Unit 1 - 1 The Straight Line

The distance formula

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

derived using Pythagoras' Theorem.

Applications:

Calculating length of a line
Useful for showing triangles are isosceles or equilateral
Use to show that two sides of a shape have the same length.
Use in circles to calculate radius or diameter

The mid-point formula

$$\mathrm{M}\left(\frac{x_1 + x_2}{2},\ \frac{y_1 + y_2}{2}\right)$$

This gives the mid-point M of A(x_1, y_1) and B(x_2, y_2).

This is a simple average of the co-ordinates of A and B.

The gradient formula

$$m_{AB} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$gradient = \frac{change\ in\ y\ from\ A\ to\ B}{change\ in\ x\ from\ A\ to\ B}$$

The gradient of AB is denoted by m_{AB}

Linking the gradient to the tangent

$$\mathbf{m_{AB} = \tan\theta}$$

where θ is the **anti-clockwise** angle from OX to AB

For a line sloping down from left to right,
θ is obtuse angle, tan θ is negative and so gradient is negative

Parallel and perpendicular lines

Parallel: gradients are equal

Perpendicular: product of gradients = –1

If the gradients of two perpendicular lines are m_1 and m_2, then

$$m_1 = -\frac{1}{m_2} \qquad if\ \ m_1 = \frac{a}{b} \qquad then \quad m_2 = -\frac{b}{a}$$

Lines parallel to OX and OY

gradient = 0 gradient = undefined

The **gradient** of a line **parallel to the x-axis** is 0

The **gradient** of a line **parallel to the y-axis** is undefined

Applications of: $\mathbf{m_1 \times m_2 = -1}$

- Finding gradients and equations of perpendicular lines
- Finding equations of tangents to circles (perpendicular to radius)
- Showing that an angle is a right angle.
- Checking out properties of quadrilaterals

  e.g. rhombus – diagonals bisect at right angles
  square, rectangle have corners at right angles
- Demonstrating symmetry.
- Finding equations of parallel and perpendicular lines.

Equation of a straight line

$$\mathbf{y = mx + c}$$

where m = gradient and c = y-intercept

If you have the equation in this form, then it is easy to sketch the graph:

You know the y-intercept: **c**

You know the gradient: **m**

Collinearity

If three points lie on the **same** straight line, they are said to be **collinear**.

To prove that 3 points, say P, Q and R are collinear –

(i) Show that the gradients PQ and QR are the same
(ii) **State** that the two line segments PQ and QR have a **common** point Q

Failure to state (ii) will cost you marks !!!

Re-arranging equations

Equations of straight lines come in many forms:

y = 4x – 5 y + 4x = -5

y + 4x + 5 = 0 2y + 8x + 10 = 0

are all the same equation.

Use the simple rules of algebra:

- Change side, change sign
- Multiply both sides by a constant
- Divide both sides by a constant

Unit 1 - 1 The Straight Line

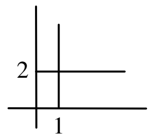
Recognising equations of straight lines

Straight lines only have **x** and **y** appearing with no higher power than 1.

If there are any terms such as x^2, x^3, x^4 or y^2, y^3, y^4 terms etc
– it is **NOT** a straight line.

e.g. $y = x^2 + 2$ is not a straight line
$x - y = 3$ is a straight line

Equations of lines parallel to OX and OY



$x = 1$ is a line parallel to the y-axis passing through $x = 1$

$y = 2$ is a line parallel to the x-axis passing through $y = 2$

The equation of a line of gradient m through a point (a, b)

Use $m = \dfrac{y-b}{x-a}$

or $y - b = m(x-a)$

where m is the gradient and (a, b) the point

When you do not have the y-intercept, but only the gradient and a point, you should proceed as follows. Recall the gradient formula, you need two points on the line to obtain the gradient.

Since you have the gradient and a point, then to obtain the equation, choose another general point (x, y). Use the gradient formula again in this form.

Example: Find the equation of the line with gradient 3, passing through P(1, 2)

$3 = \dfrac{y-2}{x-1}$ so $3(x-1) = (y-2)$ $3x - 3 = y - 2$

simplify $\Rightarrow$ $y = 3x - 1$ or $y - 3x + 1 = 0$ or $y - 3x = -1$

Intersecting Lines:

Treat the equations of the lines as simultaneous equations.

To find the co-ordinates of the point where two lines intersect,

treat the equations of the lines as simultaneous equations and solve them.

The solution (x, y) is the point of intersection

Some useful bits and pieces:

Median of a triangle - the line drawn from a vertex to the **middle** of the opposite side.

Altitude of a triangle – the line drawn from a vertex perpendicular to the opposite side. (Measures the **height**)

Perpendicular bisector of a line is a perpendicular line passing through the mid-point.

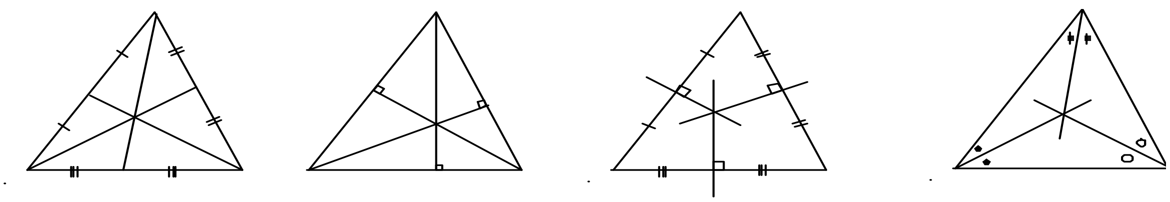
Perpendicular at right angles to

Intersect where lines cut each other

Bisect lines cut each other exactly in **half**.

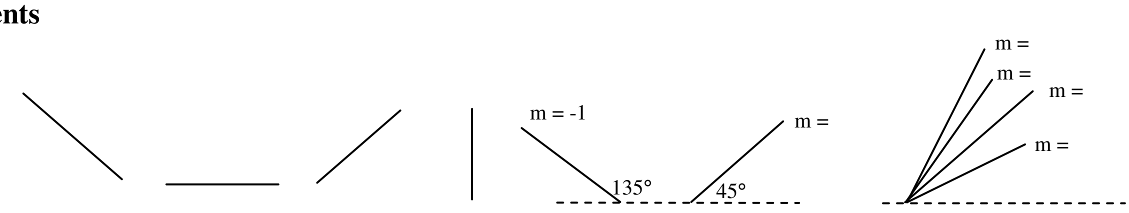
Locus the path traced out according to some specified condition.
By using this condition, you can find the equation of this path which is called the locus.

Image of a line under reflection, translation or rotation is the position where line moves to under the transformation.



Medians Altitudes Perpendicular bisectors Angle bisectors

Gradients



negative zero positive undefined lines at 45° ~ gradients +1, -1 gradients relative to m = 1

Unit 1 - 2.1 Composite and Inverse Functions

Definition of a function

A function is defined from a set A to a set B as a rule which links each member of A to exactly one member of B.

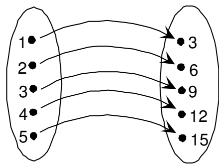
Notation:

y = f(x) or f : x → y (f maps x to y)

Domain and Range

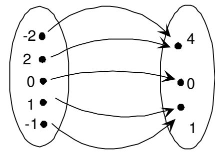
The **domain** of a function is the **input** – the variable the function operates upon.

The **range** of a function is the **output** – the value of the function.



Domain Range

$f(x) = 3x$ or $f : x \to 3x$



Domain Range

$g(x) = x^2$ or $g : x \to x^2$

Domain of $h(x)=\sqrt{x}$ and $k(x) = \dfrac{1}{x-1}$

We write this as:

domain of **h(x) is** $\{x : x \in \Re : x \geq 0\}$

and:

domain of **k(x) is:** $\{x : x \in \Re : x \neq 1\}$

You should always be aware of the danger of dividing by zero.

