients of cos x: - 4 = k sin α:- equation 2

Squaring and adding gives 4 + 16 = $k^2(\cos^2\alpha + \sin^2\alpha) = k^2$

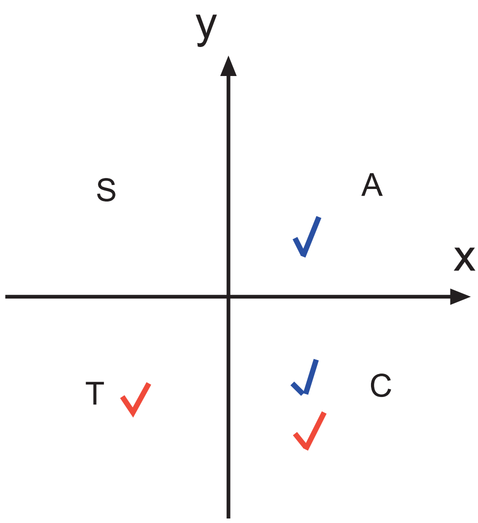
k = $\sqrt{20}$

Dividing gives tan α = $^{-4}/_2$

In the first quadrant this solves to give α = 63° to the nearest degree.

from equation 1: cos is positive

from equation 2: sin is negative



The angle lies in quadrant four.

α = 360 - 63° = 297°

2 sin x - 4 cos x can be expressed as $\sqrt{}$20 sin (x + 297)°

So 2 sin x - 4 cos x = $\sqrt{}$20 sin (x + 297)° = - 3 $\Rightarrow$

sin (x + 297)° = -3 ÷ $\sqrt{}$20

Thus the first quadrant angle α = 42° to the nearest degree.

Since sin is negative the solutions lie in quadrants three and four.

x + 297 = 180 + 42° = 222° or x + 297 = 360 - 42° = 318°

$0 \leq x < 360° \Rightarrow 297 \leq x + 297 < 657$

Check that the solutions lie within the range and whether there are others.

Thus x + 297 = 222 + 360 = 582° and x + 297 = 318°

x = 285° and x = 21°

Q23: - 3 sin x + 6 cos x can be expressed as say, k sin (x + α)

-3 sin x + 6 cos x = k sin x cos α + k cos x sin α

Equate the coefficients of sin x: -3 = k cos α:- equation 1

Equate the coefficients of cos x: 6 = k sin α:- equation 2

Squaring and adding gives 9 + 36 = $k^2(\cos^2\alpha + \sin^2\alpha) = k^2$

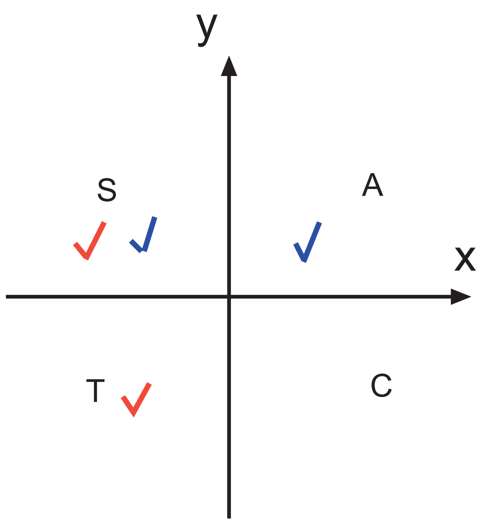
k = $\sqrt{}$45

Dividing gives tan α = $^6/_{-3}$

In the first quadrant this solves to give α = 63° to the nearest degree.

from equation 1: cos is negative

from equation 2: sin is positive



The angle lies in quadrant two.

α = 180 - 63° = 117°

- 3 sin x + 6 cos x can be expressed as $\sqrt{45}$ sin (x + 117)°

So - 3 sin x + 6 cos x = $\sqrt{45}$ sin (x + 117)° = 2 $\Rightarrow$

sin (x + 117)° = 2 $\div$ $\sqrt{45}$

Thus the first quadrant angle α = 17° to the nearest degree.

