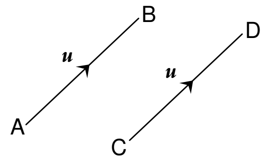
Unit 3 - 1 Vectors

A vector may be considered as a set of instructions for moving from one point to another.

A line which has both magnitude and direction can represent this vector.

The vector $\boldsymbol{u}$ can be represented in magnitude and direction by the directed line segment $\overrightarrow{AB}$.

The length of $\overrightarrow{AB}$ is proportional to the magnitude of u and the arrow shows the direction of $\boldsymbol{u}$.



AB and CD both represent the same vector $\boldsymbol{u}$

A vector does not have a position – only magnitude and direction, so many different directed line segments may represent this vector.

We say <u>directed line segment</u> because AB indicates movement <u>from</u> A <u>to</u> B whereas BA would indicate movement <u>from</u> B <u>to</u> A

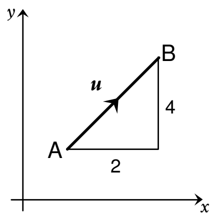
Components of a Vector in 2 dimensions:

To get from A to B you would move:

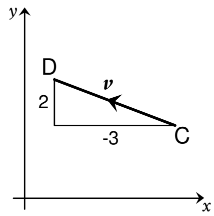
2 units in the x direction (x-component)
4 units in the y direction (y-component)

The components of the vector are these moves in the form of a column vector.

thus $\overrightarrow{AB} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$ or $\boldsymbol{u} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$



A 2-dimensional column vector is of the form $\begin{pmatrix} x \\ y \end{pmatrix}$



Similarly: $\overrightarrow{CD} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$ or $\boldsymbol{v} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$

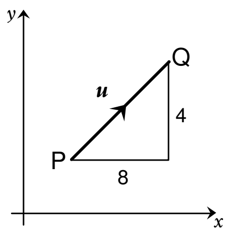
Magnitude of a Vector in 2 dimensions:

We write the magnitude of $\boldsymbol{u}$ as $|\boldsymbol{u}|$

$$\boldsymbol{u} = \begin{pmatrix} x \\ y \end{pmatrix} \quad then \quad |\boldsymbol{u}| = \sqrt{x^2 + y^2}$$

The magnitude of a vector is the length of the directed line segment which represents it.

Use Pythagoras' Theorem
to calculate the length of the vector.



The magnitude of vector $\boldsymbol{u}$ is $|\boldsymbol{u}|$ *(the length of PQ)*

The length of PQ is written as $\left|\overrightarrow{PQ}\right|$

$$\overrightarrow{PQ} = \begin{pmatrix} 8 \\ 4 \end{pmatrix} \quad \text{then} \quad \left|\overrightarrow{PQ}\right|^2 = 8^2 + 4^2$$

and so $\left|\overrightarrow{PQ}\right| = \sqrt{8^2 + 4^2} = \sqrt{80} = 8.9$

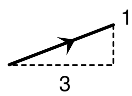
Examples:

1. Draw a directed line segment representing $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$

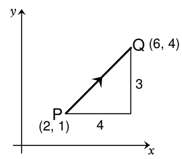
2. $\overrightarrow{PQ} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$ and P is (2, 1), find co-ordinates of Q

3. P is (1, 3) and Q is (4, 1) find $\overrightarrow{PQ}$

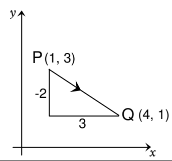
Solutions:

1. 

2. Q is (2 + 4, 1 + 3) $\rightarrow$ Q(6, 4)



3. $\overrightarrow{PQ} = \begin{pmatrix} 4-1 \\ 1-3 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$



Vector:
A quantity which has magnitude and direction.

Examples:
Displacement, force, velocity, acceleration.

Scalar:
A quantity which has magnitude only.

Examples:
Temperature, work, width, height, length, time of day.

Unit 3 - 1	**Vectors**

Vectors in 3 dimensions:

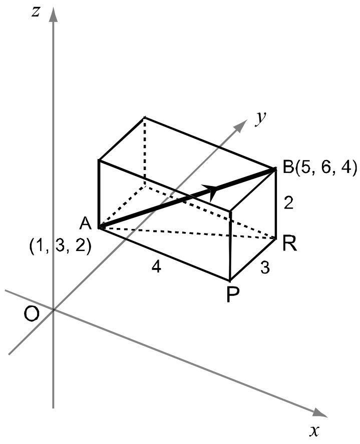
3 dimensional vectors can be represented on a set of 3 axes at right angles to each other (orthogonal), as shown in the diagram.

Note that the z axis is the vertical axis.

To get from A to B you would move:

4 units in the x-direction, (x-component)
3 units in the y-direction, (y-component)
2 units in the z-direction. (z-component)

In component form: $\overrightarrow{AB} = \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix}$



In general: $\overrightarrow{AB} = \begin{pmatrix} x_B - x_A \\ y_B - y_A \\ z_B - z_A \end{pmatrix}$,

Magnitude of a 3 dimensional vector

$$|\boldsymbol{u}| = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}$$

This is the length of the vector.

Use Pythagoras' Theorem in 3 dimensions.

$$AB^2 = AR^2 + BR^2$$
$$= (AP^2 + PR^2) + BR^2$$
$$= (x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2$$

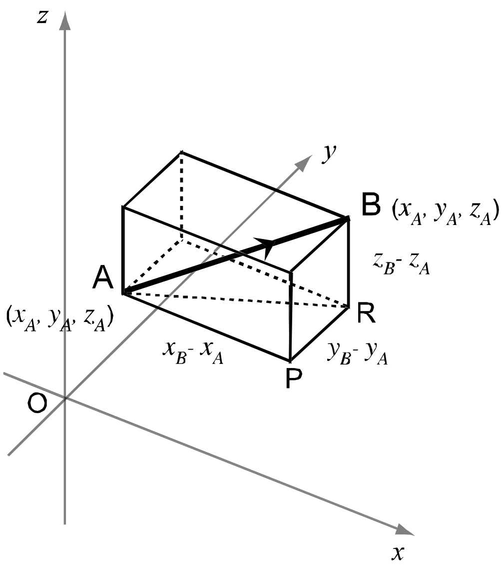
and if $\boldsymbol{u} = \overrightarrow{AB}$

then the magnitude of $\boldsymbol{u}$, $|\boldsymbol{u}|$ = length of AB

This is known as the
Distance formula for 3 dimensions

Recall that since: $\overrightarrow{AB} = \begin{pmatrix} x_B - x_A \\ y_B - y_A \\ z_B - z_A \end{pmatrix}$, then

if $\boldsymbol{u} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ then $|\mathbf{u}| = \sqrt{x^2 + y^2 + z^2}$



Since $x = x_B - x_A$ and $y = y_B - y_A$ and $z = z_B - z_A$

Example:

1. If A is (1, 3, 2) and B is (5, 6, 4)
   Find $|\overrightarrow{AB}|$

$$|\overrightarrow{\boldsymbol{AB}}| = \sqrt{(5-1)^2 + (6-3)^2 + (4-2)^2} = \sqrt{4^2 + 3^2 + 2^2} = \sqrt{29}$$

2. If $\boldsymbol{u} = \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix}$ Find $|\boldsymbol{u}|$

$$|\mathbf{u}| = \sqrt{(3)^2 + (-2)^2 + (2)^2} = \sqrt{9+4+4} = \sqrt{17}$$

Unit 3 - 1	Vectors

The components of a vector are **unique**.

i.e. a vector has only **one** set of components

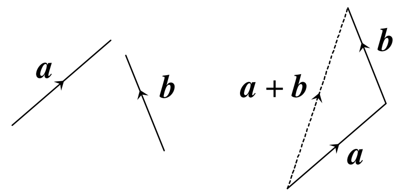
So if two vectors are equal, then their components are equal.

e.g. if $\begin{pmatrix} 2x \\ y+3 \\ z-1 \end{pmatrix} = \begin{pmatrix} 6 \\ 8 \\ 2 \end{pmatrix}$ then x = 3, y = 5 and z = 3

Addition and subtraction of vectors

Vectors are added ‘nose to tail’

This is known as the Triangle Rule.



To calculate $\boldsymbol{a} + \boldsymbol{b}$ we add the components

$$\boldsymbol{a} = \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix} \quad \boldsymbol{b} = \begin{pmatrix} 6 \\ 1 \\ 0 \end{pmatrix} \quad \boldsymbol{a}+\boldsymbol{b} = \begin{pmatrix} 3+6 \\ 2+1 \\ 5+0 \end{pmatrix} = \begin{pmatrix} 9 \\ 3 \\ 5 \end{pmatrix}$$

To calculate $\boldsymbol{a} - \boldsymbol{b}$ we subtract the components

$$\boldsymbol{a} = \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix} \quad \boldsymbol{b} = \begin{pmatrix} 6 \\ 1 \\ 0 \end{pmatrix} \quad \boldsymbol{a}-\boldsymbol{b} = \begin{pmatrix} 3-6 \\ 2-1 \\ 5-0 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ 5 \end{pmatrix}$$

The Zero Vector is $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

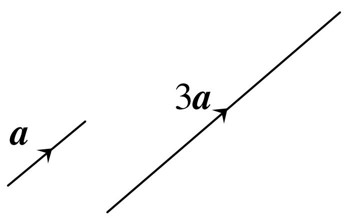
To obtain the negative of a vector – multiply all its components by –1

$$\boldsymbol{p} = \begin{pmatrix} 3 \\ -2 \\ 7 \end{pmatrix} \quad \text{then} \quad -\boldsymbol{p} = \begin{pmatrix} -3 \\ 2 \\ -7 \end{pmatrix}$$

Multiplying by a scalar

A vector can be multiplied by a number (scalar).

e.g. multiply ***a*** by 3 - written 3***a***
Vector 3***a*** has three times the length but is in the **same** direction as ***a***



In component form, each component will be multiplied by 3.

$$\boldsymbol{a} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} \quad \text{then} \quad 3\boldsymbol{a} = \begin{pmatrix} 6 \\ 3 \\ -9 \end{pmatrix}$$

We can also take a common factor out of a vector in component form.

$$\boldsymbol{v} = \begin{pmatrix} 12 \\ 16 \\ -4 \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{v} = 4\begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix}$$

Scalar Multiples

If a vector is a scalar multiple of another vector, then the two vectors are parallel, and differ only in magnitude.