The largest domain of $h(x) = \sqrt{x}$ is the set of real numbers greater than or equal to zero, since you cannot take the square root of a negative number.

The largest domain of $k(x) = \dfrac{1}{x-1}$ is the set of real numbers except x = 1

(since this would make the denominator zero ~ you cannot divide by zero).

Undefined functions

Functions may be undefined for particular values of x ~ in particular:

- where you would need to take the square root of a negative number
- where you would need to divide by 0

$f(x) = \sqrt{(x-1)}$ is undefined when x < 1

(requires square root of negative number)

$h(x) = \dfrac{1}{x-3}$ is undefined when x = 3

(results in division by zero).

Related Functions

Given f(x), what is f(x+1) or f(x^2) or f(2x) etc.

To find f(x+1),

simply replace the 'x' in f(x) with 'x+1' etc.

and simplify.

Example: $f(x) = 3x + 1$ what is $f(x+1)$

Solution: $f(x+1) = 3(x+1) + 1 \Rightarrow$ $\mathbf{3x + 4}$

Example: $h(x) = x^2 - 3x$ what is $h(2x)$

Solution: $h(2x) = (2x)^2 - 3(2x) \Rightarrow$ $\mathbf{4x^2 - 6x}$

Example: $f(x) = 2x^2 + 3x$ what is $f(x+1)$

Solution: $f(x+1) = 2(x+1)^2 + 3(x+1) \Rightarrow$ $\mathbf{2x^2 + 7x + 5}$

Evaluating functions:

Given f(x), what is f(1) or f(0) or f(-2) etc.

To evaluate a function f(x) at x = 2 (say), calculate what the value of the function is when you replace x by 2

Example: If $f(x) = x^2 + 3x - 1$ Evaluate f(-1)

Solution: $f(-1) = (-1)^2 + 3(-1) - 1 \Rightarrow$ **-3**

Example: If $f(x) = 3x^3 - 5x + 2$ What is f(a)

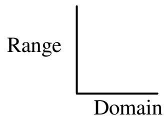
Solution: $f(a) = 3a^3 - 5a + 2$

Unit 1 - 2.1	Composite and Inverse Functions

Recognising the domain and range of a function

Domain: the input

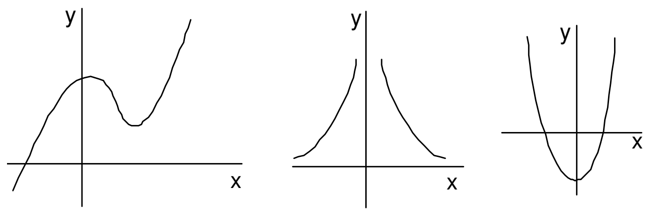
Range: the output



Domain: Look for the **input** (the variable in the function – in f(x) the **'x' axis** in g(t) the **'t' axis**). This variable is known as the **'independent variable'** since you can choose any value in the domain.

Range: Look for the **output** or **value of the function** (this is the value of **f** in f(x) or **g** in g(t)). This variable is known as the **'dependent variable'**, as once you have chosen a value for x or t then f or g is determined by the function.

Examples of Range and Domain



Domain:	$-2 \le x \le 3$	$-1 \le x \le 4$	$-1 \le x \le 3$
Range:	$-1 \le y \le 6$	$-2 \le y \le 5$	$-3 \le y \le 12$

Composite Functions

If $f(x) = x - 3$ and $g(x) = x^2$
Then what is $f(g(x))$

Start from the outside function. f(…..)
replace the 'x' with g(x)

So we have $f(g(x)) = f(x^2) = (x^2) - 3$

$\Rightarrow$ $\mathbf{f(g(x)) = x^2 - 3}$

NB: in general, f(g(x)) is **NOT** the same as g(f(x))
which you can demonstrate easily in the example on the left, since:
$\mathbf{g(f(x))} = g(x-3) = (x-3)^2 = \mathbf{x^2 - 6x + 9}$

Example:

$f(x) = x + 2$ $\quad g(x) = 2x^2$

Then $f(g(x)) = f(2x^2) = \mathbf{2x^2 + 2}$

and $g(f(x)) = g(x+2) = 2(x+2)^2 = \mathbf{2x^2 + 8x + 8}$

Example:

$f(x) = \dfrac{x}{x-1}$ find f(f(x)) [Hint: replace the x in f(x) with $\dfrac{x}{x-1}$]

so $f(f(x)) = \dfrac{\frac{x}{x-1}}{\frac{x}{x-1} - 1}$ now simplify to get $\mathbf{f(f(x)) = x}$

Functions with inverses

An inverse of a function, is a function that 'undoes' the operation of the original function.

Example: If the original function is $f(x) = 2x + 3$

The function takes a number, **doubles it** and **adds 3**

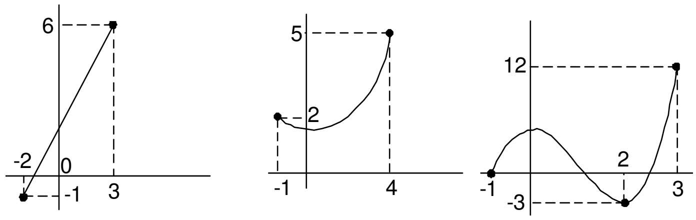
To undo this – we take the result, **subtract 3** and then **halve it**.

For a function to have an inverse

the members of its domain and range must be in **one-to-one correspondence**.

This means that for: every member of the domain, there is exactly ONE corresponding member of the range and for every member of the range there is exactly ONE corresponding member of the domain i.e. No duplicate values.

Look at the following graphs – only one of them has an inverse
– can you see which one?



Except for the straight line on the left, in each case, there are two or more values in the domain, which have the same value in the range.

A quick way to check for an inverse on a graph is to slide a ruler horizontally down. If it crosses the graph at more than one point, you have multiple domain members corresponding to one range value.

Unit 1 - 2.1 | Composite and Inverse Functions

Finding inverse functions

If a function f which maps A to B has an inverse function which we call f^{-1},
then f takes x to y and f^{-1} takes y back to x.

$$\text{So} \quad y = f(x) \quad \Leftrightarrow \quad x = f^{-1}(y)$$

NB $\Leftrightarrow$ means 'implies and is implied by'
~ each implies the other - a two way process

Recall $\sin^{-1}(x)$, $\cos^{-1}(x)$ and $\tan^{-1}(x)$ on your calculator – these are the inverse functions of sin x, cos x and tan x.

Methods for finding inverse functions

Method 1:

Establish what the function does, then undo it.

Example: Find the inverse of f(x) = 4x

The function multiplied the input by 4, so to undo this we divide by 4 ~ hence $f^{-1}(x) = \dfrac{x}{4}$

Sometimes it is more difficult to see the inverse:

Example: Find the inverse of $f(x) = \dfrac{3x-1}{2}$

In this case the number is multiplied by three, one is subtracted, and the result divided by 2.

To undo this – start by: multiplying by two

$\Rightarrow$ $f^{-1}(x) = 2x$ then add 1

$\Rightarrow$ $f^{-1}(x) = 2x+1$

finally divide by 3 $\Rightarrow$ $f^{-1}(x) = \dfrac{2x+1}{3}$

Method 2:

Use the technique:
changing the subject of the formula.

Summary:

1. Change function to y = ….
2. Change the subject of the formula
3. Switch the x and y placeholders
4. Put back into function notation – replace y with $f^{-1}(x)$

Example: Find the inverse of $f(x) = \dfrac{3x-1}{2}$

put $y = \dfrac{3x-1}{2}$ then $2y = 3x-1$ and $2y+1 = 3x$

then $x = \dfrac{2y+1}{3}$

~ the 'x' and 'y' are simply placeholders so switch the 'x' and 'y' labels and change it back to function notation.