Since sin is positive the solutions lie in quadrants one and two

x + 117= 17° or x + 117 = 180 - 17° = 163°

$0 \leq x < 360° \Rightarrow 117 \leq x + 117 < 477$

Check that the solutions lie within the range and whether there are others.

Thus x + 117 = 17 + 360 = 377° and x + 117 = 163°

x = 260° and x = 46°

Q24: $\sqrt{3}$ sin x + cos x can be expressed as say, k sin (x + α)

$\sqrt{3}$ sin x + cos x = k sin x cos α + k cos x sin α

Equate the coefficients of sin x: $\sqrt{3}$ = k cos α:- equation 1

Equate the coefficients of cos x: 1 = k sin α:- equation 2

Squaring and adding gives 3 + 1 = $k^2(\cos^2\alpha + \sin^2\alpha) = k^2$

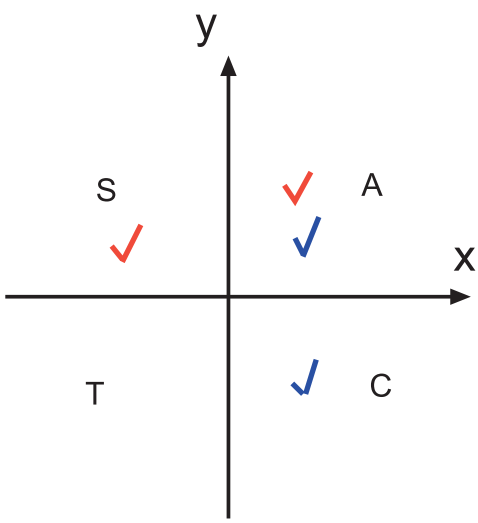
k = 2

Dividing gives tan α = 1 $\div$ $\sqrt{3}$

In the first quadrant this solves to give α = 30° to the nearest degree.

from equation 1: cos is positive

from equation 2: sin is positive



The angle lies in quadrant one $\Rightarrow \alpha = 30^\circ$

$\sqrt{3}$ sin x + cos x can be expressed as 2 sin (x + 30)°

So $\sqrt{3}$ sin x + cos x = 2 sin (x + 30)°

The maximum value is 2 when sin (x + 30)° = 1

Thus x + 30 = 90° since sin is positive.

$0 \leq x < 360^\circ \Rightarrow 30 \leq x + 30 < 390$

Check that the solutions lie within the range and that there are no others.

Thus x + 30 = 90°$\Rightarrow$ x = 60°

The minimum value is -2 when sin (x + 30)° = -1

Thus x + 30 = 270° since sin is negative.

$0 \leq x < 360^\circ \Rightarrow 30 \leq x + 30 < 390$

Check that the solutions lie within the range and that there are no others.

Thus x + 30 = 270°$\Rightarrow$

x = 240°

Problem solving for waves exercise (page 107)

Q25: Let sin t - 3 cos t = k sin (t - α)

sin t - 3 cos t = k sin t cos α - k cos t sin α

Equate the coefficients of sin t: 1 = k cos α:- equation 1

Equate the coefficients of cos t: -3 = -k sin α $\Rightarrow$ 3 = k sin α :- equation 2

Squaring and adding these two equations gives:

$1 + 9 = k^2 \cos^2\alpha + k^2 \sin^2 \alpha = k^2$

$k = \sqrt{10}$

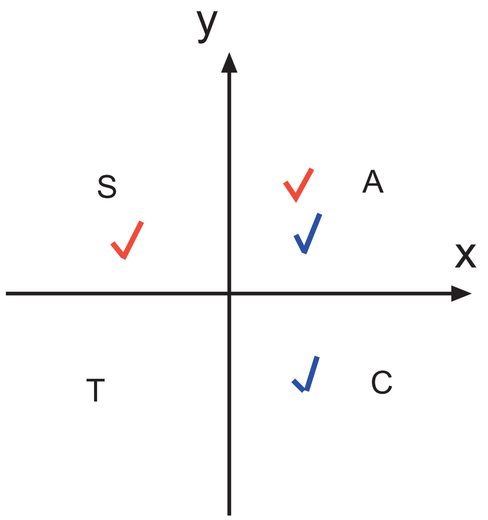
Dividing the equations ($^2/_1$) to form tan α gives:

$\tan \alpha = {}^3/_1$

In the first quadrant α = 72° to the nearest degree.