This is a useful test to see if lines are parallel.

$$\boldsymbol{u} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \quad \text{and} \quad \boldsymbol{v} = \begin{pmatrix} -6 \\ 3 \\ -9 \end{pmatrix} \quad \text{then} \quad \boldsymbol{v} = -3\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$$

v is a scalar multiple of ***u*** and so ***v*** is parallel to ***u***.

$$\boldsymbol{p} = \begin{pmatrix} 8 \\ -4 \\ -12 \end{pmatrix} \quad \text{and} \quad \boldsymbol{q} = \begin{pmatrix} -6 \\ 3 \\ 9 \end{pmatrix} \quad \text{then} \quad \boldsymbol{p} = 4\begin{pmatrix} 2 \\ -1 \\ -3 \end{pmatrix} \quad \text{and} \quad \boldsymbol{q} = -3\begin{pmatrix} 2 \\ -1 \\ -3 \end{pmatrix}$$

p and ***q*** are scalar multiples of another vector and so again are parallel

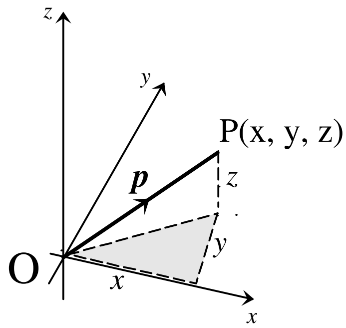
Unit 3 - 1 Vectors

Position Vectors

If P has co-ordinates P(x, y, z) then

vector $\overrightarrow{OP}$ has components $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$\overrightarrow{OP}$ is called the position vector of P and is written as $\boldsymbol{p}$



$$\overrightarrow{OP} = \begin{pmatrix} x-0 \\ y-0 \\ z-0 \end{pmatrix} = \boldsymbol{p} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

A useful result

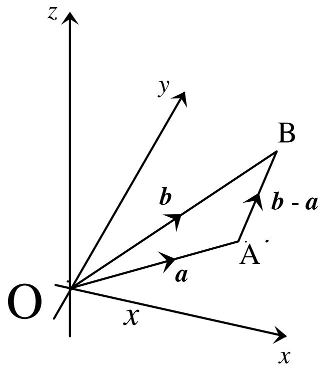
$\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$ thus $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \boldsymbol{b} - \boldsymbol{a}$

Any directed line segment may be written in terms of the position vectors of its end points.

e.g. $\overrightarrow{PQ} = \boldsymbol{q} - \boldsymbol{p}$ (note the order)

The component form of a position vector corresponds to the co-ordinates of the point.

P(x, y, z) $\Rightarrow \boldsymbol{p}$ is $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$



Collinear Points

Points are collinear if one straight line passes through all the points.

For three points A, B, C - if the line AB is parallel to BC, since B is common to both lines, A, B and C are collinear.

Test for collinearity

1. Show line segments are parallel (ie. scalar multiples)

2. Ensure there is a **COMMON** point and <u>state it</u>.

Example: A is (0, 1, 2), B is (1, 3, –1) and C is (3, 7, –7)
Show that A, B and C are collinear.

$$\overrightarrow{AB} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} \quad \overrightarrow{BC} = \begin{pmatrix} 2 \\ 4 \\ -6 \end{pmatrix} \quad \text{and} \quad \overrightarrow{BC} = 2\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} = 2\overrightarrow{AB}$$

$\overrightarrow{AB}$ and $\overrightarrow{BC}$ are scalar multiples, so AB is parallel to BC.
Since B is a <u>**common**</u> point, then A, B and C are collinear

Position vector $\boldsymbol{m}$ of mid-point of AB

M is the mid point of AB

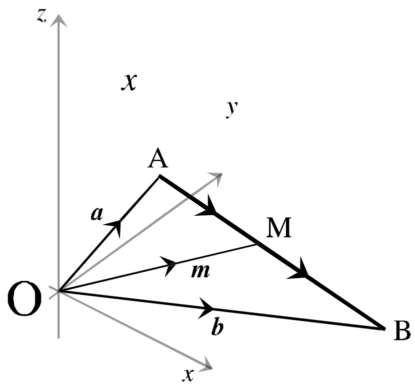
$\boldsymbol{a}$, $\boldsymbol{m}$ and $\boldsymbol{b}$ are the position vectors of A, M and B

$$\overrightarrow{AM} = \overrightarrow{MB} \text{ so}$$

$$m - a = b - m$$

$$2m = b + a$$

hence $\boldsymbol{m} = \frac{1}{2}(\boldsymbol{b} + \boldsymbol{a})$



Unit 3 - 1	Vectors

Points, Ratios and Lines

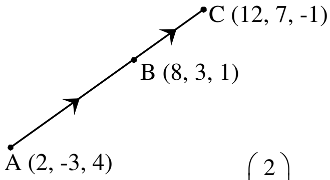
Find the ratio in which a point divides a line.

Example:

The points A(2, –3, 4), B(8, 3, 1) and C(12, 7, –1) form a straight line.

Find the ratio in which B divides AC.

Solution: B divides AC in ratio of 3 : 2



$$\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a} = \begin{pmatrix} 8-2 \\ 3-(-3) \\ 1-4 \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \\ -3 \end{pmatrix}$$

$$\overrightarrow{BC} = \boldsymbol{c} - \boldsymbol{b} = \begin{pmatrix} 12-8 \\ 7-3 \\ -1-1 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ -2 \end{pmatrix}$$

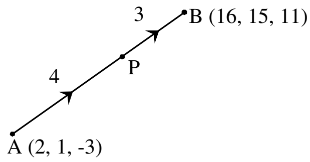
$$\overrightarrow{AB} = 3\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} \text{ and } \overrightarrow{BC} = 2\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} \quad \text{So, } \frac{\overrightarrow{AB}}{\overrightarrow{BC}} = \frac{3}{2} \ \textit{or} \ \text{AB : BC} = 3 : 2$$

Points dividing lines in given ratios.

Example:

P divides AB in the ratio 4:3. If A is (2, 1, –3) and B is (16, 15, 11), find the co-ordinates of P.

Solution: P is P(10, 9, 5)



$$\frac{\overrightarrow{AP}}{\overrightarrow{PB}} = \frac{4}{3} \quad \textit{so} \ 3\overrightarrow{AP} = 4\overrightarrow{PB}$$

$$\therefore \quad 3(\boldsymbol{p} - \boldsymbol{a}) = 4(\boldsymbol{b} - \boldsymbol{p})$$
$$3\boldsymbol{p} - 3\boldsymbol{a} = 4\boldsymbol{b} - 4\boldsymbol{p}$$
$$7\boldsymbol{p} = 4\boldsymbol{b} + 3\boldsymbol{a}$$
$$\boldsymbol{p} = \frac{1}{7}(4\boldsymbol{b} + 3\boldsymbol{a})$$

$$\boldsymbol{p} = \frac{1}{7}\left(4\begin{pmatrix} 16 \\ 15 \\ 11 \end{pmatrix} + 3\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}\right) = \frac{1}{7}\left(\begin{pmatrix} 64 \\ 60 \\ 44 \end{pmatrix} + \begin{pmatrix} 6 \\ 2 \\ -9 \end{pmatrix}\right) = \frac{1}{7}\begin{pmatrix} 70 \\ 63 \\ 35 \end{pmatrix} = \begin{pmatrix} 10 \\ 9 \\ 5 \end{pmatrix}$$

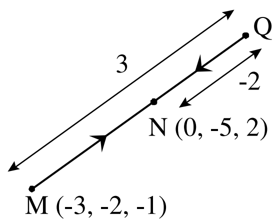
Points dividing lines in given ratios externally.

Example:

Q divides MN externally in the ratio of 3:2.
M is (–3, –2, –1) and N is (0, –5, 2).
Find the co-ordinates of Q.

Solution: Q is Q(6, –11, 8)

<u>Note</u> that QN is shown as –2 because the two line segments are MQ and QN, and QN is in the opposite direction to MQ.



$$\frac{\overrightarrow{MQ}}{\overrightarrow{QN}} = \frac{3}{-2} \quad \textit{so} \ -2\overrightarrow{MQ} = 3\overrightarrow{QN}$$

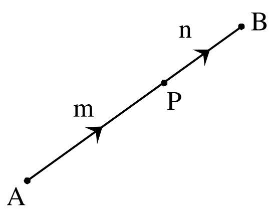
$$\therefore \quad -2(\boldsymbol{q} - \boldsymbol{m}) = 3(\boldsymbol{n} - \boldsymbol{q})$$
$$-2\boldsymbol{q} + 2\boldsymbol{m} = 3\boldsymbol{n} - 3\boldsymbol{q}$$
$$\boldsymbol{q} = 3\boldsymbol{n} - 2\boldsymbol{m}$$

$$\boldsymbol{q} = 3\begin{pmatrix} 0 \\ -5 \\ 2 \end{pmatrix} - 2\begin{pmatrix} -3 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -15 \\ 6 \end{pmatrix} - \begin{pmatrix} -6 \\ -4 \\ -2 \end{pmatrix} = \begin{pmatrix} 6 \\ -11 \\ 8 \end{pmatrix}$$

Example:

If P divides AB in the ratio m : n, show that $\boldsymbol{p}$, the position vector of P is given by:

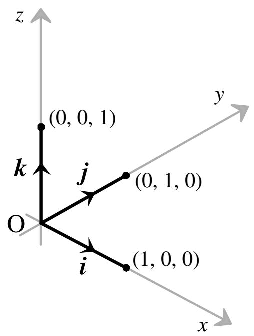
$$\boldsymbol{p} = \frac{m\boldsymbol{b} + n\boldsymbol{a}}{m + n}$$



$$\frac{\overrightarrow{AP}}{\overrightarrow{PB}} = \frac{m}{n} \quad \textit{so} \ n\overrightarrow{AP} = m\overrightarrow{PB}$$

$$\therefore \quad \text{n}(\boldsymbol{p} - \boldsymbol{a}) = \text{m}(\boldsymbol{b} - \boldsymbol{p})$$
$$\text{n}\boldsymbol{p} - \text{n}\boldsymbol{a} = \text{m}\boldsymbol{b} - \text{m}\boldsymbol{p}$$
$$\text{n}\boldsymbol{p} + \text{m}\boldsymbol{p} = \text{m}\boldsymbol{b} + \text{n}\boldsymbol{a}$$
$$(\text{n} + \text{m})\boldsymbol{p} = \text{m}\boldsymbol{b} + \text{n}\boldsymbol{a}$$
$$\boldsymbol{p} = \frac{m\boldsymbol{b} + n\boldsymbol{a}}{m + n}$$