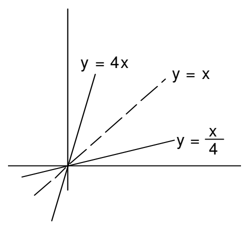
$$f^{-1}(x) = \frac{2x+1}{3}$$

This method is much easier and less prone to error.

Graphs of Inverse functions.

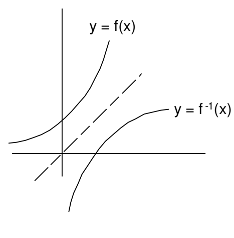
The graph $y = f^{-1}(x)$ can easily be found by reflecting the graph of $y = f(x)$ in the line y = x

Example: f(x) = 4x then $f^{-1}(x) = \dfrac{x}{4}$



Note the reflection in y = x

This technique is useful for sketching graphs of inverse functions, and indeed can also allow us to deduce what an inverse function might be.



Unit 1 - 2.2	Algebraic Functions and Graphs

Completing the square of the quadratic function

Method summary:

1. Arrange for the coefficient of x^2 to be 1
2. Form the square with an 'x' and 'half the coefficient of the x term'
3. Subtract the square of the last term **taking care to include brackets**
4. then add back the constant term

Example: Complete the square of $2x^2 + 12x - 5$

step 1. $2\left(x^2+6x\right)-5$ ensure coefficient of x^2 is 1

step 2. $2\left(x+3\right)^2$ forming the square ~ put x first and half the coefficient of x in last place.

step 3. $2\left\{\left(x+3\right)^2-9\right\}$ subtract square of last term

and don't forget the brackets !

step 4. $2\left(x+3\right)^2-18-5$ add back the constant term

then simplify $2\left(x+3\right)^2-23$

You can, of course, check by multiplying out:

$2\left(x+3\right)^2-23 \Rightarrow 2(x^2+6x+9)-23 \Rightarrow 2x^2+12x-5$

Maximum and minimum values

Complete the square and obtain a function in the form of: $f(x) = a(x + p)^2 + q$

if 'a' is positive

then since $(x + p)^2$ is always positive, the minimum value of the function will be when $(x + p)^2 = 0$

Minimum value = q when x = - p

if 'a' is negative

the function can be arranged as:

$f(x) = q + a(x + p)^2$ and since 'a' is negative, the term: $a(x + p)^2$ will be subtracted

Maximum value = q when x = - p

By completing the square, this allows us to find maximum and minimum values of quadratic functions and the value of 'x' at which this occurs.

Example: By completing the square, find the minimum value of $x^2 + 8x + 4$ and the value of x at which it occurs.

Solution: $x^2+8x+4 = (x+4)^2-16+4 = (x+4)^2-12$

Minimum value is -12 when $\left(x+4\right)^2=0$

Minimum value = -12 when x = - 4

Sketching the graph of quadratic functions

- Intersection with x, y axes
- turning point
- axis of symmetry

<u>**DO NOT PLOT**</u> the graph
– you will be awarded <u>**NO MARKS**</u> for plotting

Method: To sketch the graph of y = f(x) where f(x) is a quadratic function

1. Find the intersection with the y-axis (by putting x = 0)
2. Find the intersection with the x-axis (by putting y = 0) (by solving the equation f(x) = 0 ~ finding the roots)
3. Find the turning point by completing the square (incl. y-co-ordinate)
4. Note the turning point is on the axis of symmetry (half-way between the two roots of the equation ~ step 2)
5. Find y co-ordinate of the turning point by substitution in the equation
6. Using these points – sketch the graph – marking in co-ordinate values.
7. Show clearly any working you have done.

Sketching graphs of related functions

If you have the graph of a function such as y = f(x), then you can deduce and sketch the graph of a related function such as:

- y = - f(x)
- y = f(-x)
- $y = f(x \pm a)$
- $y = f(x) \pm k$

or any combination of these transformations.

Given y = f(x) then: (assuming a > 0 and k > 0)

- y = - f(x) This reflects the graph in the x-axis
- y = f(-x) This reflects the graph in the y-axis
- y = f(x + a) This slides the graph a units to the left
- y = f(x - a) This slides the graph a units to the right
- y = f(x) + k This slides the graph k units upwards
- y = f(x) – k This slides the graph k units downwards

These may be combined, for example:

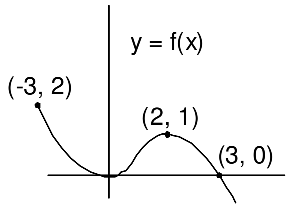
y = f(x + a) – k would move graph a units to left then k units down.

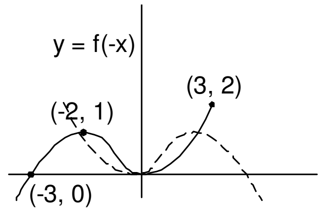
The + k or – k added onto the end is the last operation to be done

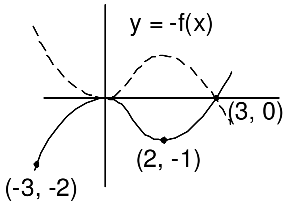
Always show the images of any marked points.

Unit 1 - 2.2 Algebraic Functions and Graphs

Examples of graphs of related functions







The exponential function and its graph

Any function of the form

$f(x) = a^x$ where $a > 0$ and $a \neq 1$

is called an *exponential function* with *base a.*

The graph of the function has equation $y = a^x$

Note: In **all cases** the graph passes through (0, 1) since $a^0 = 1$ for all values of a and the line y = 0 is an **asymptote** to the graph $y = a^x$

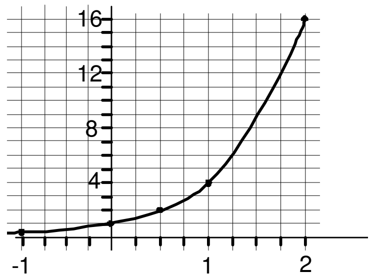
The graph of the function

$f(x) = 4^x$ is shown here

$f(2) = 4^2 = 16$ $\quad f(1) = 4^1 = 4$

$f(0) = 4^0 = 1$ $\quad f\left(\frac{1}{2}\right) = 4^{\frac{1}{2}} = 2$

$f(-1) = 4^{-1} = 0.25$

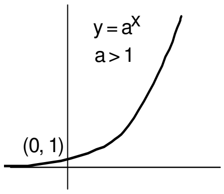


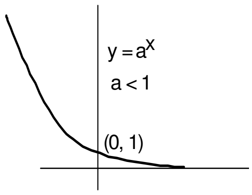
Decreasing and increasing exponential functions

For the function $f(x) = a^x$

if ***a* > 1**
then the function is an **increasing** function.

if ***a* < 1**
then the function is a **decreasing** function.





Sketching graphs of exponential functions

Method:

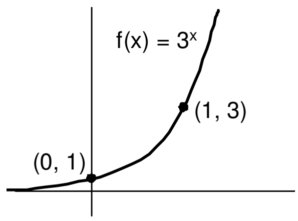
The graph must pass through (0, 1)

Is it decreasing or increasing (is $a > 1$ or $a < 1$)

Pick a suitable point (e.g. x = 1 or x = 2) to get an idea of the steepness

Example: Sketch the graph of $f(x) = 3^x$

The graph passes through (0, 1)
It is increasing, since $a = 3$
Choose x = 1 so f(x) = 3
You have 2 points ~ (0, 1) and (1, 3)
You know it is increasing



The Logarithmic Function and its Graph

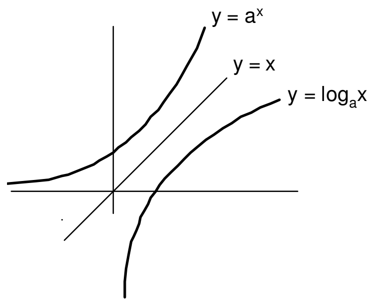
$f(x) = a^x$ has an inverse function $f(x) = \log_a x$

These are related as follows:

$$y = a^x \quad \Leftrightarrow \quad x = \log_a y$$

The graph of $y = \log_a x$ is the mirror image of the graph $y = a^x$ in the line y = x.