From equation 1: cos is positive

From equation 2: sin is positive



Then the angle required lies in the first quadrant and so α = 72° to the nearest degree.

sin t - 3 cos t = $\sqrt{10}$ sin (t - 72)°

This has a minimum of -$\sqrt{10}$ = -3.16

d = 4 + sin t - 3 cos t has a minimum of 4 - 3.16 metres = 0.84 metres.

The cruiser cannot moor safely.

Q26: Let -2 sin t + 5 cos t = k cos (t + α)

-2 sin t + 5 cos t = k cos t cos α - k sin t sin α

Equate the coefficients of sin t: -2 = - k sin $\alpha \Rightarrow$ 2 = k sin α:- equation 1

Equate the coefficients of cos t: 5 = k cos α :- equation 2

Squaring and adding these two equations gives:

4 + 25 = $k^2 \sin^2\alpha + k^2 \cos^2 \alpha = k^2$

k = $\sqrt{29}$

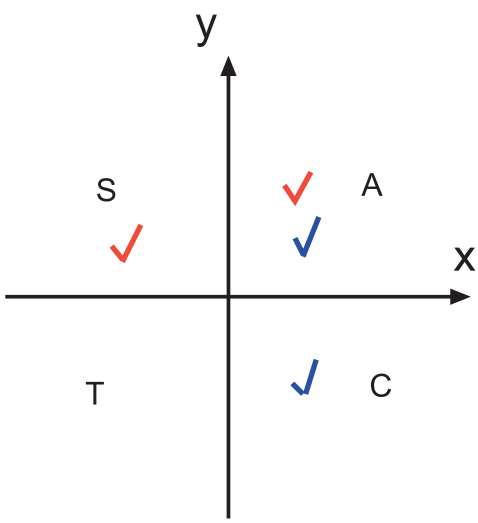
Dividing the equations ($^1/_2$) to form tan α gives:

tan α = $^2/_5$

In the first quadrant α = tan = 22° to the nearest degree.

From equation 1: sin is positive

From equation 2: cos is positive



Then the angle required lies in the first quadrant and so α = 22° to the nearest degree.

-2 sin t + 5 cos t = $\sqrt{29}$ cos (t + 22)°

This has a minimum of - $\sqrt{29}$

so m - $\sqrt{29}$ = 1 and the minimum depth requires

m = 6.38

Q27: Let 2 sin 0.5 t - 10 cos 0.5 t = r cos (0.5 t + θ)

2 sin 0.5 t - 10 cos 0.5 t = r cos 0.5 t cos θ - r sin 0.5 t sin θ

Equate the coefficients of sin 0.5 t: 2 = - r sin θ $\Rightarrow$ -2 = r sin θ:- equation 1

Equate the coefficients of cos 0.5 t: -10 = r cos θ :- equation 2

Squaring and adding these two equations gives:

4 + 100 = $r^2 \sin^2\theta + r^2 \cos^2 \theta = r^2$

r = $\sqrt{104}$

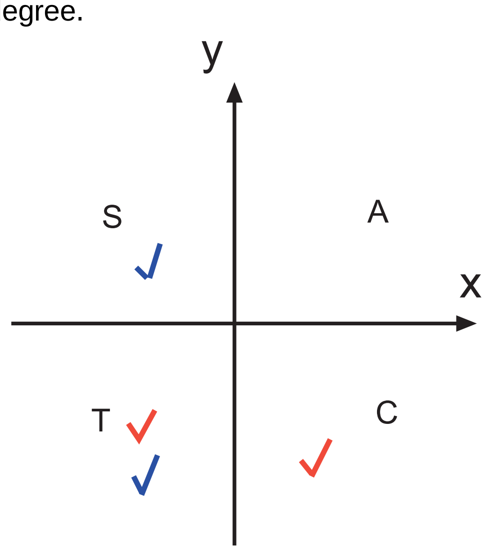
Dividing the equations ($^1/_2$) to form tan θ gives:

tan θ = $^2/_{10}$

In the first quadrant θ = 11° to the nearest degree.