q.e.d

Unit 3 - 1	**Vectors**
Unit Vectors Definition: A unit vector has a magnitude of 1	If $\overrightarrow{AB} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ then $a^2 + b^2 + c^2 = 1$
Unit Vectors $\boldsymbol{i},\ \boldsymbol{j},\ \boldsymbol{k}$ The unit vectors in the directions of the axes, OX, OY and OZ are denoted by: $\boldsymbol{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},\quad \boldsymbol{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},\quad \boldsymbol{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$	
Every vector can be expressed in terms of the unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$. The position vector $\boldsymbol{p}$ of the point P (a, b, c) is $\boldsymbol{p} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = a\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ $= a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k}$ where a, b and c are the components of the vector $\boldsymbol{p}$	
Basic Operations: If $\boldsymbol{a} = 3\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$ and $\boldsymbol{b} = 2\boldsymbol{i} - 5\boldsymbol{j} + 3\boldsymbol{k}$ Then	
1. Calculate $\boldsymbol{a} + \boldsymbol{b}$	Add the components: $\boldsymbol{a} + \boldsymbol{b} = 5\boldsymbol{i} - 3\boldsymbol{j} + 2\boldsymbol{k}$
2. Calculate $\boldsymbol{a} - \boldsymbol{b}$	Subtract the components: $\boldsymbol{a} - \boldsymbol{b} = \boldsymbol{i} + 7\boldsymbol{j} - 4\boldsymbol{k}$
3. Calculate $\lvert \boldsymbol{a} \rvert$	$\lvert \boldsymbol{a} \rvert = \sqrt{(3^2 + 2^2 + (-1)^2)} = \sqrt{(9 + 4 + 1)} = \sqrt{14}$
4. Calculate $\lvert \boldsymbol{a} + \boldsymbol{b} \rvert$	From (1): $\boldsymbol{a} + \boldsymbol{b} = 5\boldsymbol{i} - 3\boldsymbol{j} + 2\boldsymbol{k}$ So $\lvert \boldsymbol{a} + \boldsymbol{b} \rvert = \sqrt{(5^2 + (-3)^2 + 2^2)} = \sqrt{(25 + 9 + 4)} = \sqrt{38}$
5. Express $2\boldsymbol{a} + 3\boldsymbol{b}$ in component form	$2\boldsymbol{a} + 3\boldsymbol{b} = 2\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} + 3\begin{pmatrix} 2 \\ -5 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \\ -2 \end{pmatrix} + \begin{pmatrix} 6 \\ -15 \\ 9 \end{pmatrix} = \begin{pmatrix} 12 \\ -11 \\ 7 \end{pmatrix}$
6. Express $\boldsymbol{p} = \begin{pmatrix} 4 \\ 0 \\ -5 \end{pmatrix}$ in unit vector form	$\boldsymbol{p} = 4\boldsymbol{i} - 5\boldsymbol{k}$ (Note that there is no $\boldsymbol{j}$ component)
7. $a\boldsymbol{i} + b\boldsymbol{j} + \frac{1}{2}\boldsymbol{k}$ is a unit vector. Find the relation between a and b	$a^2 + b^2 + (\frac{1}{2})^2 = 1 \quad \therefore a^2 + b^2 + \frac{1}{4} = 1 \quad \therefore a^2 + b^2 = \frac{3}{4}$

Unit 3 - 1	Vectors

Scalar Product of two vectors

The scalar product results from multiplying two vectors together.

For two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$

The scalar product is written as $\boldsymbol{a}\bullet\boldsymbol{b}$

and defined as:

$$\boldsymbol{a}\,.\,\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos\theta$$

neither $\boldsymbol{a}$ nor $\boldsymbol{b}$ being zero.
where θ is the angle between the vectors.

Note: θ is the angle between the vectors pointing OUT from the vertex

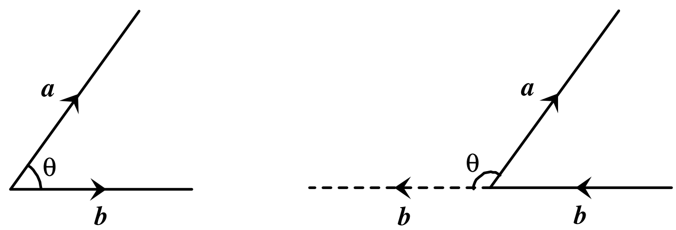
$\boldsymbol{a}\bullet\boldsymbol{b}$ is a real number, the sign of which is determined by the size of angle θ.

A practical explanation of this comes from physics.

Work done = Force x displacement = $|\mathbf{F}|\,|\boldsymbol{x}|\cos\theta$

Force and displacement are vectors (both have magnitude and direction).

The result, the work done is a **scalar** quantity.



Component form of $\boldsymbol{a}\bullet\boldsymbol{b}$

An alternative form for the scalar product can be derived using components.

$$\boldsymbol{a}\,.\,\boldsymbol{b} = x_1x_2 + y_1y_2 + z_1z_2$$

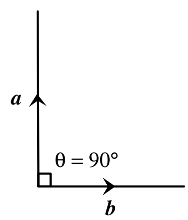
Where $\boldsymbol{a} = x_1\boldsymbol{i} + y_1\boldsymbol{j} + z_1\boldsymbol{k}$ $\quad \boldsymbol{a} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$

and $\boldsymbol{b} = x_2\boldsymbol{i} + y_2\boldsymbol{j} + z_2\boldsymbol{k}$ $\quad \boldsymbol{b} = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$

Perpendicular Vectors $\boldsymbol{a}\bullet\boldsymbol{b} = 0$

If the scalar product $\boldsymbol{a}\bullet\boldsymbol{b} = 0$
then if neither $\boldsymbol{a}$ nor $\boldsymbol{b}$ are zero,
cos θ must be zero, so θ = 90°

The vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular



Examples:

1. Calculate $\boldsymbol{a}\bullet\boldsymbol{b}$ for $|\boldsymbol{a}| = 2$, $|\boldsymbol{b}| = 5$, $\theta = \pi/6$

2. Calculate $\boldsymbol{a}\bullet\boldsymbol{b}$ for $\boldsymbol{a} = \begin{pmatrix} 2 \\ -1 \\ -3 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$

3. Calculate $\boldsymbol{p}\bullet\boldsymbol{q}$ for $\boldsymbol{p} = \begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix}$ and $\boldsymbol{q} = \begin{pmatrix} -1 \\ 4 \\ 8 \end{pmatrix}$

   What can you deduce about $\boldsymbol{p}$ and $\boldsymbol{q}$?

Solutions:

$\boldsymbol{a}\bullet\boldsymbol{b} = |\boldsymbol{a}|\,|\boldsymbol{b}|\cos\theta$ $\quad \boldsymbol{a}\bullet\boldsymbol{b} = 2 \text{ x } 5 \text{ x } \pi/6 = 10\pi/6 = 5\pi/3$

$\boldsymbol{a}\bullet\boldsymbol{b} = x_1x_2 + y_1y_2 + z_1z_2 = 2 \text{ x } 1 + (-1) \text{ x } 0 + (-3) \text{ x } (-2) = 8$

$\boldsymbol{p}\bullet\boldsymbol{q} = x_1x_2 + y_1y_2 + z_1z_2 = 4 \text{ x } (-1) + (-3) \text{ x } 4 + 2 \text{ x } 8 = 0$

Since neither $\boldsymbol{p}$ nor $\boldsymbol{q}$ are zero, then $\boldsymbol{p}$ and $\boldsymbol{q}$ are perpendicular.

Unit 3 - 1	Vectors

Angle between two vectors

The angle θ between two vectors is:

$$\cos\theta = \frac{\boldsymbol{a}.\boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}$$

Assuming that neither $\boldsymbol{a}$ nor $\boldsymbol{b}$ are zero.

Note: $\boldsymbol{a}\bullet\boldsymbol{b} = 0 \Leftrightarrow \theta = 90°$ or $\pi/2$
i.e. $\boldsymbol{a}$ is perpendicular to $\boldsymbol{b}$
assuming $\boldsymbol{a} \neq 0$, $\boldsymbol{b} \neq 0$

Remember:

θ is the angle between the vectors when they point OUT from the vertex. Choose your vectors carefully.

This is derived from the two definitions of scalar product:

$$\boldsymbol{a}\,.\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos\theta$$

$$\boldsymbol{a}\,.\boldsymbol{b} = x_1x_2 + y_1y_2 + z_1z_2$$

hence $$\cos\theta = \frac{\boldsymbol{a}.\boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|} = \frac{x_1x_2 + y_1y_2 + z_1z_2}{|\boldsymbol{a}||\boldsymbol{b}|}$$

Example:

1. Calculate the size of the angle between the vectors: $\boldsymbol{p} = \begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix}$ and $\boldsymbol{q} = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}$

Using: $\cos\theta = \dfrac{\boldsymbol{p}.\boldsymbol{q}}{|\boldsymbol{p}||\boldsymbol{q}|}$ $\quad \boldsymbol{p}\bullet\boldsymbol{q} = 6 - 3 + 10 = 13$

$|\boldsymbol{p}| = \sqrt{(3^2 + (-1)^2 + 5^2)} = \sqrt{35}$ $\quad |\boldsymbol{q}| = \sqrt{(2^2 + 3^2 + 2^2)} = \sqrt{17}$

So $\cos\theta = \dfrac{13}{\sqrt{35}\sqrt{17}} = 0.5329\ldots$ $\quad \theta = \cos^{-1}(0.5329\ldots)$

Hence $\theta = 57.8°$ (1 d.p.)

Example:

2. Calculate the size of the angle between vectors: $\boldsymbol{u} = \boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k}$ and $\boldsymbol{v} = 2\boldsymbol{i} - 3\boldsymbol{j} - 5\boldsymbol{k}$

Using: $\cos\theta = \dfrac{\boldsymbol{u}.\boldsymbol{v}}{|\boldsymbol{u}||\boldsymbol{v}|}$ $\quad \boldsymbol{u}\bullet\boldsymbol{v} = 2 - 9 + 5 = -2$

$|\boldsymbol{u}| = \sqrt{(1^2 + 3^2 + (-1)^2)} = \sqrt{11}$ $\quad |\boldsymbol{v}| = \sqrt{(2^2 + (-3)^2 + (-5)^2)} = \sqrt{38}$

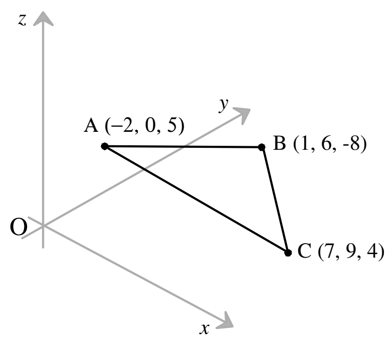
So $\cos\theta = \dfrac{-2}{\sqrt{11}\sqrt{38}} = -0.0978\ldots$ $\quad \theta = \cos^{-1}(-0.0978\ldots)$

Hence $\theta_{acute} = 84.4°$ (1 d.p.) So $\theta = 180 - 84.4° = 95.6°$

Note: $\boldsymbol{a}\bullet\boldsymbol{b} < 0 \Rightarrow \theta$ is obtuse (2nd quadrant) – because $\cos\theta < 0$

Example:

3. Calculate the size of angle ABC:



Remember – the angle is between vectors pointing OUT of the vertex.