Note that the line **x = 0** is an **asymptote** to the graph $y = \log_a x$



Unit 1 - 2.2 Algebraic Functions and Graphs

Special Logarithms

$$y = a^x \quad \Leftrightarrow \quad x = \log_a y$$

$$\log_a 1 = 0$$

(logarithm of 1 to any base is 0)

$$\log_a a = 1$$

(logarithm of a number to that base is 1)

Using the form: $y = a^x \quad \Leftrightarrow \quad x = \log_a y$

(i) $1 = a^0$ i.e. $y = 1$ when $x = 0$ $\Leftrightarrow$ $0 = \log_a 1$

(ii) $a = a^1$ i.e. $y = a$ when $x = 1$ $\Leftrightarrow$ $1 = \log_a a$

Sketching graphs of Log functions

Using **$\log_a 1 = 0$**

and **$\log_a a = 1$**

we can obtain two points on the graph.

When plotting $\log_a(x - 2)$ or similar

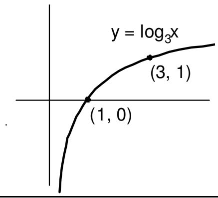
Choose a value of x to make $(x - 2)$ equal to 0

Choose a value of x to make $(x - 2)$ equal to a

Example: Sketch $y = \log_3 x$

$\log_3 1 = 0$ giving point (1, 0)

$\log_3 3 = 1$ giving point (3, 1)



Example: Sketch $y = \log_2 (x - 3)$

$\log_2 1 = 0$

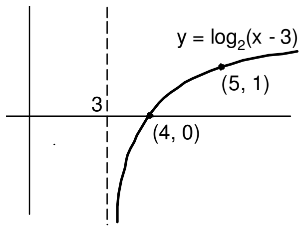
$\Rightarrow \ (x - 3) = 1$

so $x = 4$ giving point (4, 0)

$\log_2 2 = 1$

$\Rightarrow \ (x - 3) = 2$

so $x = 5$ giving point (5, 1)



Note this confirms our previous knowledge of related functions

$y = \log_2 (x - 3)$ is simply the graph of $y = \log_2 x$ shifted 3 units to the right.

$y = f(x)$ and the related function is $y = f(x - 3)$

Note also the **asymptote** at $x = 3$

In all cases use the two special logarithms

$$\log_a 1 = 0$$

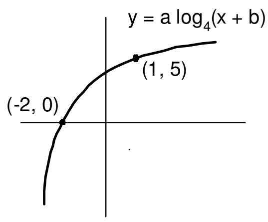
$$\log_a a = 1$$

Choose values of x as appropriate

Example:

A sketch of the graph $y = a \log4(x + b)$ is shown.

Find the values of a and b



(-2, 0) lies on the curve, so

$0 = a \log_4 (-2 + b)$

so $b - 2 = 1$, hence $\mathbf{b = 3}$

(1, 5) lies on the curve, so

$5 = a \log_4 (1 + b)$, since $b = 3$

$5 = a \log_4 (4)$, now $\log_4 4 = 1$ so $\mathbf{a = 5}$

Example:

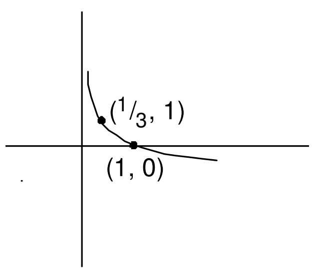
Sketch $y = \log_3 \left(\dfrac{1}{x} \right)$

Choose $x = 1$

$\Rightarrow \ \log_3 1 = 0$ giving point (1, 0)

Choose $x = \dfrac{1}{3}$

$\Rightarrow \ \log_3 3 = 1$ giving point $(^1/_3, 1)$



Note that the term $\dfrac{1}{x}$ results in a decreasing function

Consider what happens for large x and small x approaching zero

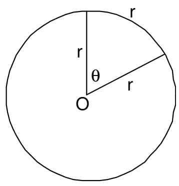
Unit 1 - 2.3	Trigonometric Functions and Graphs

Radian measure

An angle of one radian is the angle subtended at the centre of a circle by an arc of length equal to its radius.

π radians = 180°

2π radians = 360°



$\angle \theta = 1$ radian,

and $\dfrac{\theta°}{360°} = \dfrac{r}{2\pi r}$

so $\theta° = \dfrac{360}{2\pi}$ 1 radian = approximately 57.3°

Degrees	Radians
30	$\frac{\pi}{6}$
45	$\frac{\pi}{4}$
60	$\frac{\pi}{3}$
90	$\frac{\pi}{2}$
120	$\frac{2\pi}{3}$
135	$\frac{3\pi}{4}$
180	π

Changing between degrees and radians

Use proportion based on **π radians = 180°**

1 radian = $\dfrac{180}{\pi}$ degrees so multiply your radians by $\dfrac{180}{\pi}$ to get degrees

1 degree = $\dfrac{\pi}{180}$ radians so multiply your degrees by $\dfrac{\pi}{180}$ to get radians

Note the top line of the multiplier: x $\dfrac{180}{\pi}$ gives degrees

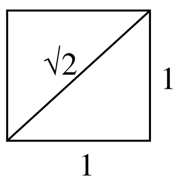
x $\dfrac{\pi}{180}$ gives radians

Exact values for sin, cos and tan

Radians	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$
Degrees	30°	45°	60°
sin	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$
cos	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$
tan	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$

You should be familiar with this table, if you cannot memorise it, learn how to create it.

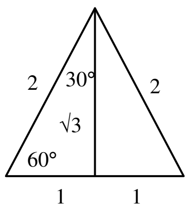
Recall the use of Pythagoras and SOHCAHTOA for obtaining exact values for 30°, 45° and 60°.



Use a square of side 1 and draw in a diagonal,
Use Pythagoras to calculate length of diagonal as √2

Hence sin 45° = $\frac{1}{\sqrt{2}}$ cos 45° = $\frac{1}{\sqrt{2}}$

and tan 45° = 1



Use an equilateral triangle of side 2 units.
Draw in the perpendicular from base to vertex.
giving 2 right angled triangles with angles of 30° and 60° and each with a base of 1 and hypotenuse of 2.

Again use Pythagoras to calculate the altitude as $\sqrt{3}$

Hence sin 30° = $\frac{1}{2}$ cos 30° = $\frac{\sqrt{3}}{2}$ tan 30° = $\frac{1}{\sqrt{3}}$ etc.

Look for the symmetry. You must be able to work in radians as well as degrees.

Maximum and minimum values of trigonometric functions

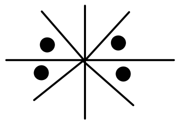
- Look when the trig. function is +1 or –1
- What value of x will cause this
- Evaluate the function for this x.

- Consider the function **f(x) = 2 + cos x**
- The maximum value will occur when **cos x is a maximum**
- The maximum value of cos x is 1 when x = 0° or 360° (or 0 and 2π radians)
- So the maximum value of the function will be 2 + 1 = 3
- Similarly the minimum value of the function will be 2 – 1 = 1 when x = 180° (or π radians).
- In all cases look at when the sin or cos part of the function is at a maximum or minimum.