From equation 1: sin is negative

From equation 2: cos is negative



Then the angle required lies in the third quadrant: θ = 191° to the nearest degree.

2 sin 0.5 t - 10 cos 0.5 t = $\sqrt{}$104 cos (0.5 t + 191)$^\circ$

The maximum frequency is $\sqrt{}$104 + 15 = 25Hz

This occurs when cos (0.5 t + 191) = 1

0.5 t + 191 = 0 or 360

0.5 t = -191° or 169° to the nearest degree.

But t is the time in seconds in the problem:
t = 2 $\times$ 169 = 338 seconds = 5 minutes and 38 seconds.

Q28: Let 5 sin 20 t + cos 20 t = k sin (20 t + α)

5 sin 20 t + cos 20 t = k sin 20 t cos α + k cos 20 t sin α

Equate the coefficients of sin 20 t: 5 = k cos α:- equation 1

Equate the coefficients of cos 20 t: 1 = k sin α :- equation 2

Squaring and adding these two equations gives:

25 + 1 = k^2 $\cos^2\alpha$ + k^2 $\sin^2$ α = k^2

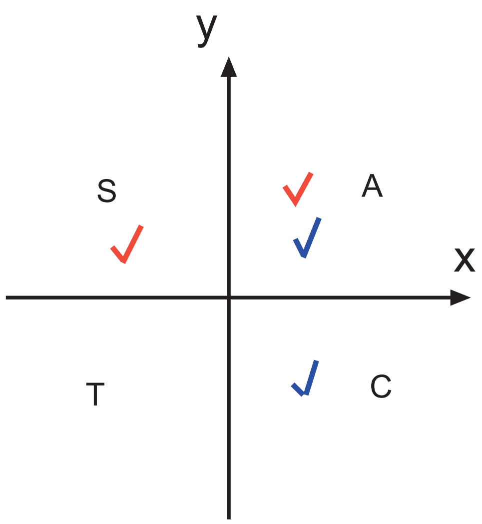
k = $\sqrt{}$26

Dividing the equations ($^2/_1$) to form tan α gives:

In the first quadrant α = 11° to the nearest degree.

From equation 1: cos is positive

From equation 2: sin is positive



Then the angle required lies in the first quadrant and so α = 11° to the nearest degree.

5 sin 20 t + cos 20 t = $\sqrt{}$26 sin (20 t + 11)$^\circ$

The minimum current is 6 - $\sqrt{}$26 milliamps = 1 milliamps to the nearest ma.

This occurs when sin (20 t + 11) = -1

20 t + 11 = 270°, 630° and so on

t = 12.95 $^\circ$, 30.95 $^\circ$

The times at which this occurs are t = 12.95 secs, 30.95 secs, etc

The minimum current occurs every 18 secs

Note however, that a shortcut can be used here. The question asks for the interval between the minimum values. This is simply the period of the graph which is easily calculated as $360^\circ \div 20 = 18^\circ$

Thus the time is every 18 seconds.

Review exercise (page 109)

Q29: Let 3 sin x + cos x = k sin (x + a) = k sin x cos a + k cos x sin a

Equate the coefficients of sin x: 3 = k cos a

Equate the coefficients of cos x: 1 = k sin a

Squaring and adding gives $9 + 1 = k^2 (\cos^2 a + \sin^2 a) = k^2$

$k = \sqrt{10}$

Dividing gives $\tan a = {}^1/_3 \Rightarrow \alpha$ (the first quadrant angle) = 18.43°

cos and sin are both positive; the answer is in the first quadrant.

a = 18.43°

3 sin x + cos x = $\sqrt{10}$ sin (x + 18)° to the nearest degree.

Q30: Let 2 sin x - 3 cos x = k cos (x - a) = k cos x cos a + k sin x sin a

Equate the coefficients of sin x: 2 = k sin a

Equate the coefficients of cos x: -3 = k cos a

Squaring and adding gives $4 + 9 = k^2 (\sin^2 a + \cos^2 a) = k^2$

$k = \sqrt{13}$

Dividing gives $\tan a = -{}^2/_3 \Rightarrow \alpha$ (in the first quadrant) = 33.69°

sin is positive and cos is negative; the answer is in the second quadrant.

a = 180 - 33.69 = 146.31°

2 sin x - 3 cos x = $\sqrt{13}$ cos (x - 146)° to the nearest degree.