We need the scalar product of $\overrightarrow{BA}$ and $\overrightarrow{BC}$

$\overrightarrow{BA} = \boldsymbol{a} - \boldsymbol{b} = \begin{pmatrix} -2-1 \\ 0-6 \\ 5-(-8) \end{pmatrix} = \begin{pmatrix} -3 \\ -6 \\ 13 \end{pmatrix}$ $\quad \overrightarrow{BC} = \boldsymbol{c} - \boldsymbol{b} = \begin{pmatrix} 7-1 \\ 9-6 \\ 4-(-8) \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \\ 12 \end{pmatrix}$

$\overrightarrow{BA}.\overrightarrow{BC} = \begin{pmatrix} -3 \\ -6 \\ 13 \end{pmatrix} . \begin{pmatrix} 6 \\ 3 \\ 12 \end{pmatrix} = -18 - 18 + 156 = 120$

$|\overrightarrow{BA}| = \sqrt{9+36+169} = \sqrt{214}$

$|\overrightarrow{BC}| = \sqrt{36+9+144} = \sqrt{189}$

$\cos\theta = \dfrac{\overrightarrow{BA}.\overrightarrow{BC}}{|\overrightarrow{BA}||\overrightarrow{BC}|} = \dfrac{120}{\sqrt{214}\sqrt{189}} = 0.5967\ldots$ $\quad$ So $\theta = 53.4°$

Hence $\angle ABC = 53.4°$

Unit 3 - 1 | **Vectors**

Some Results of the Scalar Product

$$\boldsymbol{a.a} = a^2$$

$$\boldsymbol{i.i} = \boldsymbol{j.j} = \boldsymbol{k.k} = 1$$

or

$$\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = 1$$

$$\boldsymbol{i.j} = \boldsymbol{i.k} = \boldsymbol{j.k} = 0$$

Using: $|\boldsymbol{a}|\,|\boldsymbol{b}|\cos\theta$

$\boldsymbol{a.a} = |\boldsymbol{a}|\,|\boldsymbol{a}|\cos 0° = |\boldsymbol{a}|\,|\boldsymbol{a}| \times 1 = a^2$ where $|\boldsymbol{a}| = a$

$\boldsymbol{i.i} = |\boldsymbol{i}|\,|\boldsymbol{i}|\cos 0° = |\boldsymbol{i}|\,|\boldsymbol{i}| \times 1 = 1 \times 1 \times 1 = 1$ where $|\boldsymbol{i}| = 1$

Obtain equivalent result for $\boldsymbol{j.j}$ and $\boldsymbol{k.k}$

$\boldsymbol{i.j} = |\boldsymbol{i}|\,|\boldsymbol{j}|\cos 90° = |\boldsymbol{i}|\,|\boldsymbol{j}| \times 0 = 1 \times 1 \times 0 = 0$ where $|\boldsymbol{i}| = 1$, $|\boldsymbol{j}| = 1$

Obtain equivalent result for $\boldsymbol{j.k}$ and $\boldsymbol{i.k}$

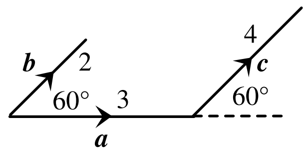
Distributive Law

$$\boldsymbol{a}.(\boldsymbol{b}+\boldsymbol{c}) = \boldsymbol{a.b} + \boldsymbol{a.c}$$

Example:

Parallel vectors $\boldsymbol{b}$ and $\boldsymbol{c}$ are inclined at 60° to vector $\boldsymbol{a}$.

$|\boldsymbol{a}| = 3$, $|\boldsymbol{b}| = 2$, $|\boldsymbol{c}| = 4$. Evaluate $\boldsymbol{a}.(\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c})$



$\boldsymbol{a}.(\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}) = \boldsymbol{a.a}+\boldsymbol{a.b}+\boldsymbol{a.c}$

$= 3^2 + 3 \times 2 \times \cos 60° + 3 \times 4 \times \cos 60°$ (since $|\boldsymbol{a}||\boldsymbol{a}| = a^2 = 3 \times 3$)

$= 9 + 6 \times ½ + 12 \times ½$

$= 18$

Example:

The vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are defined as:

$$\boldsymbol{a} = 3\boldsymbol{i} + \boldsymbol{j} + 4\boldsymbol{k}$$

$$\boldsymbol{b} = -2\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$$

$$\boldsymbol{c} = -\boldsymbol{i} + 4\boldsymbol{j} + 2\boldsymbol{k}$$

a) Evaluate $\boldsymbol{a.b} + \boldsymbol{a.c}$

b) Make a deduction about the vector $\boldsymbol{b} + \boldsymbol{c}$

Solution:

$\boldsymbol{a.b} = 3 \times (-2) + 1 \times 1 + 4 \times (-1) = -6 + 1 - 4 = -9$

$\boldsymbol{a.c} = 3 \times (-1) + 1 \times 4 + 4 \times 2 = -3 + 4 + 8 = 9$

$\boldsymbol{a.b} + \boldsymbol{a.c} = -9 + 9 = 0$ But $\boldsymbol{a.b} + \boldsymbol{a.c} = \boldsymbol{a.(b + c)}$

So $\boldsymbol{a.(b + c)} = 0$ hence $\boldsymbol{b} + \boldsymbol{c}$ is perpendicular to $\boldsymbol{a}$

Example:

Evaluate:

1. $\boldsymbol{i.(i + j)}$
2. $\boldsymbol{j.(i + k)}$
3. $\boldsymbol{i}^2 + \boldsymbol{j}^2 + \boldsymbol{k}^2$
4. $\boldsymbol{i.(i + j + k)}$

Solutions:

1. $\boldsymbol{i.(i + j) = i.i + i.j} = 1 + 0 = 1$
2. $\boldsymbol{j.(i + k) = j.i + j.k} = 0 + 0 = 0$
3. $\boldsymbol{i}^2 + \boldsymbol{j}^2 + \boldsymbol{k}^2 = 1 + 1 + 1 = 3$
4. $\boldsymbol{i.(i + j + k) = i.i + i.j + i.k} = 1 + 0 + 0 = 1$

Unit 3 - 2	Further Differentiation and Integration

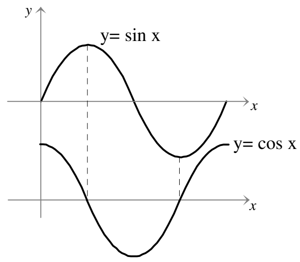
Derivative of sin x and cos x

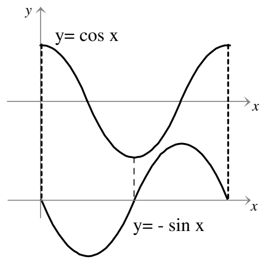
$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

If we consider the graph of y = sin x and then sketch below it, the graph of the derived function, we can deduce that the graph of the derived function is y = cos x.

Similarly we can deduce that the graph of the derived function from y = cos x is y = - sin x





y = sin x $\Rightarrow$ d/dx (sin x) = cos x   y = cos x $\Rightarrow$ d/dx (cos x) = –sin x

We can of course prove this using the limit formula.

The same rules of differentiation apply as to algebraic functions:

y = 3sin x	dy/dx = 3cos x	multiplying by a constant
y = 2cos x + sin x	dy/dx = -2sin x + cos x	y = f(x) + g(x)
y = x^2 – 4sin x	dy/dx = 2x – 4cos x	

Straight line form

The same rule applies as before when fractions are involved – get into straight line form

Example:

$$y = \frac{x^3 + x^2 \sin x}{x^2}$$

$$y = \frac{x^3}{x^2} + \frac{x^2 \sin x}{x^2} = x + \sin x \qquad \frac{dy}{dx} = 1 + \cos x$$

Examples:	**Examples:**
1. y = 2sin x	1. dy/dx = 2cos x
2. y = 1 – sin x	2. dy/dx = – cos x
3. y = 1 + cos x	3. dy/dx = – sin x
4. y = ½ cos x	4. y = – ½ sin x
5. y = sin x – cos x	5. dy/dx = cos x + sin x
6. y = 3sin x + 2cos x	6. dy/dx = 3cos x – 2sin x
7. y = x + cos x	7. dy/dx = 1 – cos x
8. y = √x – cos x	8. y = $x^{½}$ – cos x dy/dx = ½ $x^{-½}$ + sin x
9. y = x^2 + 2x – 3sin x	9. dy/dx = 2x + 2 – 3cos x
10. $y = \frac{1 - x\cos x}{x}$	10. $y = \frac{1}{x} - \frac{x\cos x}{x} = x^{-1} - \cos x \qquad \frac{dy}{dx} = -x^{-2} + \sin x$

Unit 3 - 2	Further Differentiation and Integration

Chain Rule – Algebraic functions

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

The Chain Rule applies to composite functions

or 'Functions of a function'

These will always be of the form $(....)^n$

so: $\frac{d}{dx}(........)^n = n(........)^{n-1}\frac{d}{dx}(........)$

It is important to be clear in your mind as to what the different functions are.

In function notation:

If y = f(g(x)), a composite function,

then y = f(u) and u = g(x)

and $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$$\frac{d}{dx}(f(g(x)) = f'(u) \times \frac{d}{dx}(g(x)) = f'(g(x))\frac{d}{dx}(g(x))$$

that is $\frac{d}{dx}f(........) = f'(........)\frac{d}{dx}(........)$

Note (…….) is the same function in each case – the contents of the bracket.

This is just another way of stating the rule above.

Example of composite function:

$y = (3x + 1)^3$

$f(x) = x^3$ $g(x) = 3x + 1$ $f(g(x)) = f(3x+1) = (3x + 1)^3$

Using different variables for each function we can write this as:

$y = u^3$ $u = 3x + 1$

so $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} \Rightarrow \frac{dy}{du} = 3u^2 \quad \frac{du}{dx} = 3$

$$\frac{dy}{dx} = 3u^2 \times 3 = 3(3x+1)^2 \times 3 = 9(3x+1)^2$$

With practice, we do not need to go over all these steps – it will become intuitive what you have to do.

Practical Application

Differentiate the bracket – with respect to the bracket
then multiply by the derivative of the bracket with respect to x.

$\frac{d}{d(....)} \times \frac{d(....)}{dx}$ d by d (bracket) times d (bracket) by dx

In the above example:

$y = (3x + 1)^3$

$dy/dx = 3(3x + 1)^2 \times 3 = 9(3x + 1)^2$

This will become clear and obvious with practice.

Examples:

1. $y = (x - 1)^4$
2. $y = (5x + 1)^2$
3. $y = (4 - u^2)^3$
4. $y = (t^3 - 5)^{-3}$
5. $y = \frac{1}{2x+3}$
6. $y = (x^2 + 2x)^{-1}$
7. $y = \sqrt{(t-2)(t+1)}$

Solutions:

1. $dy/dx = 4(x - 1)^3 \times 1 = 4(x - 1)^3$
2. $dy/dx = 2(5x + 1)^1 \times 5 = 10(5x + 1)$
3. $dy/du = 3(4 - u^2)^2 \times (-2u) = -6(4 - u^2)^2$
4. $dy/dt = -3(t^3 - 5)^{-4} \times 3t^2 = -9t^2(t^3 - 5)^{-4}$
5. $y = (2x + 3)^{-1}$ $dy/dx = -1(2x + 3)^{-2} = -(2x + 3)^{-2}$
6. $dy/dx = -1(x^2 + 2x)^{-2} \times (2x + 2) = -(2x + 2)(x^2 + 2x)^{-2}$
7. $y = (t^2 - t - 2)^{½}$ $dy/dt = ½ (t^2 - t - 2)^{-½} \times (2t - 1)$

   $dy/dt = ½ (2t - 1) (t^2 - t - 2)^{-½}$

Unit 3 - 2	Further Differentiation and Integration

Chain Rule – Trigonometric functions

The Chain Rule also applies to trigonometric functions. These will appear in **two** forms:

1. $y = \sin(\ldots\ldots.)$ or $y = \cos(\ldots\ldots.)$
2. $y = (\ldots.\sin x)^n$ or $y = (\ldots.\cos x)^n$

These are dealt with in exactly the same way as for algebraic functions.