Unit 1 - 2.3 Trigonometric Functions and Graphs

Angles greater than 90°

S	A
T	C



shows where sine, cosine and tangent are positive.

Recall '**A**ll **S**inners **T**ake **C**are'

When considering angles in the 2^{nd} 3^{rd} and 4^{th} quadrants, remember the acute angle is always between the **rotating arm** and the **x-axis**.

Example:

sin 135° related acute angle = 45° ~ sin 135° = + sin 45° = $\frac{1}{\sqrt{2}}$

tan $\frac{11\pi}{6}$ related acute angle = $\frac{\pi}{6}$ ~ tan $\frac{11\pi}{6}$ = – tan $\frac{\pi}{6}$ = $\frac{1}{\sqrt{3}}$

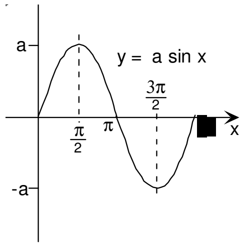
Sketching Trigonometric Graphs

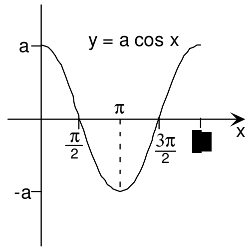
y = a sin nx **y = a cos nx**

a = amplitude (max and min values of y)

n = number of waves in 360° or 2π

period of the graph is $\frac{360}{n}$ ° or $\frac{2\pi}{n}$ radians





Solving Trigonometric Equations

All these equations can ultimately be resolved into the form

sin (…..) = constant

cos (……..) = constant

tan (……..) = constant

Once you have reached this form, you can ***generally*** find 2 solutions using 'ASTC'.

Reminder: when you have for example: cos θ = -0.5 or tan θ = -0.7

Ignore the negative sign when getting the acute angle on your calculator.

Use the negative sign to determine which **quadrants** the solutions are in.

However – first you have to get the equation into this form !

See below for strategies for the different types of equations.

Type 1: Solve **2 sin x = 1** 0 ≤ x ≤ 360°

Divide by 2 ⇒ $\sin x = \frac{1}{2}$ obtain two solutions (30° and 150°)

Type 2: Solve **√2 cos θ + 1 = 0** 0 ≤ x ≤ 2π

Re-arrange √2 cos θ = -1 hence $\cos\theta = -\frac{1}{\sqrt{2}}$

Type 3: Solve **sin 3x = -1** 0 ≤ x ≤ 360°

The range becomes 0 ≤ 3x ≤ 1080° Now sin( ) = -1 at 270°
but to cover the range we need 270°, 270° + 360°, 270° + 720°
solutions are: x = 90°, 210°, 330°

In general if you have sin *nx*, cos *nx*, tan *nx* then multiply your range by *n*

Type 4: Solve **2sin² x = 1** 0 ≤ x ≤ 360°

Re-arrange to get: $\sin^2 x = \frac{1}{2}$ Taking square roots gives $\sin x = \pm\frac{1}{\sqrt{2}}$

Note now there are **2 equations to solve** and you will obtain 4 solutions.

solutions are: x = 45°, 135° and x = 225° and 315°

Type 5: Solve **4sin² x + 11 sin x + 6 = 0**
0 ≤ x ≤ 2π

A **quadratic equation** in sin x : Factorising ⇒ (4sin x + 3)(sin x + 2) = 0
reduces to 2 simpler equations. **Solutions are: x = 3.99 or 5.43 radians**
Note that sin x + 2 = 0 has **no solutions** so discard it.

Type 6: Solve **sin² x – cos x = 1**
0 ≤ x ≤ 360°

Use sin² x + cos² x = 1 (see table right)
Replace sin² x with 1 - cos² x
Now a quadratic in cos x: 1 - cos² x - cos x = 1
Re-arrange and factorise: cos x (cos x + 1) = 0
solutions: x = 90° and 270° or **x = 180°**

$$\boxed{\begin{array}{l}\sin^2 x + \cos^2 x = 1\\ \sin^2 x = 1 - \cos^2 x\\ \cos^2 x = 1 - \sin^2 x\end{array}}$$

Type 7: Solve **sin (2x – 20) ° = 0.5**
0 ≤ x ≤ 360°

Range becomes 0 ≤ x ≤ 720°
Now sin (….) = 0.5 ⇒ an acute angle of 30°
so we have (…….) = 30° or 150° or 390° or 510° (giving four solutions)
This gives four equations: like 2x – 20 = 30, 2x – 20 = 150, etc.
Solutions: **x = 25°, 85°, 205°, 265°**

Unit 1 - 3.1	**Introduction to Differentiation**

The limit formula

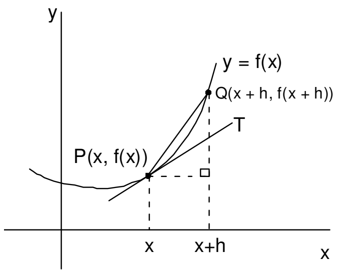
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

We define this limit as f′(x) this is known as the gradient function or derived function

We have differentiated f(x) and obtained f ′(x).

f ′(x) is the derivative of f(x).

This is a function that allows us to calculate the gradient at any point x on the curve.



Differentiation relates primarily to the **gradient** of a graph or function (generally a **curve**).

A graph is a pictorial representation of a function.

We are interested in the gradient of a curve (function), because the gradient is a measure of the **rate of change** of the function.

The **gradient** of a curve is continually changing as you move along the curve.

The **gradient** of the curve at point P is defined as the gradient of the **tangent** to the curve at P.

Calculating the gradient at a point P (x, y) on the curve y = f(x)

- Differentiate the function f(x) to get f ′(x)
- Evaluate function f ′(x) at point P(x, y)

Example: Find the gradient on the curve $f(x) = 2x^2 + 3x + 5$ at P(-2, 1)

Solution: Differentiate $\Rightarrow$ $f'(x) = 4x + 3$

Evaluate $f'(-2) = 4(-2) + 3 \Rightarrow f'(-2) = -5$

Gradient at P(-2, 1) = –5

Rules for differentiation:

f(x)	f ′(x)
x^n	nx^{n-1}
c (constant)	0
ax (a is a constant)	a
$a\,x^n$	$a\,nx^{n-1}$
f(x) + g(x)	f ′(x) + g ′(x)
$3x^2 + 2x + 1$	$6x + 2$

These rules work for any power of n - positive or negative, whole number or fractional.

General Rule:
Put the power in front (multiply), and **decrease the power by 1**.

IMPORTANT:

You must have the function f(x) as a polynomial, a series of powers of x.

You cannot differentiate fractions, brackets or anything else directly at present.

Indices:

Rules of indices
$a^m \times a^n = a^{m+n}$
$a^m \div a^n = a^{m-n}$
$(a^n)^m = a^{nm} = (a^m)^n$

Recall rules of indices:

Also recall meaning of fractional and negative indices.

$$x^{-1} = \frac{1}{x} \qquad x^{-n} = \frac{1}{x^n} \qquad x^{\frac{m}{n}} = \left(\sqrt[n]{x}\right)^m = \sqrt{x^m}$$

$$x^{\frac{1}{n}} = \sqrt[n]{x} \qquad x^{-\frac{1}{n}} = \frac{1}{\sqrt[n]{x}} \qquad x^{-\frac{m}{n}} = \frac{1}{\left(\sqrt[n]{x}\right)^m} = \frac{1}{\sqrt[n]{x^m}}$$

$$\frac{1}{2x^n} = \frac{1}{2}\cdot\frac{1}{x^n} = \frac{1}{2}x^{-n} \qquad \frac{3}{4x^n} = \frac{3}{4}\cdot\frac{1}{x^n} = \frac{3}{4}x^{-n}$$

Leibnitz Notation: $\dfrac{dy}{dx}$ or $\dfrac{d}{dx}((f(x))$ or $\dfrac{df}{dx}$

Newton's notation: f ′(x) or y ′(x) or y ′

Use Leibnitz or Newtons notation depending upon the wording of the question.