Q31: Let 4 sin x + 3 cos x = k cos (x + a) = k cos x cos a - k sin x sin a

Equate the coefficients of sin x: 4 = -k sin a $\Rightarrow$ -4 = k sin a

Equate the coefficients of cos x: 3 = k cos a

Squaring and adding gives $16 + 9 = k^2 (\sin^2 a + \cos^2 a) = k^2$

k = 5

Dividing gives $\tan a = -{}^4/_3 \Rightarrow \alpha$ (first quadrant) = 53.13°

sin is negative and cos is positive; the answer is in the fourth quadrant.

a = 360 - 53.13 = 306.87°

4 sin x + 3 cos x = 5 cos (x + 307)° to the nearest degree.

The maximum height occurs when cos (x + 307)° = 1

That is the maximum height is 5 + 6m = 11m

Q32: Let -3 sin x + 2 cos x = k sin (x - a) = k sin x cos a - k cos x sin a

Equate the coefficients of sin x: -3 = k cos a

Equate the coefficients of cos x: 2 = -k sin a $\Rightarrow$ -2 = k sin a

Squaring and adding gives 9 + 4 = k^2 ($\cos^2$ a + $\sin^2$ a) = k^2

k = $\sqrt{13}$

Dividing gives tan a = $^{-2}/_{-3}$ $\Rightarrow$ α (the first quadrant) = 33.69°

sin and cos are both negative; the answer is in the third quadrant.

a = 180 + 33.69 = 213.69°

-3 sin x + 2 cos x = $\sqrt{13}$ sin (x + 214)° to the nearest degree.

The amplitude of the wave is $\sqrt{13}$

Advanced review exercise (page 110)

Q33: Let -5 sin θ - 3 cos θ = r cos (θ - α) = r cos θ cos α + r sin θ sin α

Equate the coefficients of sin θ: -5 = r sin α

Equate the coefficients of cos θ: -3 = r cos α

Squaring and adding gives 25 + 9 = r^2 ($\sin^2$ α + $\cos^2$ α) = r^2

r = $\sqrt{34}$

Dividing gives tan α = $^5/_3$ $\Rightarrow$ angle in quadrant one = 59.04° (do not confuse with α if α has been used in another context as has been done here).

sin and cos are negative; the answer is in the third quadrant.

α = 180 + 59.04 = 239.04°

-5 sin θ - 3 cos θ = $\sqrt{34}$ cos (θ - 239)° to the nearest degree.

For the second part: 0 $\leq \theta <$ 360° $\Rightarrow$ -239 $\leq \theta$ - 239 $<$ 121°

-5 sin θ - 3 cos θ = 3 $\Rightarrow$ $\sqrt{34}$ cos (θ - 239) = 3

cos (θ - 239) = 0.5145 $\Rightarrow$ first quadrant angle = 59.04°

But the cos is positive and so the answer is in either the first or fourth quadrants.

θ - 239 = 59° or θ - 239 = -59°

Check the values in the range :

θ - 239 = 59, -59

θ = 298° or θ = 180°

Q34:

a) Let f (x) = k sin (x - α)
   4 cos x - 2 sin x = k sin x cos α - k cos x sin α
   Equate the coefficients of sin x: -2 = k cos α
   Equate the coefficients of cos x: 4 = -k sin $\alpha \Rightarrow$ -4 = k sin α
   Squaring and adding gives 4 + 16 = k^2 ($\cos^2$ α + $\sin^2$ α) = k^2
   k = $\sqrt{20}$

Dividing gives tan α = $^4/_2$ $\Rightarrow$ angle in quadrant one = 63.43°
sin and cos are negative; the answer is in the third quadrant
α = 180 + 63.43 = 243.43°
4 cos x - 2 sin x = $\sqrt{}20$ sin (x - 243)$^\circ$ to the nearest degree.