1. $y = \sin(\ldots)$ $\quad \dfrac{dy}{dx} = \cos(....)\dfrac{d}{dx}(....)$

   $y = \cos(\ldots)$ $\quad \dfrac{dy}{dx} = -\sin(....)\dfrac{d}{dx}(....)$

2. $y = (\ldots.\sin x)^n$ $\quad \dfrac{dy}{dx} = n(\ldots\sin x)^{n-1}\dfrac{d}{dx}(\ldots\sin x)$

   $y = (\ldots.\cos x)^n$ $\quad \dfrac{dy}{dx} = n(\ldots\cos x)^{n-1}\dfrac{d}{dx}(\ldots\cos x)$

As with algebraic functions, it is important to be clear in your mind what the two functions are.

With practice it becomes intuitive as to what you do.

Example:

1. $y = \sin 2x$ — This is $\sin(\ldots.)$ where $(\ldots) = 2x$

   So, $dy/dx = \cos 2x \times 2$

   $\mathbf{dy/dx = 2\cos 2x}$

2. $y = (1 + \cos x)^3$ — This is $(\ldots\cos x)^3$

   where $(\ldots) = 1 + \cos x$

   So, $dy/dx = 3(\ldots)^2 \times (-\sin x)$

   $\mathbf{dy/dx = -3\sin x\,(1 + \cos x)^2}$

3. $y = \sin^3 x$ — This is $y = (\sin x)^3$

   $dy/dx = 3(\sin x)^2 \times \cos x$

   $\mathbf{dy/dx = 3\cos x \sin^2 x}$

There will only be two functions at most, all you have to do is identify them, and use the above rules.

Examples:

1. $y = \cos 5x$
2. $y = \sin(2x - 3)$
3. $y = \cos(x^2 - 1)$
4. $y = \sqrt{(\sin x)}$ Hint: write as $y = (\sin x)^{½}$
5. $y = \cos^2 x$ Hint: write as $y = (\cos x)^2$
6. $y = \dfrac{1}{\sin t}$ Hint: write as $y = (\sin t)^{-1}$
7. $y = \dfrac{3}{4\cos t}$ Hint: write as $¾(\cos t)^{-1}$
8. $y = \sin 2x + \cos 3x$
9. $y = \sqrt{(1 + \cos x)}$ Hint: write as $y = (1 + \cos x)^{½}$
10. $y = \dfrac{1}{x} - \dfrac{1}{\sqrt{\sin x}}$ Hint: write as $y = x^{-1} - (\sin x)^{-½}$
11. $y = 2\sin x\cos x$ Hint: write as $y = \sin 2x$

Solutions:

1. $dy/dx = -5\sin 5x$
2. $dy/dx = \cos(2x - 3) \times 2 = 2\cos(2x - 3)$
3. $dy/dx = -\sin(x^2 - 1) \times 2x = -2x\sin(x^2 - 1)$
4. $dy/dx = ½(\sin x)^{-½} \times \cos x = ½\cos \times (\sin x)^{-½}$
5. $dy/dx = 2(\cos x)^1(-\sin x) = -2\sin x\cos x = -\sin 2x$
6. $dy/dx = -1(\sin t)^{-2} \times \cos t = -\cos t\,(\sin t)^{-2}$
7. $dy/dx = ¾(-1)(\cos t)^{-2} \times (-\sin t) = ¾\sin t\,(\cos t)^{-2}$
8. $dy/dx = 2\cos 2x - 3\sin 3x$
9. $dy/dx = ½(1 + \cos x)^{-½} \times (-\sin x) = -½\sin x\,(1 + \cos x)^{-½}$
10. $dy/dx = -x^{-2} - (-½)(\sin x)^{-3/2}(\cos x) = -½x^{-2}\cos x(\sin x)^{-3/2}$
11. $dy/dx = 2\cos 2x$

Unit 3 - 2	Further Differentiation and Integration

Integration – Standard Integrals - 1

We will be able to integrate functions that we recognise as the result of a Chain Rule differentiation.

$$\int (ax+b)^n \, dx = \frac{(ax+b)^{n+1}}{(n+1)a} + c$$

Rather than remember the formula, it is better to understand how it is derived.

In principle –

1. Recognise the function as a Chain Rule derivative.
2. Work out what it must have come from.
3. Put in any necessary multipliers/divisors
4. Check the result by differentiation.

Sounds complicated, but again, with a little practice, it becomes second nature.

Consider:

$$\frac{d}{dx}(ax+b)^{n+1} \;=\; (n+1)(ax+b)^n \, a$$

So working backwards:

$\int (ax+b)^n \, dx$ must have come from $(ax+b)^{n+1}$

but, upon differentiation we would get the multipliers ***n + 1*** from the ***index*** and ***a*** from the ***bracket derivative***.

So we need to have these two multipliers in the **denominator** of the integrated function, in order to **cancel out** upon differentiation.

Examples:

Integrate these functions: ***(Don't forget the constant)***

1. $(x+1)^4$
2. $(3x-2)^2$
3. $(x-5)^{-2}$
4. $(5-2x)^{-3}$
5. $(2x+1)^{½}$
6. $(1-4x)^{-3/2}$
7. $\sqrt{(v+4)}$ Straight line form is: $(v+4)^{½}$
8. $\dfrac{3}{(2t+3)^4}$ Straight line form is: $3(2t+3)^{-4}$
9. $\dfrac{1}{\sqrt{(2x+3)}}$ Straight line form is $(2x+3)^{-½}$
10. $\dfrac{2}{\sqrt[3]{(1-t)}}$ Straight line form is $2(1-t)^{-1/3}$

Solutions: (Check by differentiation)

In each case consider what function it came from:

1. $(x+1)^5 \times (^1/_5) = {}^1/_5\,(x+1)^5 + c$
2. $(3x-2)^3 \times (^1/_3) \times (^1/_3) = {}^1/_9\,(3x-2)^3 + c$
3. $(x-5)^{-1} \times (-1) = -(x-5)^{-1} + c$
4. $(5-2x)^{-2} \times (-^1/_2) \times (-^1/_2) = {}^1/_4\,(5-2x)^{-2} + c$
5. $(2x+1)^{3/2} \times {}^2/_3 \times ½ = {}^2/_6\,(2x+1)^{3/2} = {}^1/_3\,(2x+1)^{3/2} + c$
6. $(1-4x)^{-1/2} \times (-2) \times (-¼) = ½\,(1-4x)^{-1/2} + c$
7. $(v+4)^{3/2} \times {}^2/_3 = {}^2/_3\,(v+4)^{3/2} + c$
8. $3(2t+3)^{-3} \times (-^1/_3) \times ½ = -½\,(2t+3)^{-3} + c$
9. $(2x+3)^{½} \times 2 \times ½ = (2x+3)^{½} + c$
10. $2(1-t)^{2/3} \times {}^3/_2 \times (-1) = -3(1-t)^{2/3} + c$

Unit 3 - 2	Further Differentiation and Integration

Integration – Standard Integrals - 2

Integration of trigonometric functions, is just the reverse of differentiation:

$$\int \cos x \; dx = \sin x + c$$

and

$$\int \sin x \; dx = -\cos x + c$$

Since:

$$\frac{d}{dx}(\sin x) = \cos x$$

and

$$\frac{d}{dx}(\cos x) = -\sin x$$

We can also integrate trigonometric functions that we recognise as the result of a Chain Rule differentiation.

$$\int \cos(ax+b) \; dx = \frac{1}{a}\sin(ax+b) + c$$

and

$$\int \sin(ax+b) \; dx = -\frac{1}{a}\cos(ax+b) + c$$

Again, by considering what it must have come from:

$$\frac{d}{dx}\sin(ax+b) = a\cos(ax+b)$$

and

$$\frac{d}{dx}\cos(ax+b) = -a\sin(ax+b)$$

Rather than remember the formula, again, it is better to understand how it is derived.

In principle –

1. Recognise the function as a Chain Rule derivative.
2. Work out what it must have come from.
3. Put in any necessary multipliers/divisors
4. Check the result by differentiation.

So we need to have the multiplier in the **denominator** of the integrated function, in order to **cancel out** upon differentiation.

Sounds complicated, but again, with a little practice, it becomes second nature.

Examples:

Integrate these functions: ***(Don't forget the constant)***

1. 3cos x
2. 5sin x
3. sin 4x
4. 5 cos 2x
5. 3 sin ½ x
6. cos(x + 2)
7. sin(3x + 4)
8. sin 2x + cos 3x
9. t^2 + 2cos 2t

Solutions: (Check by differentiation)

In each case consider what function it came from:

1. 3sin x + c
2. –5cos x + c
3. –cos 4x × (¼) = – ¼ cos 4x + c
4. 5 sin 2x × ½ = $^5/_2$ sin 2x + c
5. –3 cos ½ x × 2 = –6 cos ½ x + c
6. sin(x + 2) + c
7. –cos (3x + 4) × $^1/_3$ = –$^1/_3$ cos (3x + 4) + c
8. –cos 2x × ½ + sin 3x × $^1/_3$ = –½ cos 2x + $^1/_3$ sin 3x + c
9. $^1/_3$ t^3 + 2sin 2t × ½ = $^1/_3$ t^3 + sin 2t + c

Unit 3 - 2 Further Differentiation and Integration

Definite Trigonometric Integrals

Definite integrals of trigonometric functions are handled in exactly the same way as definite integrals of algebraic functions.

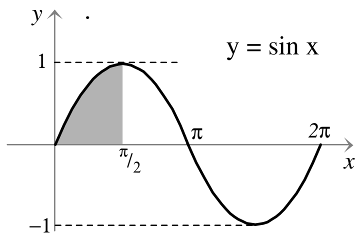
The limits are **ALWAYS** in **radians**.

The integral represents the area between the curve and the x-axis.

Areas below the x-axis are NEGATIVE.

$$\int_0^{\pi/2} \sin x \; dx$$

The above integral represents the shaded area on the graph.



$$\int_0^{\pi/2} \sin x \; dx = [-\cos x]_0^{\pi/2} = \left(-\cos\frac{\pi}{2}\right) - (-\cos 0)$$

= – 0 – (–1) = 1

Examples:

1. $\displaystyle\int_{\pi/6}^{\pi/4} (1+\sin 2x)\, dx$

$$= \left[x - \frac{1}{2}\cos 2x\right]_{\pi/6}^{\pi/4} = \left(\frac{\pi}{4} - \frac{1}{2}\cos\frac{\pi}{2}\right) - \left(\frac{\pi}{6} - \frac{1}{2}\cos\frac{\theta}{3}\right)$$

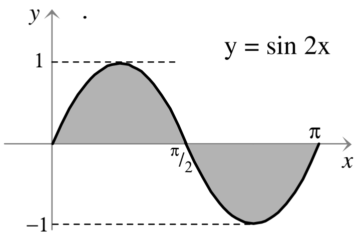
$$= \left(\frac{\pi}{4} - 0\right) - \left(\frac{\pi}{6} - \frac{1}{2}\times\frac{1}{2}\right) = \frac{\pi}{12} + \frac{1}{4}$$

2. $\displaystyle\int_0^{\pi} (\sin t + \cos t)\, dt$

$$= [-\cos t + \sin t]_0^{\pi} = (-\cos\pi + \sin\pi) - (-\cos 0 + \sin 0)$$

$$= (-(-1)+0) - (-1+0) \;=\; 1 + 1 = 2$$

3. Calculate the total area of the shaded region.



We cannot integrate between 0 and π because the areas above and below the x-axis will cancel out to zero.