Both notations are equivalent.

Unit 1 - 3.1 Introduction to Differentiation

Changing functions to straight line form – or simple index form.

Fractional functions:
either:
- express in index notation, or
- put into separate fractions

Examples:

Type 1: f (x) = $x^2 + \dfrac{3}{x}$ change to f(x) = $x^2 + 3x^{-1}$ to differentiate

Type 2: f(x) = $\dfrac{x^4 + 2x^2 + 3}{x}$ change to f(x) = $\dfrac{x^4}{x} + \dfrac{2x^2}{x} + \dfrac{3}{x}$

then simplify to $x^3 + 2x + 3x^{-1}$ (straight line form)

Finding the gradient of the tangent to a curve at P(a, b):

- Differentiate to get $\dfrac{dy}{dx}$ or f′(x)
- Evaluate gradient function $\dfrac{dy}{dx}$ at P

Example: Find the gradient of the tangent to f(x) = $3x^3 - 5x + 2$ at P(2, 1)

Solution: Differentiate ⇒ f′(x) = $9x^2 - 5$

Evaluate at P(2, 1)

f′(2) = $9(2)^2 - 5 = 36 - 5 = 31$

Hence gradient of tangent at P(2,1) = 31

Finding the equation of the tangent:

- Find gradient.
- Find the y co-ordinate of point if not given.
- Use gradient formula for equation: $\dfrac{y - y_1}{x - x_1} = m$

Find gradient by differentiation and evaluation.
Find the y co-ordinate of the point by putting x co-ordinate into original equation.

Use formula for the equation: $\dfrac{y - y_1}{x - x_1} = m$ or $y - a = m(x - b)$

Example: Find equation of the tangent to y = $x^2 + 3$ at x = 2

Solution: Gradient of tangent is $\dfrac{dy}{dx} = 2x$ when x = 2 gradient = 4

When x = 2, y = $(2)^2 + 3 = 7$

Hence equation is: y – 7 = 4 (x – 2) ⇒ **y – 4x + 1 = 0**

Find point on curve where tangent has a given gradient (say gradient to be 3)

- Find gradient function by differentiation
- Put gradient function = 3; solve equation.
- Get y co-ordinate from original equation.

Example: Find the point on the curve y = $2x^2 + 1$ where gradient = 5

Solution: Gradient function of curve is $\dfrac{dy}{dx} = 4x + 1$

When gradient = 5, $\dfrac{dy}{dx} = 5$ thus, 4x + 1 = 5, so x = 1

When x = 1, y = $2(1)^2 + 1 = 3$ **Hence point is (1, 3)**

Graphs of derived functions

The derived function f ′(x) is the function resulting from differentiating f(x)

To sketch the derived function:

Method:

Step 1. Mark the zeros on the x-axis

Note sign of gradient either side of zeros.

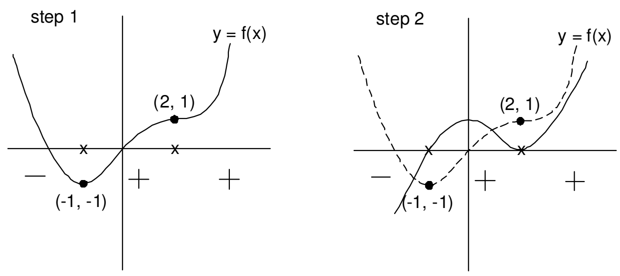
Step 2. Sketch the derived function – it must be **below** the x axis where the gradient was negative and **above** the x axis where the gradient was positive –

What will fit ?

Step 1. Locate the point(s) where the gradient of the function is **ZERO** (i.e. a turning point) – mark these points on the x-axis.
Note either side of the point whether gradient is **positive** or **negative**

Step 2. Sketch the derived function – it must be **below** the x axis where you deduced the gradient was negative and **above** the x axis where the gradient was positive – What will fit ?

Consider the form of the derived function by differentiation
– is it a straight line, quadratic, cubic etc.



Unit 1 - 3.2	Using Differentiation

Finding Stationary points and their nature

- Differentiate to find the gradient function
- For a stationary point, dy/dx or f'(x) = 0
- Solve the equation to find the x co-ordinate(s)
- Substitute into the original equation y = or f(x) = to get the y co-ordinate(s)
- Determine the nature of each stationary point using a table of signs as shown in the example.

  When checking for the nature – **use any factorisation** that you have for $\frac{dy}{dx}$

If you try to deduce the signs from a complicated expression, you will probably get it wrong.

In each case you are looking either to the **left** or the **right** of the stationary point.

Minimum S.P. have signs: – 0 +

Maximum S.P. have signs: + 0 –

Points of Inflexion have signs: – 0 – or + 0 +

Example: Find the stationary points of $y = x^3 + 3x^2 - 9x + 1$ and determine their nature.

Differentiate to get: $\frac{dy}{dx} = 3x^2 + 6x - 9$

For a s.p. $\frac{dy}{dx} = 0$ so $3x^2 + 6x - 9 = 0$ or $x^2 + 2x - 3 = 0$

hence $(x - 1)(x + 3) = 0$ so s.p. occur when $x = 1$ or $x = -3$

now find y co-ordinates: when $x = 1$ $y = (1)^3 + 3(1)^2 - 9(1) + 1 = 6$

when $x = -3$ $y = (-3)^3 + 3(-3)^2 - 9(-3) + 1 = 28$

So stationary points are **(1, 6)** and **(-3, 28)**

Now check for their nature. Using the factorisation in dy/dx

x	→	-3	→	1	→
(x-1)	-	↓	-	↓	+
(x+3)	-	↓	+	↓	+
$\frac{dy}{dx}$	+	0	-	0	+

max min

Hence stationary points are: **(1, 6) minimum** and **(-3, 28) maximum**

The interval on which the function is increasing or decreasing.

To determine the interval on which the function is increasing or decreasing, you need to look at the stationary points and the gradient on each side of them.

Where the gradient is **positive**, the function is **increasing**.

Where the gradient is **negative**, the function is **decreasing**.

Where the gradient is **zero**, then the function is **stationary!**

In the above example:

The function is **increasing** for: $x < -3$ and $x > 1$

and **decreasing** for: $-3 < x < 1$

You need the stationary points to determine the length of the interval.

Maximum and minimum value on a closed interval

If a closed interval is specified for a graph, then the maximum and minimum value of the function will either be at a **stationary** point **OR** one of the **end points** of the graph.

To determine the maximum and minimum value on a closed interval:

- Find the stationary points of the function
- If any lie outside the interval, discard them
- Check the S.V. of each stationary point (i.e. y co-ordinate)
- Check the value of the function at each end of the interval.
- State the maximum and minimum value of the function on this interval.

Example: Find the maximum and minimum value of $y = x^3$ on [1, 3]

$\frac{dy}{dx} = 3x^2$ for a S.P. $\frac{dy}{dx} = 0$ so $3x^2 = 0$ hence $x = 0$ (outside of interval)

Now check ends of interval [1, 3] $y(1) = 1$ and $y(3) = 27$

Hence on the interval [1, 3], $y = x^3$ has max value of 27 and min value of 1

Unit 1 - 3.2 | Using Differentiation

Curve sketching

A practical application of maximum and minimum.

- Points of intersection with x and y axes:
- Find stationary points using differentiation
- Find y co-ordinates by substitution
- Nature of S.P. using table of signs.
- Behaviour for large values of + and – x
- Any useful points on the graph.
- Sketch **(DO NOT PLOT)** the graph.