b) f (x) - 0.3 = 0 $\Rightarrow$ 4 cos x - 2 sin x - 0.3 = 0 $\Rightarrow$ $\sqrt{}20$ sin (x - 243)$^\circ$ - 0.3 = 0
sin (x - 243) = 0.067 $\Rightarrow$ the first quadrant angle solution α = 3.84°
Sin is positive and thus lies in quadrants one or two
x - 243 = 3.84° or x - 243 = 176.16°
Check the range: 0 $\leq$ x $<$ 360° $\Rightarrow$ -243 $\leq$ x - 243 $<$ 117°
x - 243 = 176.16 - 360 = -183.84°: there are no further solutions.
x = 246.84° or x = 59.16° (x = 176.16 + 243 = 419.16 is too large).

c) When f (x) cuts the x-axis, f (x) = 0 $\Rightarrow$ 4 cos x - 2 sin x = $\sqrt{}20$ sin (x - 243)$^\circ$ = 0
sin (x - 243) = 0 $\Rightarrow$ x = 0°, $\pm$180°, $\pm$360°
x = 243°, 423°, 63°, 603°, 117°
The value nearest the origin is x = 63°

Q35:

a) Let f (x) = k cos (x - α)
-2 cos x + 5 sin x = k cos x cos α + k sin x sin α
Equate the coefficients of sin x: 5 = k sin α
Equate the coefficients of cos x: -2 = k cos α
Squaring and adding gives 25 + 4 = k^2 (sin^2 α + cos^2 α) = k^2
k = $\sqrt{}29$
Dividing gives tan α = -$^5/_2$ $\Rightarrow$ angle in quadrant one = 68.2°
sin is positive and cos is negative; the answer is in the second quadrant.
α = 180 - 68.2 = 111.8°
-2 cos x + 5 sin x = $\sqrt{}29$ cos (x - 112)$^\circ$ to the nearest degree.

b) f (x) - 1 = 0 $\Rightarrow$ -2 cos x + 5 sin x - 1 = 0 $\Rightarrow$ $\sqrt{}29$ cos (x - 112)$^\circ$ - 1 = 0
cos (x - 112) = 0.186 $\Rightarrow$ the first quadrant angle solution α = 79.3°
cos is positive and thus lies in quadrants one or four
x - 112 = 79.3° or x - 112 = 360 - 79.3° = 280.7°
Check the range: 0 $\leq$ x $<$ 360° $\Rightarrow$ -112 $\leq$ x - 112 $<$ 248°
x - 112 = 280.7 - 360 = -79.3°: there are no further solutions.
x = 191.3° or x = 32.7°

c) When f (x) cuts the x-axis, f (x) = 0 $\Rightarrow$ -2 cos x + 5 sin x = $\sqrt{}29$ cos (x - 112)$^\circ$ = 0
cos (x - 112) = 0 $\Rightarrow$ x = $\pm$90°, $\pm$270°
x = 22°, 202°, -158°, 382$^{\circ\circ}$
The value nearest the origin is x = 22°

Q36: cos x = 3 sin x $\Rightarrow$ 3 sin x - cos x = 0

Let 3 sin x - cos x = r sin (x + θ) = r sin x cos θ + r cos x sin θ

Equate coefficients of sin: 3 = r cos θ

Equate coefficients of cos: -1 = r sin θ

Squaring and adding gives $9 + 1 = r^2(\cos^2\theta + \sin^2\theta) = r^2$

$r = \sqrt{10}$

Dividing gives $\tan\theta = -{}^1/_3 \Rightarrow \alpha = 18.4^\circ$

cos is positive and sin is negative so the solution is in quadrant four.

$\theta = -18.4^\circ = 341.6^\circ$

Thus $3 \sin x - \cos x = \sqrt{10} \sin (x + 342)^\circ$ to the nearest degree and the phase shift is -342°

The amplitude is $\sqrt{10}$

Set review exercise (page 110)

Q37: This answer is only available on the web.

Q38: This answer is only available on the web.

Q39: This answer is only available on the web.

Q40: This answer is only available on the web.