We split the integral into two parts: from **0 to π/2** and from **π/2 to π.**

The second integral will be negative (below the x-axis) so we ignore the negative sign (since an area is always positive).

We then add the two areas together. However, by symmetry, the area below the x-axis is the same as that above the x-axis, apart from the sign.

Area above x-axis is $\displaystyle\int_0^{\pi/2} \sin 2x \; dx = \left[-\frac{1}{2}\cos 2x\right]_0^{\pi/2}$

$$= \left(-\frac{1}{2}\cos 2\times\frac{\pi}{2}\right) - \left(-\frac{1}{2}\cos 2\times 0\right) = \left(-\frac{1}{2}\cos\pi\right) - \left(-\frac{1}{2}\cos 0\right)$$

$$= \left(-\frac{1}{2}(-1)\right) - \left(-\frac{1}{2}(1)\right) = \frac{1}{2} - \left(-\frac{1}{2}\right) = \frac{1}{2} + \frac{1}{2} = 1$$

So total area is twice this. **Hence total shaded area = 2**

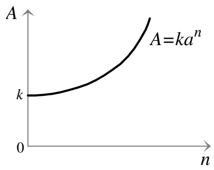
Unit 3 - 3 The Exponential and Logarithmic Functions

Growth Function

This is of the form:

$$A(n) = ka^n$$

with a > 1



Examples of growth functions:

Bank Account – compound interest

£200 at 7% for 6 years. Amount after 6 years A = 200×1.07^6

Population growth

Now 47,000 growth 3% per year. Population after 9 years A = $47\,000 \times 1.03^9$

Appreciation

House cost £55 000 when purchased. It appreciates at 4% for 25 years.

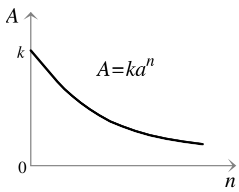
Value after 25 years A = $55\,000 \times 1.04^{25}$

Decay Function

This is of the form:

$$A(n) = ka^n$$

with a < 1



Examples of decay functions:

Evaporation

Initially 10 litres – evaporates at 5% per hour (**NB** loses 5% means 95% remains)

After 15 hours amount left is: A = 10×0.95^{15}

Population decline

Was 20,000 declines 3% per year. (NB declines 3% means 97% remains)

Population after 20 years A = $20\,000 \times 0.97^{20}$

Depreciation

Car cost £23 000 depreciates 20% each year. (NB loses 20% means worth 80%)

Value after 3 years A = $23\,000 \times 0.8^3$

Examples:

1. An open can is filled with 2 litres of cleaning fluid, which evaporates at the rate of 30% per week. Construct a function for the amount of fluid (in millilitres) left after *t* weeks.

   Calculate how much fluid remains after 6 weeks.

2. A population of 100 cells increases by 60% per hour. Construct a function to show the number of cells after after *h* hours.

   Calculate how many cells there would be after 12 hours

3. Radium has a half life of 1600 years. This means that a given mass of radium will decay steadily and be halved in 1600 years.

   Check that, starting with 5g of radium, the decay function for the mass after *t* years is

   $$R(t) = 5(0.5)^{t/1600}$$

   Calculate the mass remaining after 400 years.

4. Construct a decay function for Carbon-14 which has a half-life of 5720 years. Using C_0 for the initial amount of carbon-14 present.

Solutions:

1. 30% evaporation, means that 70% remains

   After 1 week A = 2000×0.7 mls remain

   After t weeks **A(t) = $\mathbf{2000 \times 0.7^t}$** mls remain.

   After 6 weeks A(6) = 2000×0.7^6 mls = **235.3 mls remain**

2. After one hour number of cells N = 100×1.6

   After h hours number of cells **N(*h*) = $\mathbf{100 \times 1.6^{h}}$**

   After 12 hours number of cells N(12) = 100×1.6^{12} = **28, 147 cells**

3. If $R(t) = 5(0.5)^{t/1600}$ then put t = 1600 (half life)

   which gives R(t) = 5×0.5^1 = 2.5 g which is correct.

   After 400 years $R(400) = 5(0.5)^{400/1600} = 5(0.5)^{0.25}$ = 4.2 g

4. $C(t) = C_0(0.5)^{t/5720}$

<table>
<tr><td>Unit 3 - 3</td><td>The Exponential and Logarithmic Functions</td></tr>
<tr><td>The exponential function
An exponential function is of the form
a^x
where a is a constant.
If $a > 0$, the function is increasing (growth)
If $a < 0$, the function is decreasing (decay)
a may take any positive value
depends on situation function is modelling.</td><td>Note: In general an exponential function will take the form:
$A(x) = ab^x$
where both a and b are constants.
a will represent an initial value
b will represent the multiplier
x will represent the variable</td></tr>
<tr><td>A special exponential function ~ e^x
e^x
e is a special constant
– a never ending decimal like π.
e = 2.718 282 828 ….</td><td>The number e crops up on many occasions in the natural world.
It is: $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$
You can find this by pressing the e^x key on your calculator followed by ‘1 =’
This effectively is evaluating e^1.</td></tr>
<tr><td>Linking the exponential and logarithmic functions.
$y = a^x \Leftrightarrow \log_a y = x$
$1 = a^0 \Leftrightarrow \log_a 1 = 0$
$a = a^1 \Leftrightarrow \log_a a = 1$</td><td>Use this relationship to switch between log and exponential forms.
Use these two relationships to simplify and evaluate logarithmic and exponential functions and expressions.</td></tr>
<tr><td>Examples:
1. Write in log form: $81 = 3^4$
2. Write in log form: $y^4 = 20$
3. Write in log form: $^1/_9 = 3^{-2}$
4. Write in log form: $z^{½} = 10$
5. Write in exp. form: $\log_2 4 = 2$
6. Write in exp. form: $\log_{10} 100 = 2$
7. Write in exp. form: $\log_9 3 = ½$
8. Write in exp. form: $\log_8 4 = {}^2/_3$
9. Write in exp. form: $\log_a c = b$
10. Solve: $\log_x 9 = 2$
11. Solve: $\log_4 x = 0.5$
12. Solve: $\log_3 81 = x$
13. Solve: $\log_x 7 = 1$
14. Solve: $\log_{10} x = 0.5$</td><td>Solutions:
1. $\log_3 81 = 4$
2. $\log_y 20 = 4$
3. $\log_3 {}^1/_9 = -2$
4. $\log_z 10 = ½$
5. $2^2 = 4$
6. $10^2 = 100$
7. $9^{½} = 3$
8. $8^{2/3} = 4$ i.e. $(^3\sqrt{8})^2 = 4$
9. $a^b = c$
10. $x^2 = 9$ so $x = 3$
11. $4^{0.5} = x$ so $4^{½} = x$ $\sqrt{4} = x$ $x = 2$
12. $3^x = 81$ so $x = 4$
13. $x^1 = 7$ $x = 7$
14. $10^{0.5} = x$ Use calculator 10 y^x 0.5 = 3.162… x = 3.16 (2 d.p)</td></tr>
</table>

Unit 3 - 3	The Exponential and Logarithmic Functions

Rules of Logarithms

$$\log_a xy = \log_a x + \log_a y$$

$$\log_a \frac{x}{y} = \log_a x - \log_a y$$

$$\log_a x^p = p \log_a x$$

when working with logs, always be on the lookout for powers of the base,

this will enable you to simplify expressions

These are derived from the corresponding Rules of Indices

$$a^m \times a^n = a^{m+n}$$

$$a^m \div a^n = a^{m-n}$$

$$\left(a^m\right)^p = a^{mp}$$

Proofs:

Let $\log_a x = m$ $\log_a y = n$ then $a^m = x$ and $a^n = y$

1. $xy = a^m \times a^n = a^{m+n}$ so $xy = a^{m+n}$ $\Rightarrow$ $\log_a xy = m + n$
   $\log_a xy = m + n$ $\Rightarrow$ $\mathbf{\log_a xy = \log_a x + \log_a y}$
2. $x/y = a^m \div a^n = a^{m-n}$ so $xy = a^{m-n}$ $\Rightarrow$ $\log_a x/y = m - n$
   $\log_a x/y = m - n$ $\Rightarrow$ $\mathbf{\log_a x/y = \log_a x - \log_a y}$
3. $x^p = (a^m)^p = a^{mp}$ so $x^p = a^{mp}$ $\Rightarrow$ $\log_a x^p = mp$
   $\log_a x^p = mp$ $\Rightarrow$ $\mathbf{\log_a x^p = p \log_a x}$

Examples:

Simplify – assume same base

1. log 7 + log 2
2. log 12 – log 2
3. log 6 + log 2 – log 3
4. log 2 + 2 log 3
5. 2 log 3 + 3 log 2

Simplify and evaluate

6. $\log_8 2 + \log_8 4$
7. $\log_5 100 - \log_5 4$
8. $\log_4 18 - \log_4 9$
9. $2 \log_{10} 5 + 2 \log_{10} 2$
10. $3 \log_3 3 + ½ \log_3 9$
11. $5 \log_8 2 + \log_8 4 - \log_8 16$
12. $\log_2 (½) - \log_2 (¼)$

Solve for x:

13. $\log_a x + \log_a 2 = \log_a 10$
14. $\log_a x - \log_a 5 = \log_a 20$
15. $\log_a x + 3\log_a 3 = \log_a 9$

Solutions:

1. $\log 7 \times 2 = \log 14$
2. $\log 12 \div 2 = \log 6$
3. $\log (6 \text{ x } 2 \div 3) = \log 4$
4. $\log 2 + \log 3^2 = \log 2 \times 3^2 = \log 18$
5. $\log 3^2 + \log 2^3 = \log 9 + \log 8 = \log 9 \times 8 = \log 72$

6. $\log_8 2 \times 4 = \log_8 8 = 1$
7. $\log_5 (100 \div 4) = \log_5 25 = \log_5 5^2 = 2 \log_5 5 = 2$
8. $\log_4 (18 \div 9) = \log_4 2 = \log_4 4^{½} = ½$
9. $\log_{10} 5^2 + \log_{10} 2^2 = \log_{10} (25 \times 4) = \log_{10} 100 = \log_{10} 10^2 = 2$
10. $\log_3 27 + \log_3 9^{½} = \log_3 27 \times 3 = \log_3 81 = \log_3 3^4 = 4$
11. $\log_8 32 + \log_8 4 - \log_8 16 = \log_8 (32 \times 4 \div 16) = \log_8 8 = 1$
12. $\log_2 2^{-1} - \log_2 (^1/_{2^2}) = \log_2 2^{-1} - \log_2 2^{-2} = -1 - (-2) = 1$

13. $\log_a 2x = \log_a 10$ $\therefore$ $2x = 10$ $\therefore$ $x = 5$
14. $\log_a (x/5) = \log_a 20$ $\therefore$ $x/5 = 20$ $\therefore$ $x = 100$
15. $\log_a x + 3\log_a 3 = \log_a 9$ $\therefore$ $\log_a x + \log_a 27 = \log_a 9$
    $\log_a 27x = \log_a 9$ $\therefore$ $27x = 9$ $\therefore$ $x = {}^1/_3$

Unit 3 - 3	The Exponential and Logarithmic Functions

Calculator keys

log - means $\log_{10}$ (common log)

ln - means $\log_e$ (natural log)

y^x - means raise to the power of

e^x - means e raised to the power of

10^x - means 10 raised to the power of

You will need to use the above keys, when solving exponential or logarithmic equations.