Detailed explanation:

Find points of intersection with x and y axes:
for intersection with y axis ~ put x = 0
for intersection with x axis ~ put y = 0 and solve the equation.

Find the stationary points and their nature
Differentiate and put dy/dx = 0, solve the equation to get x co-ordinates
Substitute into original equation to get y co-ordinates
Determine nature of stationary point(s) using table of signs.

Behaviour for large positive and negative x
Look at behaviour of the graph for large values of positive and negative x
Approximate – what does the curve behave like
e.g. $y = x^3 + 2x + {}^1/_x$ behaves as x^3 for large x and ${}^1/_x$ as $x \to 0$

Consider any other particular points on the graph. e.g. x = 0 or x = 1 (say)

Problem solving

- Mathematical modelling.
- Use constraints to make an equation.
- Look for a maximum or minimum.

Method:

Make an equation to represent the model – you will have two unknowns at this stage.

Your constraint will connect these two unknown variables.

Using your constraint, obtain a function with only one variable.

Differentiate and find any stationary points – generally there will only be one.

Find the nature of the stationary point (max or min) using table of signs.

The value of stationary point will cause the model to have a maximum or minimum value.

Interpret your solution into the form of the question.

Example:

A rectangle has length x cm and breadth y cm and perimeter p cm. Its area is 100 cm^2. Find the length and breadth of the rectangle with the smallest perimeter.

Solution:

Since we want to find the smallest perimeter, we need an expression for the perimeter.

P = 2x + 2y This is our model with two unknowns.

We now need to use the constraint to reduce it to one unknown.

Area = xy 100 = xy so $y = \frac{100}{x}$ now replace y in our equation for P

$P = 2x + \frac{200}{x}$ we want to find a value for x to make perimeter P a minimum.

$P = 2x + 200x^{-1}$ differentiating $\Rightarrow \frac{dP}{dx} = 2 - 200x^{-2}$

For a s.p. $\frac{dy}{dx} = 0$ hence $0 = 2 - \frac{200}{x^2} \Rightarrow 2x^2 = 200 \Rightarrow x^2 = 100$

so x = +10 or x = -10 (this last solution is not possible, so discard it).

hence our S.P. is when x = 10. Using our constraint, we find that y = 10 also.

We should verify that this is a minimum by using the table of signs.

Hence dimensions of the rectangle are **10cm × 10cm for minimum perimeter**.

Rate of Change

The rate of change of y with respect to x is

given by: $\frac{dy}{dx}$

Negative rate of change means function is decreasing

Positive rate of change means that it is increasing.

Velocity and acceleration

A point P moving along the x axis has a displacement x (OP) from the origin O at time t. We can model this as: x = f(t)

The **velocity** of the point P is the rate of change of its displacement x at time t, given by $v = \frac{dx}{dt}$

The **acceleration** of P is the rate of change of its velocity v at time t,

given by: $a = \frac{dv}{dt}$

Note: When using velocity and acceleration remember that they are vectors and have **direction** as well as magnitude.

When using them vertically as in height problems, the **greatest height reached** is when the **velocity = 0**

When the **acceleration = 0**, the object is moving at **constant velocity.**

Unit 1 - 4	Sequences
Formula for n^{th} term Given a formula for the n^{th} term, we can calculate all the terms. Conversely given a sequence, we can find a formula for its n^{th} term.	**e.g.** $u_n = 3n + 2$ start with n = 1 giving $u_1 = 5$, $u_2 = 8$, $u_3 = 11$ etc. e.g. 5, 9, 13, 17, 21 …. It goes up in multiples of 4 (we add 4 on each time) so start off with $u_n = 4n$ However it is not the 4 times table – it is offset by 1 more So the nth term is given by $u_n = 4n + 1$ Now check to see if this generates the sequence.
Recurrence Relations If we are given the first term of a sequence and a rule for calculating u_{n+1} from u_n, we can calculate all of its terms. The **recurrence relation** is the rule for calculating the $n+1^{th}$ term from the n^{th} term.	**e.g.** If the recurrence relation is $u_{n+1} = u_n + 7$ all this means is:- add 7 to the any term in the sequence to get the next term in the sequence. If the 2^{nd} term is 12 i.e. $u_2 = 12$ then $u_3 = u_2 + 7$ or $u_3 = 12 + 7 = 19$ In general - given the first term and the recurrence relation, we can generate all the terms of the sequence: **e.g.** $u_1 = 5$ and $u_{n+1} = 2u_n + 3$ This will generate the sequence: 5, 13, 29, 61, ….. Conversely, given the sequence, it may be possible to define it by giving the first term and the recurrence relation (the relationship between u_{n+1} and u_n). **e.g.** 13, 10, 7, 4, ….. first term is 13 rule is: subtract 3 to get next term. so: $u_1 = 13$ $u_{n+1} = u_n -3$
Forming recurrence relationships modelling a real-life situation.	**Example**: A mushroom bed has 60 mushrooms. Each morning the number has doubled, and the gardener picks 50 mushrooms. Start from day n – there are u_n mushrooms Look at the next day - there are twice as many $2u_n$ but the gardener has picked 50 so there will be $2u_n – 50$ mushrooms. So: $u_{n+1} = 2u_n – 50$ **Example**: There are 3 trees in Jim's garden. He plants 2 more trees each day for the next 6 days. Take u_n trees to be the number of trees after n days, (This means that that original number of 3 trees is u_0 ~ $u_0 = 3$) Then: $u_{n+1} = u_n +2$
Linear Recurrence Relations These are of the form: $\mathbf{u_{n+1} = mu_n + c}$ where m and c are constants Special sequences are obtained if m = 1 or c = 0	(compare with y = mx + c). if m = 1 you get an arithmetic sequence if c = 0 you get a geometric sequence
Arithmetic Sequences If m = 1 in the recurrence relation $u_{n+1} = mu_n + c$ then $\mathbf{u_{n+1} = u_n + c}$	the difference between successive terms in the sequence is the constant c. This is an arithmetic sequence. We are just adding on c each time. e.g. $u_{n+1} = u_n + 2$ if $u_1 = 3$ then the sequence generated is 3, 5, 7, 9, …. Note the constant 2 being added on for each successive term.

Unit 1 - 4	**Sequences**
Geometric Sequences If $c = 0$ in the recurrence relation $u_{n+1} = mu_n + c$ then $\mathbf{u_{n+1} = mu_n}$ ~ each term is multiplied by a constant **m**	In other words the ratio of successive terms is constant. This is an geometric sequence. We are just multiplying by m each time.. e.g. $u_{n+1} = 3u_n$ if $u_1 = 2$ then the sequence generated is 2, 6, 18, 54, ……
Examples of Geometric Sequences **Example 1:** Every year a typical bag of groceries rises in price by 5%. Its initial value V_0 is £20 Describe the price by a recurrence relation.	Current years price is 5% more than last year $V_{n+1} = 1.05V_n$ and $V_0 = 20$
Example 2: There are 40 fish in a pond, 10% are eaten but 3 new ones are born every day. Describe this by a recurrence relation.	If 10% are eaten then 90% are left for the next day The recurrence relation is $u_{n+1} = 0.9u_n + 3$ where $u_0 = 40$ (u_0 in this case because it is an initial condition).
Example 3: A sequence is generated by the recurrence relation, the n^{th} term being u_n $u_{n+1} = 4u_n - 2$ $u_1 = 2$	The sequence generated is: 2, 6, 10, 14, ….. Note that a term u_0 would not make sense in this example.
When expressing a recurrence relation – write down the relation **AND** the first term u_0 or u1 as appropriate.	In general u_0 is an initial condition before the recurrence relation starts. u_1 is the first term of the recurrence relation. Effectively it depends upon how you define u_n
Examples forming recurrence relations:	
u_n is the number of bacteria in a culture after n hours. At present there are 100 but their number doubles after each hour.	$\mathbf{u_{n+1} = 2u_n}$ $\mathbf{u_0 = 100}$
An office plant is 150cm tall and its height increases each month by 5% of its height at the beginning of the month. H_n is the height after n months.	$\mathbf{H_{n+1} = 1.05u_n}$ $\mathbf{H_0 = 150}$ **cm**
Jim is a salesman, travelling 300 km per week. His mileometer reads 9350 when he begins R_n is the reading after n weeks.	$\mathbf{R_{n+1} = R_n + 300}$ $\mathbf{R_0 = 9350}$

Unit 1 - 4 Sequences

Finite and Infinite Sequences

A finite sequence has a finite number of terms – there are a fixed number of terms in the sequence.