Evaluate:

$\log_{10} 2 \Rightarrow$	log	2	=	0.3010….		
$\log_e 5 \Rightarrow$	ln	5	=	1.6094 ….		
$5^{0.2} \Rightarrow$	5	y^x	0.2	=	1.3797….	
$e^{1.7} \Rightarrow$	2^{nd} Fn	ln/e^x	1.7	=	5.4739…	
$10^{0.3010} \Rightarrow$	2^{nd} Fn	log/10^x	0.3010	=	1.99986…	
or $\Rightarrow$	10	y^x	0.3010	=	1.99986…	

Solving exponential equations.

Solving equations of the type:

1. $5^x = 4$

   Take $\log_{10}$ of both sides.

   Changes to log form: $\log_{10} 5^x = \log_{10} 4$

   $x \log_{10} 5 = \log_{10} 4$

   $x = \log_{10} 4 \div \log_{10} 5 = 0.8613 \ldots$

2. $20 = e^t$

   Take $\log_e$ of both sides.

   Changes to log form: $\log_e 20 = \log_e e^t$

   $\log_e 20 = t \log_e e$ (but $\log_e e = 1$)

   $t = \log_e 20 = 2.9957\ldots$

(Always choose $\log_e$ when dealing with growth or decay functions with e as the base because $\log_e$ e, makes calculation simpler)

In both the above cases other constants many be involved.

Examples:

1. Solve: $8 \times 0.6^x = 16$
2. $D(t) = 500 (0.65)^t$

   For what value of t does D(t) = 2
3. Solve: $e^{3t} = 120$
4. $S(t) = 225 e^{-0.36t}$

   For what value of t is S(t) = 70

Solutions:

1. $0.6^x = 2$ $\quad \log_{10} 0.6^x = \log_{10} 2$ $\quad x \log_{10} 0.6 = \log_{10} 2$

   $x = \log_{10} 2 \div \log_{10} 0.6$ $\quad$ **$x = -1.36$**

2. $2 = 500 (0.65)^t$ $\quad 0.004 = 0.65^t$ $\quad \log_{10} 0.004 = \log_{10} 0.65^t$

   $\log_{10} 0.004 = t \log_{10} 0.65$ $\quad t = \log_{10} 0.004 \div \log_{10} 0.65$

   $t = 12.8$

3. $\log_e e^{3t} = \log_e 120$ $\quad 3t \log_e e = \log_e 120$ $\quad 3t = \log_e 120$

   $t = (\log_e 120) \div 3$ (careful here) $\quad$ **$t = 1.596$**

4. $70 = 225 e^{-0.36t}$ $\quad 70 \div 225 = e^{-0.36t}$ $\quad 0.3111 = e^{-0.36t}$

   $\log_e 0.3111 = \log_e e^{-0.36t}$ $\quad \log_e 0.3111 = -0.36t \log_e e$

   $\log_e 0.3111 = -0.36t$ $\quad t = \log_e 0.3111 \div (-0.36)$ $\quad$ **$t = 3.243$**

Unit 3 - 3	The Exponential and Logarithmic Functions

Experiment and Theory

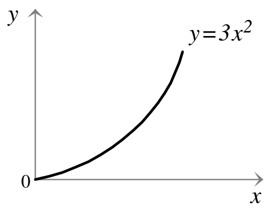
In experimental work, data can often be modelled by equations of the form:

$$y = ax^n \quad \text{(polynomial)}$$

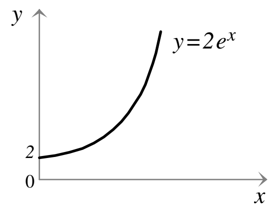
or

$$y = ab^x \quad \text{(exponential)}$$

both are similar.



polynomial graph



exponential graph

By taking logs of both sides of the above equations we find that the graph of each is a straight line.

A **polynomial** graph is a straight line when **log x** is plotted against **log y**

An **exponential** graph is a straight line when **x** is plotted against **log y**

So when we have a graph or a table of data, we find the **gradient** and the **y-intercept** of the straight line.

You will be given the relationship in the question.

Take logs of both sides of the given relationship (base 10 or base e according to the question)

Equate **log a** to the **y-intercept**.
Equate **n** or **log b** to the **gradient**

Solve these equations to calculate the constants.

Proof:

$y = ax^n$	$y = ab^x$
$\log y = \log ax^n$	$\log y = \log ab^x$
$\log y = \log a + \log x^n$	$\log y = \log a + \log b^x$
$\log y = \log a + n \log x$	$\log y = \log a + x \log b$
This looks like:	This looks like:
$Y = \log a + n X$	$Y = \log a + X \log b$
where n is the gradient and log a is the y-intercept.	where log b is the gradient and log a is the y-intercept.

Example:

The following data was obtained from an experiment

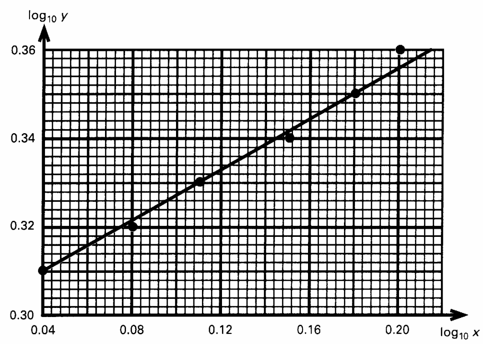
x	1.1	1.2	1.3	1.4	1.5	1.6
y	2.06	2.11	2.16	2.21	2.26	2.30

Logs were taken of data as shown below

$\log_{10} x$	0.04	0.08	0.11	0.15	0.18	0.20
$\log_{10} y$	0.31	0.32	0.33	0.34	0.35	0.36

a graph was plotted – the line of best fit showing a straight line. An equation of the form $y = ax^n$ is suggested.

Find the values of a and n



Suggested relation is $y = ax^n$

Take $\log_{10}$ of both sides $\log_{10} y = \log_{10} ax^n$

$\Rightarrow \log_{10} y = \log_{10} a + \log_{10} x^n$

$\Rightarrow \log_{10} y = \log_{10} a + n \log_{10} x$ …….. (1)

This is a straight line with:

y-intercept = $\log_{10} a$

gradient = n

From the graph y-intercept = 0.31 and gradient = 0.29

i.e. $\log_{10} a = 0.31$ So $a = 10^{0.31} = 2.0$ (1 d.p.)

$n = 0.29 = 0.3$ (1 d.p.) So relationship is: $y = 2x^{0.3}$

OR pick two points on the line i.e. (0.04, 0.31) and (0.18, 0.35)

Substituting into (1) above: $0.31 = 0.04n + \log_{10} a$

$0.35 = 0.18n + \log_{10} a$

Subtracting gives $n = 0.29$, $\log_{10} a = 0.3$ $\therefore a = 10^{0.3} = 2$ (1 d.p.)

Again this gives the relationship of: $y = 2x^{0.3}$

Example

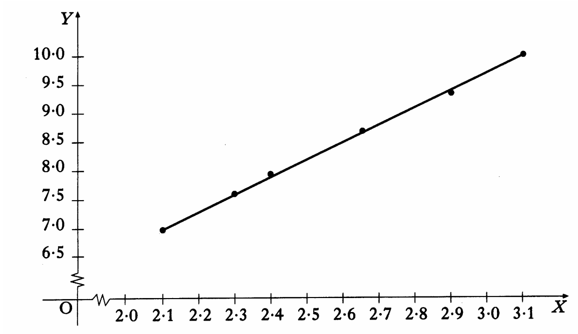
Six spherical sponges were dipped in water and weighed to see how much water each could absorb.
The diameter (x millimetres) and gain in weight (y grams) were measured and recorded for each sponge.

It is thought that x and y are connected by a relationship of the form $y = ax^b$

By taking logarithms of the values of x and y, this table was constructed.

$X(=\log_e x)$	2.10	2.31	2.40	2.65	2.90	3.10
$Y(=\log_e y)$	7.00	7.60	7.92	8.70	9.38	10.00

A graph was drawn and is shown here.



a) Find the equation of the line in the form $Y = mX + c$

b) Hence find the values of the constants a and b in the relationship $y = ax^b$

Solution:

a)

$$y = ax^b$$
$$\log_e y = \log_e ax^b$$
$$\log_e y = \log_e a + \log_e x^b$$
$$\log_e y = \log_e a + b\log_e x$$

This is of the form Y = mX + c where m = b and $\log_e a = c$

b) Choose two points on the line of best fit. (2.1, 7.0) and (3.1, 10.0)

Substitute into $\log_e y = \log_e a + b\log_e x$

giving: $7.0 = \log_e a + 2.1\,b$ ….. (1)

$10.0 = \log_e a + 3.1\,b$ ….. (2)

subtracting: (2) – (1) $\Rightarrow$ 3.0 = b substituting $\Rightarrow$ $\log_e a = 0.7$ so $a = e^{0.7}$ a = 2.01…

Hence relationship is: $y = 2x^3$ i.e. a = 2.0 and b = 3.0 (1 d.p.)

Note: You should be confident in applying the method in part (b) rather than relying on the gradient and y-intercept, as in this case, you cannot determine the y-intercept.

Example

Find the relation $y = ab^x$ for this data

x	2.15	2.13	2.00	1.98	1.95	1.93
y	83.33	79.93	64.89	62.24	59.70	57.26

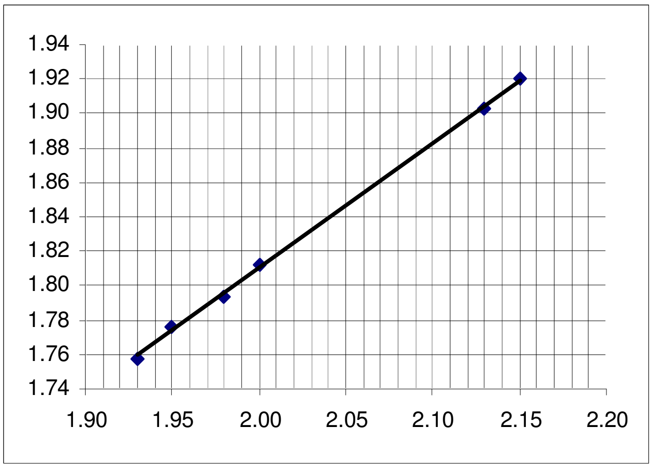
Solution:

$y = ab^x$

$\log_{10} y = \log_{10} ab^x$

$\log_{10} y = \log_{10} a + \log_{10} b^x$

$\log_{10} y = \log_{10} a + x \log_{10} b$



Add a row to the table showing $\log_{10}$ y

Plot data **$\log_{10}$ y** against **x**
(because relationship is exponential)
to determine **line of best fit** which will indicate which points to use.

x	**2.15**	**2.13**	**2.00**	**1.98**	**1.95**	**1.93**
y	83.33	79.93	64.89	62.24	59.70	57.26
$\log_{10}$ y	**1.92**	**1.90**	**1.81**	**1.79**	**1.78**	**1.76**

From graph, choose points (1.93, 1.76) and (2.15, 1.92) corresponding to (x, $\log_{10}$ y)

Substituting into $\log_{10} y = \log_{10} a + x \log_{10} b$

gives: $1.92 = \log_{10} a + 2.15 \log_{10} b$ ….. (1)

and: $1.76 = \log_{10} a + 1.93 \log_{10} b$ ….. (2)

Subtracting: (1) – (2) $0.16 = 2.15 \log_{10} b - 1.93 \log_{10} b$

$0.16 = 0.22 \log_{10} b$

$\log_{10} b = 0.727$

$b = 10^{0.727} = 5.3$ (1 d.p.)

Substituting into (1) $\Rightarrow$ $\log_{10} a = 1.92 - 2.15 \log_{10} 5.3$

$\log_{10} a = 1.92 - 1.56$

$\log_{10} a = 0.36$

$a = 10^{0.36} = 2.29 = 2.3$ (1 d.p.)

Hence relationship is: $y = 2.3\,(5.3)^x$

Unit 3 - 4 | The Wave Function a cos x + b sin x

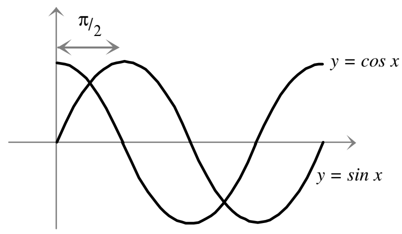
When two waves of the form a cos x + b sin x are combined together, the result is a sine or cosine wave that is shifted in phase from the original waves.

The wave function

We can express a cos x + b sin x in the form of a single wave.

This can be a sine or a cosine wave, since a cosine wave is simply a sine shifted 90° to the left.

This single wave is called the wave function.



R cos (x ± α) and R sin (x ± α)

There are four different forms we can use – all of these are equivalent – we choose whatever is convenient. You will always be given the appropriate form in the question.

Expressing a cos x + b sin x as R cos (x ± α) or R sin (x ± α)

Example:

Express 3 cos x + 5 sin x
in the form R cos (x - α)

Step 1.
Expand R cos (x - α)

$R \cos(x - \alpha) = R \cos x \cos \alpha + R \sin x \sin \alpha$

Step 2.
Compare coefficients of sin x and cos x

$R \sin \alpha = 5 \quad \ldots\ldots (1)$
$R \cos \alpha = 3 \quad \ldots\ldots (2)$

Step 3.
Square and add to obtain R

$R^2 \sin^2 \alpha + R^2 \cos^2 \alpha = 5^2 + 3^2$
$R^2(\sin^2 \alpha + \cos^2 \alpha) = 5^2 + 3^2$

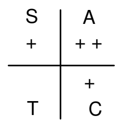
Note: $\sin^2 \alpha + \cos^2 \alpha = 1$ so, $R^2 = 5^2 + 3^2$ $R^2 = 34$ $\mathbf{R = \sqrt{34}}$

Step 4.
Divide the sin α equation by the cos α equation. This gives you tan α.

$\frac{R \sin \alpha}{R \cos \alpha} = \frac{5}{3}$ note that $\tan \alpha = \frac{\sin \alpha}{\cos \alpha}$ so $\tan \alpha = \frac{5}{3}$

Step 5.
Identify the quadrant for α by looking at the two equations obtained in step 2.

From equation (1) and (2) look at the signs of sin α and cos α ; sin α is +, cos α is +
These conditions both apply in 1st quadrant only.



Step 6.
Calculate α

$\tan \alpha = \frac{5}{3} \Rightarrow \alpha = 59.036\ldots \quad \alpha = 59°$

Step 7.
Put it all together

$\therefore$ 3 cos x + 5 sin x = $\mathbf{\sqrt{34} \cos (x - 59)}°$

Always use this method of setting out your working.

Do NOT try to remember formulae for this. Work it out !

Unit 3 - 4 The Wave Function a cos x + b sin x

We have shown that:

$3\cos x + 5\sin x = \sqrt{34}\cos(x - 59)°$

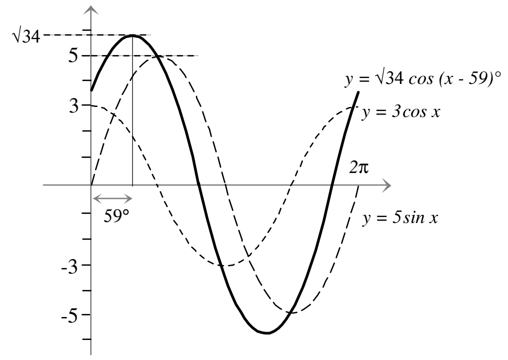
The combined waveform is a cosine wave, of

amplitude $\sqrt{34}$
periodicity – same as original waves (2π)
phase shift is 59° to the right.

This procedure allows us to:

i) investigate maximum and minimum values and where they occur.

ii) solve the equation
$3\cos x + 5\sin x = \text{constant}$



Maximum and minimum values

Example:

Find the maximum and minimum values of:

$3\cos x + 5\sin x$

for $0 \le x \le 360°$

and state the values of x at which they occur.

Solution:

Express the two functions as a single function
– in the form of $R\cos(x \pm \alpha)$ or $R\sin(x \pm \alpha)$

Since we have already done this above, we shall use the above result:
and express $3\cos x + 5\sin x$ as $\sqrt{34}\cos(x - 59)°$

The cosine has a maximum value of 1 and a minimum value of –1
The maximum occurs when cos (…) = 0° and 360° (0 or 2π radians)
The minimum occurs when cos (…) = 180° (π radians)

$\therefore$ max value of $\sqrt{34}\cos(x - 59)°$ is $\sqrt{34}$
this occurs when $x - 59 = 0$ and $x - 59 = 360$
i.e. $x = 59°$ or $x = 419°$ (discard 419° as out of range)

$\therefore$ min value of $\sqrt{34}\cos(x - 59)°$ is $-\sqrt{34}$
this occurs when $x - 59 = 180$ i.e. $x = 239°$

Hence maximum value is $\sqrt{34}$ when $x = 59°$
and minimum value is $-\sqrt{34}$ when $x = 239°$

This result also tells us that there is
a **maximum turning point at (59°, √34)**
and a **minimum turning point at (239°, –√34)**.

Solving Equations

Example:

Solve the equation: $3\cos x + 5\sin x - 2 = 0$

for $0 \le x \le 360°$

Solution:

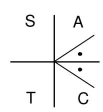
Express $3\cos x + 5\sin x$ in the form of $R\cos(x \pm \alpha)$ or $R\sin(x \pm \alpha)$

Since we have already done this above, we shall use the above result:
and express $3\cos x + 5\sin x$ as $\sqrt{34}\cos(x - 59)°$

The equation we have to solve becomes:

$$\sqrt{34}\cos(x - 59) = 2$$
$$\therefore \cos(x - 59) = 2/\sqrt{34}$$
$$\therefore \cos(x - 59) = 0.3430$$
$$\therefore \text{acute } (x - 59) = 69.9°$$

cosine is positive, so angle lies in 1st or 4th quadrants.



so $x - 59 = 69.9$ or $x - 59 = 360 - 69.9$

Hence x = 128.9° or 349.1°

Unit 3 - 4	The Wave Function a cos x + b sin x

Examples:

1. Solve for $0 \le x \le 180$ $\quad 6 \cos(3x + 60) - 3 = 0$

$6 \cos(3x + 60) = 3$

$\cos(3x + 60) = 0.5$ so, acute $(3x + 60) = 60°$

The range for x is: $0 \le x \le 180$ so the range for 3x is: $0 \le x \le 540$

The cosine is positive, so the required quadrants are 1st, 4^{th} and 5^{th} (1^{st} quadrant – second time around)

$\therefore\ 3x + 60 = 60 \qquad 3x + 60 = 360 - 60 \qquad 3x + 60 = 360 + 60$

$\therefore$ **x = 0°, 80° or 120°**

2. i) Express $\sqrt{3} \cos x - \sin x$ in the form $k \sin(x - \alpha)$
   ii) and hence solve the equation $\sqrt{3} \cos x - \sin x = 0$ for $0 \le x \le 360$

i) $k \sin(x - \alpha) = k \sin x \cos \alpha - k \cos x \sin \alpha$

comparing coefficients:	$-k \sin \alpha = \sqrt{3}$	$k \sin \alpha = -\sqrt{3}$	… (1)
	$k \cos \alpha = -1$	$k \cos \alpha = -1$	… (2)
squaring and adding:	$k^2 = (\sqrt{3})^2 + 1^2$	$k^2 = 3 + 1 = 4$	$k = 2$
dividing:	$\tan \alpha = \sqrt{3}$	acute $\alpha = 60°$	
from (1) and (2)	$\sin \alpha$ and $\cos \alpha$ both negative, so α lies in 3^{rd} quadrant		
	$\therefore\ \alpha = 180 + 60° = 240°$		

Hence: $\mathbf{\sqrt{3} \cos x - \sin x = 2 \sin(x - 240)}$

ii) Using $2 \sin(x - 240) = 0 \qquad \sin(x - 240) = 0 \qquad (x - 240) = -180°,\ 0°,\ 180°$, or $360°$

$\therefore$ **x = 60° or x = 240°**

(because we are adding 240°, we need to make sure we cover all the range, so we need to consider the solution –180° as well, we do not need to go any further back, since we would be then out of the range)

3. Using $R \cos(2x - \alpha)$, find the maximum and minimum values of : $4 \cos 2x + 3 \sin 2x + 5$
   and the corresponding values for x in $0 \le x \le 2\pi$.

$R \cos(2x - \alpha) = R \cos 2x \cos \alpha + R \sin 2x \sin \alpha$

compare coefficients:	$R \sin \alpha = 3$		
	$R \cos \alpha = 4$		
squaring and adding:	$R^2 = 3^2 + 4^2$	$R^2 = 25$	$R = 5$
dividing:	$\tan \alpha = ¾$	acute $\alpha = 0.643$ rad	

$\sin \alpha$ and $\cos \alpha$ both positive, so α is in first quadrant,

Hence: $4 \cos 2x + 3 \sin 2x + 5$ can be expressed as: $5 \cos(2x - 0.643) + 5$

Maximum value is: **10** when $(2x - 0.643) = 0,\ 2\pi$, or 4π (since we have 2x and not x)
when x = 0.32 rad, 3.46 rad (6.60 rad – discard – out of range)

Minimum value is: **0** when $(2x - 0.643) = \pi$ or 3π (since we have 2x and not x)
when x = 1.89 rad or 5.03 rad.