An infinite sequence has an unlimited number of terms, they continue on forever.

If the n^{th} term tends to a limiting value as n gets very large (i.e. n tends to infinity $n \to \infty$) the sequence is convergent – it converges to a limit.

Examples:

1. $u_n = 2 - \frac{1}{n^2}$ as $n \to \infty$ $\frac{1}{n^2} \to 0$ and so $u_n \to 2$

2. $u_n = 1 - (0.5)^n$ as $n \to \infty$ $(0.5)^n \to 0$ and so $u_n \to$

3. $u_n = \frac{2n+1}{n}$ we need to re-arrange this to:

   $u_n = \frac{2n}{n} + \frac{1}{n} \Rightarrow u_n = 2 + \frac{1}{n}$ and as $n \to \infty$ $u_n \to 2$

4. $u_n = \frac{n}{n+1}$ slightly trickier here the following technique is useful:

   the denominator n may be written as $n = n + 1 - 1$

   so we get: $u_n = \frac{n+1-1}{n+1} \Rightarrow u_n = \frac{n+1}{n+1} - \frac{1}{n+1} \Rightarrow u_n = 1 - \frac{1}{n+1}$

   and so as $n \to \infty$ $u_n \to 1$

The Geometric Sequence

If a geometric sequence is written $u_{n+1} = ru_n$ where r is the multiplier each time

and u_1 = **a** (the **first term**)
{we call **r** the **common ratio**}

We can write the sequence out as:
a, ar, ar^2, ar^3, ………. ar^{n-1} (n^{th} term)

Define the sum to n terms, S_n as
$S_n = a + ar + ar^2 + ar^3 + \ldots\ldots\ldots + ar^{n-1}$

Then: $S_n = \frac{a(1-r^n)}{(1-r)}$

If $-1 < r < 1$ as $n \to \infty$ $r^n \to 0$
and so the sum S_n will tend to a limit.

The limit of S_n as $n \to \infty$ will be $S_n = \frac{a}{1-r}$

Can we find an expression to allow us to calculate S_n easily ?
This time we use a different technique. Multiply the series by r, giving us:

$rS_n = ar + ar^2 + ar^3 + \ldots\ldots\ldots + ar^n$

Now subtract this series from the original one, term by term,

$S_n - rS_n = a + \underline{ar} + \underline{ar^2} + \underline{ar^3} + \ldots\ldots\ldots + \underline{ar^{n-1}}$

$- \{ \underline{ar} + \underline{ar^2} + \underline{ar^3} + \ldots\ldots\ldots + ar^n \}$

Notice that all the underlined terms cancel out, leaving us with:

$S_n - rS_n = a - ar^n$ which we can factorise as $S_n(1-r) = a(1-r^n)$

re-arrange to get: $S_n = \frac{a(1-r^n)}{(1-r)}$ **a** is the **first term** and **r** is the **common ratio**.

Unit 1 - 4	Sequences

The linear recurrence relation
$\mathbf{u_{n+1} = mu_n + c}$ with $m \neq 1$ and $c \neq 0$

Example:

A loch contains 10 tonnes of toxic waste. Tidal action removes 50% of the waste each week, but a local factory discharges 8 tonnes of waste into the loch at the end of each week. u_n is the amount of waste in the loch after n weeks.

The recurrence relation modelling this is: $\mathbf{u_{n+1} = 0.5u_n + 8}$ $\mathbf{u_0 = 10}$

If we write down the first few terms in the sequence we find:

$u_0 = 10$, $u_1 = 13$, $u_2 = 14.5$, $u_3 = 15.25$, $u_4 = 15.625$, $u_5 = 15.8125$ ….

It seems to be levelling off at 16 tonnes.

Limits: If $-1 < m < 1$ then u_{n+1} and u_n will each tend to a limit L

So in $u_{n+1} = 0.5u_n + 8$ we can say $L = 0.5L + 8$

and so $L - 0.5L = 8$ $L(1 - 0.5) = 8$

$L = \frac{8}{1-0.5} \Rightarrow L = \frac{8}{0.5}$ L = 16 as deduced above.

In general:

For a recurrence relation $\mathbf{u_{n+1} = mu_n + c}$

Provided m is a fraction i.e. $-1 < m < 1$

then **a limit L exists**

and we can replace u_{n+1} and u_n by L

$L = mL + c$ re-arranging gives $L - mL = c$

hence $L(1 - m) =$ and so $L = \frac{c}{1-m}$

This is an important result.

Examples:

A mushroom bed has 1000 mushrooms ready for picking. Each morning 60% of the crop are picked. Each night another 200 are ready for picking. Let M_n be the number ready for picking after n days.

Write down the recurrence relation, find the limit of the sequence explaining what it means in the context of this question.

Recurrence relation: $\mathbf{M_{n+1} = 0.4M_n + 200}$ (60% picked $\Rightarrow$ 40% left)

multiplier m is a fraction so a limit exists i.e. M_{n+1} and $M_n \rightarrow L$

So $L = 0.4L + 200$ $L - 0.4L = 200$ $0.6L = 200$ $\mathbf{L = 333.33….}$

In the long term, the number of mushrooms ready for picking will settle out at around 333.

(In this case it will drop down to 333 since the initial condition was 1000 mushrooms).

Example:

Dr Sharma is studying a flock of 200 birds. Every minute 10% of the birds leave the flock and 30 birds return. Let B_n be the number of birds in the flock at the end of minute n.

Write down the recurrence relation, find the limit of the sequence explaining what it means in the context of this question.

Recurrence relation: $\mathbf{B_{n+1} = 0.9B_n + 30}$ (10% leaving $\Rightarrow$ 90% left)

multiplier m is a fraction so a limit exists i.e. B_{n+1} and $B_n \rightarrow L$

This time we will use the result: $L = \frac{c}{1-m}$ and so $L = \frac{30}{1-0.9}$ L = 300

In the long term, the number of birds in the flock will settle out at around 300

(In this case it will rise to 300 since the initial condition was 200 birds)

It is important to be very careful in how you phrase an answer to the common question:

"Explain what the limit means in the context of this question"

The safest wording is that "The number of …… will settle out at around ….."

1. Since it will never actually get there (the limit is only achieved when n is infinite)
2. If you do not know the initial condition, then you do not know whether the sequence is decreasing down to the limit or increasing up to the limit.
3. If the multiplier is negative, then the sequence will oscillate on both sides of the limit.

Example:

Consider the recurrence relation: $u_{n+1} = 0.5u_n + 100$ and $u_0 = 500$

Limit will be: $L = \frac{c}{1-m}$ and $L = \frac{100}{1-0.5}$ L = 200

in this case the sequence **drops down** to this limit.

If we have $u_{n+1} = 0.5u_n + 100$ and $u_0 = 50$ i.e. the initial value is 50

then the sequence would **rise up** to the limit.

In all cases:

Think through the implications of the questions, these are practical examples modelling real-life situations.